



The relativistic bound states of a non-central potential

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Abstract. We investigate the relativistic effects of a moving particle in the field of a pseudoharmonic oscillatory ring-shaped potential under the spin and pseudospin symmetric Dirac wave equation. We obtain the bound-state energy eigenvalue equation and the corresponding two-components spinor wave functions by using the formalism of supersymmetric quantum mechanics (SUSYQM). Furthermore, the non-relativistic limits are obtained by simply making a proper replacement of parameters. The thermodynamic properties are also studied. Our numerical results for the energy eigenvalues are also presented.

Keywords. Dirac wave equation; supersymmetric quantum mechanics formalism; pseudoharmonic oscillatory ring-shaped potential.

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1. Introduction

It is known that the study of the relativistic wave equations plays an important role in different fields of modern physics. This started, by solving the spin-1/2 Dirac equation, as one of the most challenging wave equations in the past 80 years. The Dirac wave equation is mainly used for describing particle dynamics in relativistic quantum mechanics, the behaviour of nucleons in nuclei and the relativistic collisions of heavy ions and interaction of laser with matter. Recently, many researchers have been working on the exact solution of the Dirac equation with different non-central potentials [1–5]. The near realization of these symmetries may explain degeneracies in some heavy meson spectra (spin symmetry) or in single-particle energy levels in nuclei (pseudospin symmetry) [6–8]. The spin and pseudospin symmetries are $SU(2)$ -type symmetries of a Dirac Hamiltonian. They have been studied since 1969 in quasidegeneracy. Besides, these symmetries were considered in the context of deformed nuclei [9], the superdeformation [10], the magnetic moment interpretation [11,12], the identical bands [13–16] and the effective shell-model coupling scheme [17]. The

pseudospin symmetry occurs in the Dirac equation when $S(r) = -V(r)$, and the spin symmetry which is relevant to mesons appears in the Dirac equation when the subtraction of the scalar potential component $S(r)$ from time-like vector potential component $V(r)$ is equal to zero [18–20].

Non-central potentials are applied mainly in quantum chemistry and atomic physics. It is extensively used to describe many properties of some of the ring-shaped organic molecules (such as benzene molecules) and also to study the interactions between deformed nucleons, that is, they are widely used in quantum chemistry and nuclear physics. Therefore, it is interesting and necessary to study the solutions of the Schrödinger, Klein–Gordon and Dirac wave equations with such non-central potentials [21–34].

The ring-shaped pseudoharmonic oscillatory (RSPHO) potential is one such physical potential. The spherical interaction potential takes the most general form:

$$V(r, \theta) = \frac{1}{2}Kr^2 + \frac{A}{r^2} + \frac{B}{r^2 \sin^2 \theta} + C \frac{\cos^2 \theta}{r^2 \sin^2 \theta}, \quad (1)$$

where K , A , B and C are constant parameters [35]. Figures 1–4 show plots of the RSPHO potential (1) for

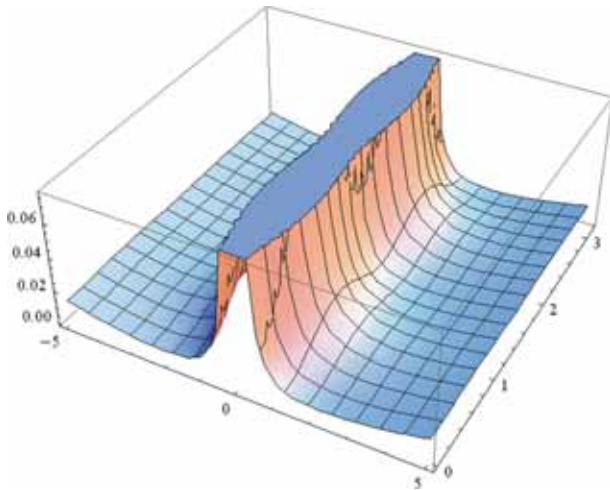


Figure 1. A plot of the RSPHO potential (1) in the range $r = [-5, 5]$ and $\theta = [0, \pi]$ when $A = B = 0.01$, $C = 0.01$ and $K = 0.001$.

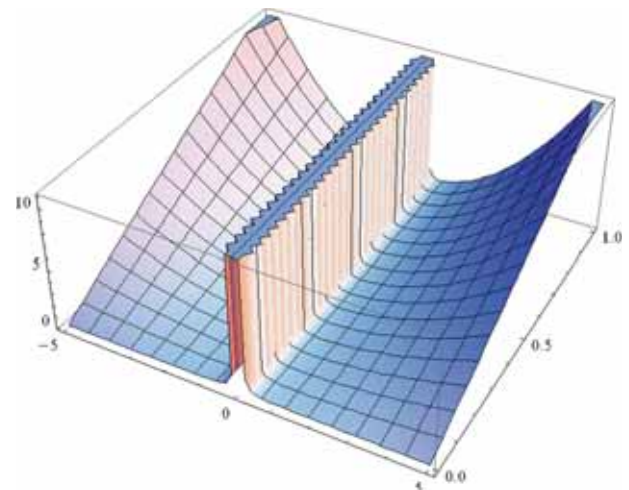


Figure 3. A plot of the RSPHO potential (1) in the range $r = [-5, 5]$ and $K = [0, 1]$ when $\theta = \pi/4$, $A = B = C = 0.01$.

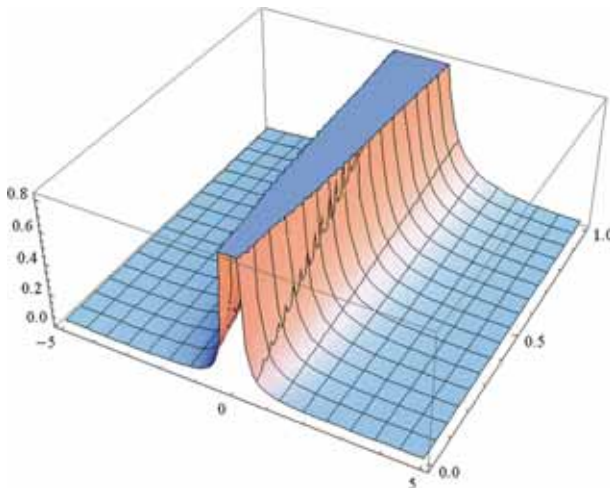


Figure 2. A plot of the RSPHO potential (1) in the range $r = [-5, 5]$ and $B = [0, 1]$ when $\theta = \pi/4$, $A = 0.01$, $C = 0.01$ and $K = 0.001$.

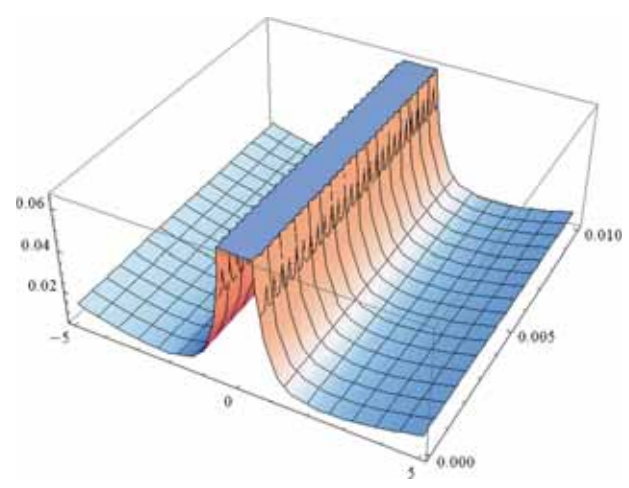


Figure 4. A plot of the RSPHO potential (1) in the range $r = [-5, 5]$ and $C = [0, 0.01]$ when $\theta = \pi/4$, $A = 0.01$, $B = 0.01$ and $K = 0.001$.

the given set of parameters values: $A = B = 0.01$, $C = 0.1$ and $K = 0.001$ as explained in each figure.

This paper is organized as follows. In §2, we solve the Dirac equation with the RSPHO potential (1) in the presence of pseudospin and spin symmetries. We obtain the energy eigenvalue equations and the corresponding spinor wave functions by using the supersymmetric quantum mechanics (SUSYQM) method. In addition, we calculate some numerical results for the energy eigenvalue equation. Further, we find the non-relativistic limits of our solution by a proper replacement of parameters. We briefly discuss the thermodynamic properties of this non-central RSPHO potential. Finally, we present our conclusions in §3.

2. Solution of the Dirac equation

The Dirac Hamiltonian in the natural units of $\hbar = c = 1$, is [36,37]

$$H = \vec{\alpha} \cdot \vec{p} + \beta(M + S(\vec{r})) + V(\vec{r}), \quad (2)$$

where $V(\vec{r})$ and $S(\vec{r})$ stand for scalar and time-like vector non-central RSPHO potential, respectively, $\vec{\alpha}$ and β are Dirac matrices and M denotes the composite fermionic mass. Thus, the Dirac equation can be written as

$$[\vec{\alpha} \cdot \vec{p} + \beta(M + S(\vec{r})) + V(\vec{r})] \Psi(\vec{r}) = E \Psi(\vec{r}), \quad (3)$$

where E denotes the binding energy and the momentum $\vec{p} = -i\vec{\nabla}$. In the Pauli–Dirac representation, let us define the two spinor-component wave functions as

$$\Psi(\vec{r}) = \begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix}, \quad (4)$$

so that we can obtain the following two coupled equations:

$$(\vec{\alpha} \cdot \vec{p}) \chi(\vec{r}) = [E - M - V(\vec{r}) - S(\vec{r})] \varphi(\vec{r}) \quad (5)$$

and

$$(\vec{\alpha} \cdot \vec{p}) \varphi(\vec{r}) = [E + M - V(\vec{r}) + S(\vec{r})] \chi(\vec{r}). \quad (6)$$

The spin symmetry demands that the scalar potential should be equal to the time-like vector potential, that is, $S(\vec{r}) = V(\vec{r})$. So, we have the following decoupled equations for the upper and lower spinor components of the wave function:

$$[p^2 + 2(E + M)V(\vec{r})]\varphi(\vec{r}) = (E^2 - M^2)\varphi(\vec{r}) \quad (7)$$

and

$$\chi(\vec{r}) = \frac{(\vec{\sigma} \cdot \vec{p})}{(E + M)} \varphi(\vec{r}), \quad (8)$$

respectively.

After substituting eq. (1) into eq. (7), we can obtain a second-order Schrödinger-like differential equation for the upper-spinor component as

$$\left\{ -\nabla^2 + 2(E + M) \left[\frac{K}{2} r^2 + \frac{A}{r^2} + \frac{B}{r^2 \sin^2 \theta} + C \frac{\cos^2 \theta}{r^2 \sin^2 \theta} \right] \right\} \varphi(\vec{r}) = (E^2 - M^2)\varphi(\vec{r}). \quad (9)$$

Further, we need to separate the variables by inserting the following form of the upper component of the wave function given by

$$\varphi(\vec{r}) = \frac{R(r)}{r} G(\theta) F(\phi), \quad (10)$$

into eq. (9) and this leads to the following set of second-order differential equations:

$$\left\{ -\frac{d^2}{dx^2} + K(E + M)r^2 + [2A(E + M) + \lambda] \frac{1}{r^2} \right\} R(r) = (E^2 - M^2)R(r) \quad (11)$$

$$\left\{ -\frac{d^2}{d\theta^2} - \cot \theta \frac{d}{d\theta} - \lambda + \frac{2(E + M)(B + C \cos^2 \theta) + m^2}{\sin^2 \theta} \right\} G(\theta) = 0 \quad (12)$$

and

$$\frac{d^2 F(\phi)}{d\phi^2} + m^2 F(\phi) = 0, \quad (13)$$

where $\lambda = \ell(\ell + 1)$ and m^2 are two separation constants. The general solution to eq. (13) is

$$F(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m \in Z = 0, \pm 1, \pm 2, \dots \quad (14)$$

2.1 Solution of the angular part

Now, we seek to find a solution for the angular part of the wave function $G(\theta)$. Now, by letting

$$G(\theta) = H(\theta)/\sqrt{\sin \theta}, \quad (15)$$

and inserting it into eq. (12), one can obtain the Schrödinger-like equation:

$$\left(\frac{d^2}{d\theta^2} + \tilde{V} \cot^2 \theta \right) H(\theta) = \tilde{E} H(\theta), \quad 0 \leq \theta \leq \pi, \quad (16)$$

where we have identified

$$\begin{aligned} \tilde{E} &= -\lambda - \frac{1}{2} + 2(E + M)B + m^2, \\ \tilde{V} &= -2(E + M)(B + C) - m^2 + \frac{1}{4}, \end{aligned} \quad (17)$$

and the function $H(\theta = 0) = H(\theta = \pi) = 0$, must vanish at the end points.

Now, we need to solve eq. (16) by using the basic concepts of SUSYQM formalism [38–41]. We can start by writing down the ground-state lower spinor component $G_0(\theta)$ as

$$G_0(\theta) = \exp\left(-\int W(\theta) d\theta\right), \quad (18)$$

with $W(\theta)$ being the superpotential in the SUSYQM formalism. Hence, the substitution of eq. (18) into eq. (16) leads to the following equation satisfying $W(\theta)$ as

$$W^2(\theta) - W'(\theta) = \tilde{V} \cot^2 \theta - \tilde{E}_0. \quad (19)$$

Taking the superpotential form as $W(\theta) = -Q \cot \theta$ and substituting it into eq. (19) we obtain the conditions,

$$Q = \tilde{E}_0 \quad (20)$$

and

$$Q^2 - Q = \tilde{V}, \quad (21)$$

where

$$Q = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \tilde{V}}. \quad (22)$$

Thus, the SUSYQM partner potentials $V_+(\theta)$ and $V_-(\theta)$ are given by

$$\begin{aligned} V_+(\theta) &= W^2(\theta) + W'(\theta) \\ &= (Q^2 + Q) \operatorname{cosec}^2 \theta - Q^2 \\ &= (Q^2 + Q) \cot^2 \theta - Q \end{aligned} \quad (23)$$

and

$$\begin{aligned} V_-(\theta) &= W^2(\theta) - W'(\theta) \\ &= (Q^2 - Q) \operatorname{cosec}^2 \theta - Q^2 \\ &= (Q^2 - Q) \cot^2 \theta - Q, \end{aligned} \quad (24)$$

respectively. Equations (23) and (24) demonstrate that $V_+(\theta)$ and $V_-(\theta)$ are varied in similar shapes.

If the condition $V_+(\theta, a_0) = V_-(\theta, a_1) + R(a_1)$ is to be satisfied, the partner Hamiltonians are called shape-invariant, where a_1 is a new set of parameters uniquely determined from the old set via the mapping $F: a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include the independent variable θ . In such a case $E_n = \sum_{k=1}^n R(a_k)$. Now, we can write out $V_+(\theta)$ as

$$\begin{aligned} V_+(\theta) &= W^2(\theta) + W'(\theta) \\ &= (Q + 1) \times [(Q + 1) - 1] \operatorname{cosec}^2 \theta \\ &\quad - (Q + 1)^2 + (Q + 1)^2 - Q^2. \end{aligned} \quad (25)$$

So the potential $V_-(\theta)$ is a shape-invariant potential as defined with

$$a_0 = Q, \quad (26)$$

$$a_1 = F(a_0) = a_0 + 1 = Q + 1, \quad (27)$$

$$a_n = a_0 + n = Q + n. \quad (28)$$

Hence, the partner potentials $V_+(\theta)$ and $V_-(\theta)$ satisfy the relationship $V_+(\theta, a_0) = V_-(\theta, a_1) + R(a_1)$, and $R(a_n)$ can be obtained from the following relation as

$$R(a_1) = -a_0^2 + a_1^2, \quad (29)$$

$$R(a_2) = -a_1^2 + a_2^2, \quad (30)$$

$$R(a_3) = -a_2^2 + a_3^2, \quad (31)$$

.

$$R(a_n) = -a_{n-1}^2 + a_n^2. \quad (32)$$

For example, we can obtain, from eq. (29), the above relation $R(a_1) = 1 + 2Q$ and so forth. The ground-state energy of $V_-(\theta)$ is zero. For the partner potential $V_-(\theta)$, the energy spectrum is given by refs [42–44]

$$\begin{aligned} \tilde{E}_n^{(-)} &= \sum_{k=1}^n R(a_k) = R(a_1) + R(a_2) + \dots + R(a_n) \\ &= -a_0^2 + a_n^2 = n^2 + 2nQ, \end{aligned} \quad (33)$$

$$\tilde{E}_0^{(-)} = 0. \quad (34)$$

Hence, we can obtain the energy spectra as

$$\tilde{E} = \tilde{E}_n^{(-)} + \tilde{E}_0 = n^2 + 2nQ + Q = (n + Q)^2. \quad (35)$$

To check the accuracy of our results, we may set $B = 0$ and $\alpha = 1$ into Rosen–Morse (trigonometric) potential of figure 6 [45]. Our results of eqs (24), (26), (27) and (35) turn out to be identical to those obtained before in ref. [45]. That is, the present results look exactly the same as in [45].

Using eqs (35), (22) and (17), we can obtain λ as follows:

$$\begin{aligned} \lambda &= \left[n + \frac{1}{2} \pm \sqrt{\frac{1}{2} - 2(E + M)(B + C) - m^2} \right]^2 \\ &\quad + 2(E + M)(B + C) + m^2 - \frac{1}{2}, \end{aligned} \quad (36)$$

where $\lambda = \ell(\ell + 1)$.

The significance of finding λ in eq. (36) is that it is the key factor in finding the energy levels of the system in terms of the orbital angular momentum and the potential parameters. The eigenvalue equation (36) will be essential in finding energy states when the radial part of the Schrödinger equation is solved in the next section.

2.2 Solution of the radial part

Now we seek to find a solution for the radial part of the wave function in eq. (11) by identifying

$$\Delta = K(E + M), \quad (37)$$

$$\delta = 2A(E + M) + \lambda \quad (38)$$

and

$$\tilde{E} = E^2 - M^2. \tag{39}$$

Then, the radial part of the equation can be rewritten in a more compact form:

$$\left[-\frac{d^2}{dr^2} + V_{\text{eff}}(r) \right] R(r) = \tilde{E} R(r), \tag{40}$$

where

$$V_{\text{eff}}(r) = \frac{\delta}{r^2} + \Delta r^2. \tag{41}$$

Considering eq. (40), one may introduce the following operators [39,40]:

$$\hat{A} = \frac{d}{dr} - W(r), \quad \hat{A}^+ = -\frac{d}{dr} - W(r), \tag{42}$$

where $W(r)$ is the radial superpotential. In the SUSYQM formalism [38–41], we can write down the radial part of the ground-state lower spinor component $R_0(r)$ as

$$R_0(r) = \exp\left(-\int W(r)dr\right) \tag{43}$$

which is inserted in eq. (40) to provide the Riccati equation

$$W^2(r) - W'(r) = V_{\text{eff}}(r) - \tilde{E}_0 \tag{44}$$

for which we assume the superpotential of the simple form

$$W(r) = \Delta r + \frac{\delta}{r}. \tag{45}$$

For a solution satisfying the Riccati equation, the following restrictions on the ansatz parameters

$$\begin{aligned} \Delta^2 &= K(E + M), \\ \delta^2 + \delta &= 2A(E + M) + \lambda, \\ 2\delta\Delta - \Delta &= -\tilde{E}_0, \end{aligned} \tag{46}$$

are to be satisfied. After solving the set of equations in (46), the parameters δ , Δ and \tilde{E}_0 are found to have the forms

$$\begin{aligned} \delta &= -\frac{1}{2}\left(1 + \sqrt{1 + 4(2A(E + M) + \lambda)}\right), \\ \Delta &= \sqrt{K(E + M)}, \quad \tilde{E}_0 = \Delta(1 - 2\delta). \end{aligned} \tag{47}$$

Obviously, we have chosen the negative solution as the appropriate solution of the quadratic equation in δ so that we can get a positive physical energy state \tilde{E}_0 .

We can now construct the two supersymmetric partner potentials as

$$V_+(r) = W^2(r) + W'(r) = \frac{\delta(\delta - 1)}{r^2} + \Delta^2 r^2 + 2\delta\Delta + \Delta \tag{48}$$

and

$$V_-(r) = W^2(r) - W'(r) = \frac{\delta(\delta + 1)}{r^2} + \Delta^2 r^2 + 2\delta\Delta - \Delta. \tag{49}$$

The above two partner potentials possess the following relationship:

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1), \tag{50}$$

where $a_0 = \delta$, $a_1 = f(a_0) = a_0 - 1 = \delta - 1$ and the remainder can be followed with equation $R(a_1) = 4\Delta(a_0 - a_1)$. From eq. (50), we know that the two partner potentials $V_+(r)$ and $V_-(r)$ are shape-invariant potentials in the sense of ref. [42] and they have similar shapes. Using the shape-invariance approach [42] to determine the energy spectra, the ground-state energy of $V_-(r)$ is zero, that is, $\tilde{E}_0^{(-)} = 0$. For the partner potential $V_-(r)$, the energy spectrum is given by [28,42–44]

$$\begin{aligned} \tilde{E}_n^{(-)} &= \sum_{k=1}^n R(a_k) \\ &= R(a_1) + R(a_2) + \dots + R(a_n) \\ &= 4\Delta(a_0 - a_1) + 4\Delta(a_1 - a_2) + 4\Delta(a_2 - a_3) \\ &\quad + \dots + 4\Delta(a_{n-1} - a_n) = 4\Delta[\delta - (\delta - n)] \\ &= 4n\Delta, \quad n = 0, 1, 2, \dots \end{aligned} \tag{51}$$

This leads us to the expression

$$\begin{aligned} \tilde{E}_n &= \tilde{E}_0 + \tilde{E}_n^{(-)} = \Delta(1 - 2\delta) + 4n\Delta, \\ E^2 - M^2 &= \sqrt{K(E + M)} \\ &\quad \times \left[2 + \sqrt{1 + 4(2A(E + M) + \lambda)} \right] \\ &\quad + 4n\sqrt{K(E + M)}. \end{aligned} \tag{52}$$

Therefore, we can obtain the energy eigenvalue equation as

$$\begin{aligned} E^2 - M^2 &= 2\sqrt{K(E + M)} \\ &\quad \times \left[2n + 1 + \sqrt{\frac{1}{4} + 2A(E + M) + \lambda} \right], \end{aligned} \tag{53}$$

with the quantum number $n = 0, 1, 2, \dots$

It is worthy to note that when $\lambda = \ell(\ell + 1)$, eq. (53) reduces to eq. (41) of ref. [35] which was obtained

before for the potential (1) using the standard associated Legendre differential equation. Therefore, after making use of eqs (36) and (53), the energy states of the potential (1) can be easily found as

$$E - M = 2 \frac{\sqrt{K}}{\sqrt{E + M}} \left\{ 2n + 1 + \left\{ \frac{1}{4} + 2A(E + M) + \left[n + \frac{1}{2} \pm \sqrt{\frac{1}{2} - 2(E + M)(B + C) - m^2} \right]^2 + 2(E + M)(B + C) + m^2 - \frac{1}{2} \right\}^{1/2} \right\}. \quad (54)$$

In figures 5–8, we show the behaviour of the spin-symmetric energy eigenvalues vs. the potential parameters A and K for various states of the quantum number $n = 1, 2, 3$. Figure 5 shows the influence of parameter K on the energy spectrum E_s . It is obvious that energy approximately increases with the increase in the parameter A for $K = 5.0, 5.5, 6.0$ and when we take the state $n = 1$. We also see that the energy increases with the increase in the quantum number n from 1 to 3 when $K = 5$.

In figure 6, we also show the influence of parameter B on the energy spectrum E_n . It is obvious that the energy increases vs. the parameter A with the decrease in azimuthal quantum number $m = 2, 0, -2$ and the energy decrease with the decreasing values of B , i.e., $-0.48, -0.49, -0.50$ for $m = 2$. We also see

that the energy has pseudolinear decreases with the increase in the azimuthal quantum number, i.e., for $m = -2, 0, 2$. Furthermore, in figure 7, we show the influence of the parameter A on the energy spectrum E_n . We see that the energy increases with the increase in the value of parameter A for increasing values of A , i.e., $5.0, 5.5, 6.0$ when $n = 1$. We also see that the energy increases linearly with the quantum number $n = 1, 2, 3$ when $A = 5$. Finally, in figure 8, we

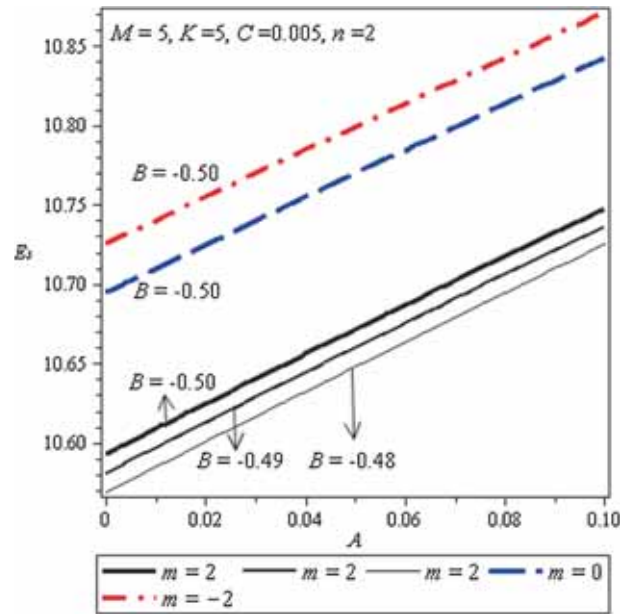


Figure 6. The spin-symmetric energy states of the RSPHO potential vs. A for different values of m and B .

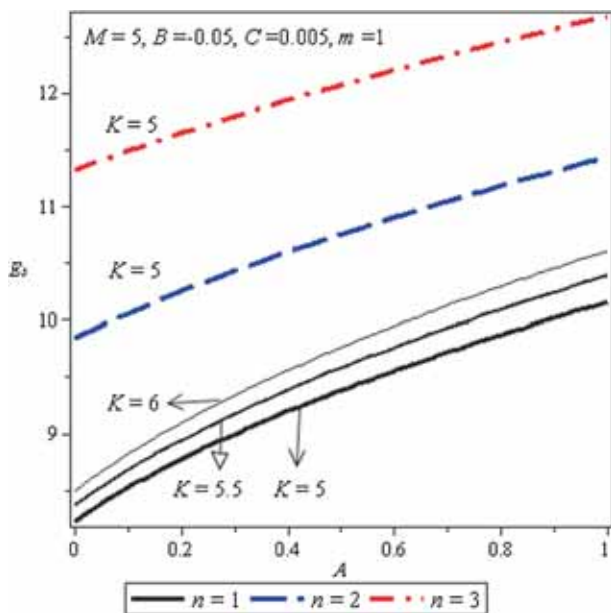


Figure 5. The spin-symmetric energy states of the RSPHO potential vs. A for different values of n and K .

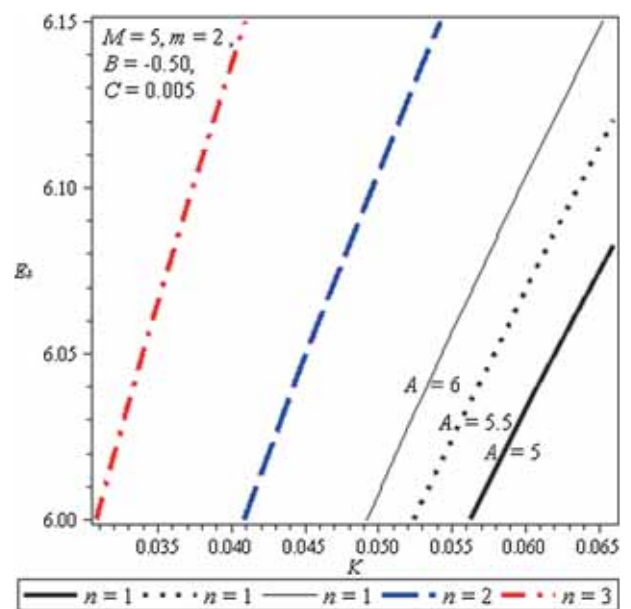


Figure 7. The spin-symmetric energy states of the RSPHO potential vs. K for different values of n and A .

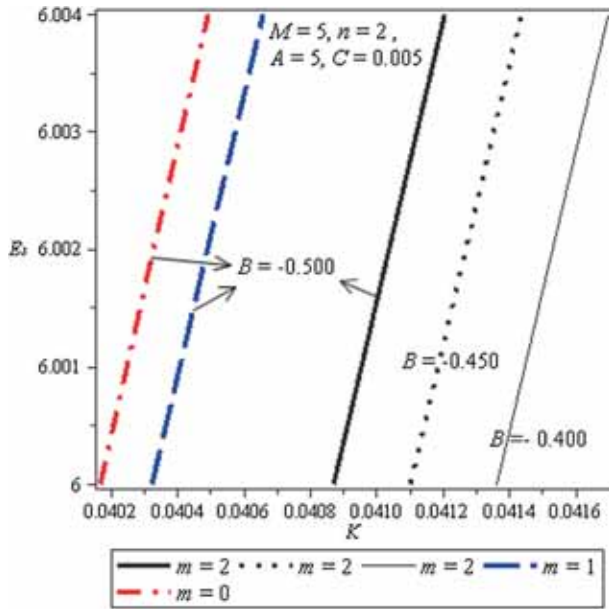


Figure 8. The spin-symmetric energy states of the RSPHO potential vs. K for different values of m and B .

show the influence of parameter B on the energy levels E_n . We see that the energy decreases linearly with the increase in parameter K for different values of B , i.e., $-0.40, -0.45, -0.50$ when $m = 2$. We also notice that the energy increases with the decrease of azimuthal quantum number $m = 0, 1, 2$ when $B = -0.50$.

We give some numerical results for spin-symmetric energy state in table 1.

Table 1 presents the calculated energy states with the change in parameter while fixing the other parameters as $B = -0.05, K = 5, C = 0.005$ and $M = 0.5 \text{ fm}^{-1}$.

Table 1. The spin-symmetric energy states for various values of A when $B = -0.05, K = 5.0, C = 0.005$ and $M = 5.0 \text{ fm}^{-1}$.

n	m	A	E_s	n	m	E_s
1	0	6	14.38516214	1	1	14.36707671
		6.5	14.68410842			14.66709513
		7	14.97241387			14.95634685
2	0	7.5	15.25115354	2	1	15.23592854
		6	15.43922930			15.41239928
		6.5	15.72755149			15.70222204
		7	16.00603750			15.98203989
3	0	7.5	16.27564844	3	1	16.25284192
		6	16.46852268			16.43488023
		6.5	16.74677536			16.71490807
		7	17.01595171			16.98566875
		6.5	17.27690633			17.24804736

At first, the energy splitting increases with increasing A and when we set $m = 0$, we find that the energy splitting increases with increasing n values and it is becoming slightly smaller by increasing the value of m . In fact, the energy splitting decreases with increasing m . We conclude that when $m = 0$ the angular part is related only to the change in parameter values of B and C . Hence, it does not have much effect on our calculations of energy states. However, when $m > 0$, the energy states decrease by a smaller amount than the ones obtained when $m = 0$.

We have noticed a similar effect on the energy states while changing the parameter A . However, the essential feature is that the effect of this change is quite smaller than that made while changing the parameter K . This is due to the fact that the parameter K is the coefficient of harmonic oscillatory part while the parameter A is the coefficient of pseudoharmonic oscillatory part in RSPHO potential (1). That is, the contribution of K is larger than A on the energy. On the other hand, to find a non-relativistic solution, we make the following simple mapping of parameters as $E - M \approx E_{NR}, E + M \approx 2\mu$, and in accordance with the relativistic energy equation (54) and choosing $\xi = \mu (C + B)$, we have

$$E_{NR} = \frac{\sqrt{2K}}{\sqrt{\mu}} \left\{ 2n + 1 + \left\{ \frac{1}{4} + 4A\mu + \left[n + \frac{1}{2} \pm \sqrt{\frac{1}{2} - 4\xi - m^2} \right]^2 + 4\xi + m^2 - \frac{1}{2} \right\}^{1/2} \right\}. \tag{55}$$

On the other hand, to investigate the thermodynamic properties, we need to calculate the energy so that all thermodynamic quantities of the present nonrelativistic model can be obtained in a systematic way. So we should first calculate the partition function Z at a finite temperature T , through the Boltzmann factor given by $Z = \sum_{n=0}^{\infty} e^{-\beta E_{n\ell}}$ where $\beta = 1/k_B T$ with k_B the Boltzmann constant. Hence, the partition function Z reads as

$$Z = \sum_{n=0}^{\infty} \exp \left(-\frac{\beta \sqrt{2K}}{\sqrt{\mu}} \left\{ 2n + 1 + \left\{ \frac{1}{4} + 4A\mu + \left[n + \frac{1}{2} \pm \sqrt{\frac{1}{2} - 4\xi - m^2} \right]^2 + 4\xi + m^2 - \frac{1}{2} \right\}^{1/2} \right\} \right). \tag{56}$$

The other thermodynamic properties of the system can be easily found from the partition function. In fact, any other parameter that might contribute to the energy should also appear in the argument of Z [46], such as Helmholtz free energy F which is alternatively defined as $F = -\ln(Z)^N/\beta$, the mean energy $U = -\partial \ln Z/\partial \beta$. The entropy is related to other quantities via $S = -\partial F/\partial T$ and the specific heat for higher temperatures can be found through $C = \partial U/\partial T$. Therefore, using the above equations, we can easily find the other thermodynamic quantities. We can also find the wave functions of the radial part, in the spin symmetric case, through the relation [35]

$$R(r) = N e^{-\eta^2/2} \eta^{L+1} \times {}_1F_1\left(-n, L + \frac{3}{2}, \eta^2\right), \quad (57)$$

where N is the normalization constant, $\eta = r \sqrt[4]{(E + M)K}$ and $L(L + 1) = 2(E + M)A + \lambda$.

The Dirac wave equation in the pseudospin symmetry when $S(r) = -V(r)$ takes the form

$$[p^2 + 2(E - M)V(r)]\varphi(\vec{r}) = [E^2 - M^2]\varphi(\vec{r}), \quad (58)$$

with the lower spinor component of the Dirac equation,

$$\varphi(\vec{r}) = \frac{(\vec{\sigma} \cdot \vec{p})}{E - M} \chi(\vec{r}). \quad (59)$$

This pseudospin symmetry can be easily found by simply making the mapping transformations:

$$\begin{aligned} \varphi(r) &\rightarrow \chi(r), & \chi(r) &\rightarrow -\varphi(r) \\ V(r) &\rightarrow -V(r), & E &\rightarrow -E. \end{aligned} \quad (60)$$

Hence, the pseudospin symmetric radial wave functions can be obtained by substituting $\eta = r \sqrt[4]{(E - M)K}$ and $L(L + 1) = 2(E - M)A + \lambda$ into eq. (57). Further, the eigenvalue equation of the potential (1) can be obtained as

$$\begin{aligned} E - M = 2\sqrt{\frac{-K}{E + M}} \left\{ 2n + 1 + \left[\frac{1}{4} - 2A(E + M) \right. \right. \\ \left. \left. + \left[n + \frac{1}{2} \pm \sqrt{\frac{1}{2} + 2(E + M)(B + C) - m^2} \right]^2 \right. \right. \\ \left. \left. - 2(E + M)(B + C) + m^2 - \frac{1}{2} \right]^{1/2} \right\}. \quad (61) \end{aligned}$$

For the sake of completeness, we plot figures 9 and 10 to show the relationship between the pseudospin energy states with the potential parameters and the two quantum numbers n and m .

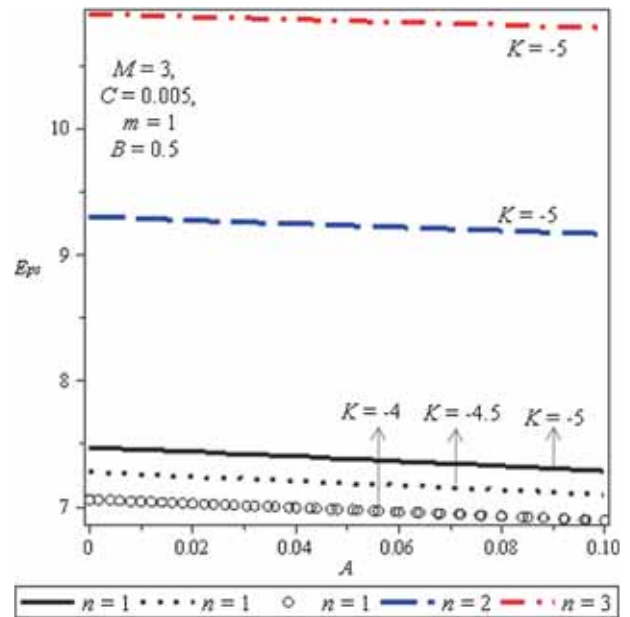


Figure 9. The pseudospin-symmetric energy states of the RSPHO potential vs. A for different values of n and K .

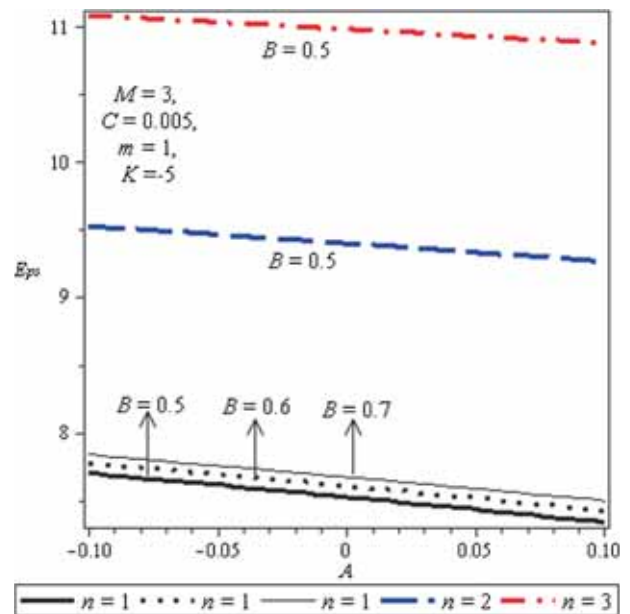


Figure 10. The pseudospin-symmetric energy states of the RSPHO potential versus A for different values of n and B .

Further, we show the plot of the energy states against B for different values of A and n in figure 11 and in figure 12 the plot of energy states vs. B for different values of m and K with m .

Also, in table 2, we calculate the energy states by changing the parameter while fixing the other parameters as $B = 0.50$, $K = -5.0$, $C = 0.005$ and $M = 3.0 \text{ fm}^{-1}$.

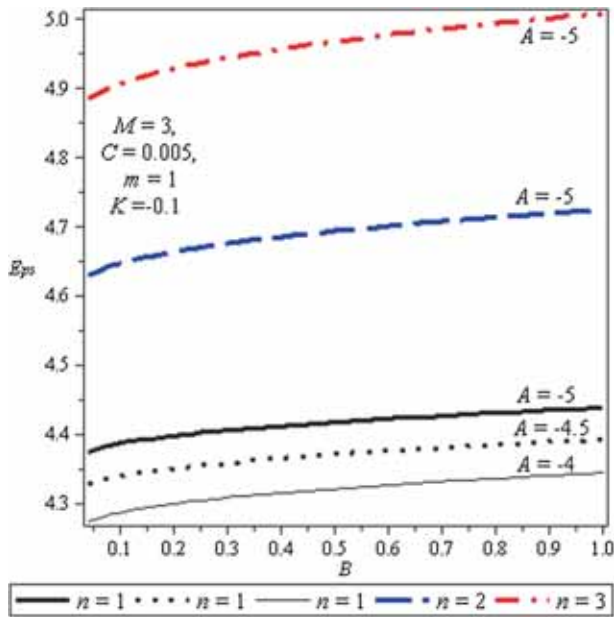


Figure 11. The pseudospin-symmetric energy states of the RSPHO potential vs. B for different values of n and A .

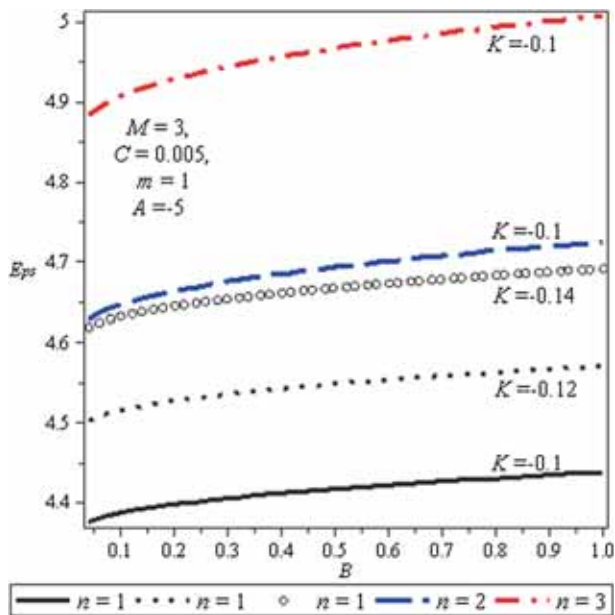


Figure 12. The pseudospin-symmetric energy states of the RSPHO potential vs. B for different values of m and K .

We see that, the energy splitting decreases with increasing A and when $m = 0$, we find that the energy splitting increases with increasing n and it is becoming slightly smaller by increasing the value of m . In fact, the energy splitting decreases by increasing the value of m . The first look demonstrates the approximately similar behaviour with the spin-symmetric case. The spacing between states becomes dense or far apart depending on the parameter values and the type of symmetry studied.

Table 2. Pseudospin symmetric energy states for various values of A when $B = 0.5$, $K = -5.0$, $C = 0.005$ and $M = 3.0 \text{ fm}^{-1}$.

n	m	A	E_{ps}	n	m	E_{ps}
1	0	-5	12.12523736	1	1	12.11721311
		-4.5	11.80243939			11.79377306
		-4	11.46422548			11.45480142
		-3.5	11.10808771			11.09775436
		-3	10.73074788			10.71930066
		-2.5	10.32777781			10.31492990
2	0	-5	13.39533062	2	1	13.38420577
		-4.5	13.09302649			13.08111418
		-4	12.77772112			12.76489504
		-3.5	12.44751045			12.43360952
		-3	12.09996935			12.08478306
		-2.5	11.73192618			11.71517108
2	2	-5	12.09120093	2	3	13.34829750
		-4.5	11.76561159			13.04258166
		-4	11.42409353			12.72330610
		-3.5	11.06397676			12.38840990
		-3	10.68174211			12.03524372
		-2.5	10.27258506			11.66030252
3	0	-5	14.60737113	3	1	14.59415692
		-4.5	14.32247694			14.30842578
		-4	14.02646655			14.01145778
		-3.5	13.71786529			13.70174843
		-3	13.39483629			13.37741997
		-2.5	13.05504166			13.03607652
3	2	-5	14.55167438			
		-4.5	14.26316729			
		-4	13.96301237			
		-3.5	13.64960113			
		-3	13.32091179			
		-2.5	12.97434270			

3. Discussions and conclusions

In this work, we solved approximately the Dirac equation with spin and pseudospin symmetries for the RSPHO potential (1) by using the SUSYQM formalism. Approximate bound-state energy eigenvalues and their associated two-component spinors of the Dirac particle are obtained in the presence of the spin and pseudospin symmetries. Our relativistic solution can be reduced to its non-relativistic limits once we make some appropriate mapping of parameters. Further, we also briefly discussed the thermodynamic properties of the resulting non-relativistic model.

Our numerical energy eigenvalues are obtained by taking some arbitrary numerical values of the parameters K and A and fixing the other parameters in the potential (1) for various principal and quantum

numbers n and m respectively. These results are displayed in tables 1 and 2.

In the spin-symmetric energy states, it is noted that if A and n increase then E increases, and if m increases then E decreases. However, in the pseudosymmetric energy states, if A increases then E decreases. If magnetic quantum number m increases then E decreases, and if principal quantum number n increases then E increases.

We have plotted the spin and pseudospin symmetries and shown the approximate similarity of energy in the presence of these two symmetries.

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