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A study of fractional Schrödinger equation composed of Jumarie fractional derivative

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Abstract. In this paper we have derived the fractional-order Schrödinger equation composed of Jumarie fractional derivative. The solution of this fractional-order Schrödinger equation is obtained in terms of Mittag–Leffler function with complex arguments, and fractional trigonometric functions. A few important properties of the fractional Schrödinger equation are then described for the case of particles in one-dimensional infinite potential well. One of the motivations for using fractional calculus in physical systems is that the space and time variables, which we often deal with, exhibit coarse-grained phenomena. This means infinitesimal quantities cannot be arbitrarily taken to zero – rather they are non-zero with a minimum spread. This type of non-zero spread arises in the microscopic to mesoscopic levels of system dynamics, which means that, if we denote x as the point in space and t as the point in time, then limit of the differentials dx (and dt) cannot be taken as zero. To take the concept of coarse graining into account, use the infinitesimal quantities as $(\Delta x)^\alpha$ (and $(\Delta t)^\alpha$) with $0 < \alpha < 1$; called as 'fractional differentials'. For arbitrarily small Δx and Δt (tending towards zero), these 'fractional' differentials are greater than Δx (and Δt), i.e. $(\Delta x)^\alpha > \Delta x$ and $(\Delta t)^\alpha > \Delta t$. This way of defining the fractional differentials helps us to use fractional derivatives in the study of dynamic systems.

Keywords. Jumarie fractional derivative; Mittag-Leffler function; fractional Schrödinger equation; fractional wave function.

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1. Introduction

Fractional-order derivatives are extensively used to study different natural processes and physical phenomena [1–14]. The mathematicians of this era are trying to construct general form of calculus by modifying the classical order derivative to an arbitrary order. Riemann–Liouville [11] definition of fractional derivative admits non-zero value for fractional differentiation of a constant. This contradicts the basic properties of classical calculus. To overcome this problem, Jumarie [9, 13, 15] modified Riemann–Liouville definition of fractional derivative to obtain zero value for the fractional derivative of a constant. This type of derivative is also applicable for continuous but non-differentiable functions. Mittag-Leffler [10] function was initially introduced in classical sense but it has several applications in the field of fractional calculus. The Mittag-Leffler function with complex argument gives the fractional sine and cosine functions [9]. Ghosh *et al* [11,16,17] discussed about the solutions of various types of linear fractional differential equations (composed of Jumarie fractional derivative) in terms of Mittag-Leffler functions. On the other hand, researchers are using different fractional differential equations by incorporating the fractional order in place of classical order derivatives. Question arises as to what will be the actual equation in fractional sense if we start with the basic fractional equations. This is a challenging task to mathematicians.

The coordinate x in x-space, originates from the differential dx, as its integration, i.e. $\int_0^x \mathrm{d}y = x$. Now with a differential $(\mathrm{d}x)^\alpha$, with $0 < \alpha < 1$, we have $(\mathrm{d}x)^\alpha >$ dx, where $\int_0^x (\mathrm{d}y)^\alpha \sim x^\alpha$ [13,18,19]. That is, the space is transformed to a fractal space where the coordinate x is transformed to x^α. This is coarse graining phenomenon in particular scale of observation. Here we come across the fractal space–time, where the normal classical differentials dx and dt cannot be taken arbitrarily to zero. Thus, in these cases the concept of classical differentiability is lost [20]. The fractional-order α is related to roughness character of the space–time, i.e. the fractal dimension [18,19]. In this paper, fractional differentiation of order α is used to study the dynamic systems defined by the function $f(x^\alpha, t^\alpha)$. In [14,18,19] the demonstration of fractional calculus on the fractal subset of real line is the Cantor set, and the order α is taken accordingly.

In this paper we have developed fractional Schrödinger equation and tried to understand the nature of quantum mechanics for the fractal region. This formulation also leads to normal quantum mechanics at limiting condition, of fractional order α tending to unity. The nature of the solution at various values of fractional orders α of fractional differentiation signifies the underlying behaviour of quantum mechanics, in fractal space–time. We have modified de Broglie's and Planck's hypothesis in the fractional sense such that they remain intact if limiting conditions of α tending to unity are used.

This paper is divided into separate sections and subsections. In §2 some definitions of fractional calculus are described. In §3 we discuss about fractional-order wave equation. Section 4 is about solution of fractional wave equation. Section 5 deals with fractional Schrödinger equation. In §6 we develop 'equation of continuity'. In §6.1 we discuss about properties of fractional wave function. Section 6.2 discusses further study on fractional wave function, §6.3 is about 'orthogonal' and 'normal' conditions of wave functions. Section 7 is about operators and expectation values. In §8 and 8.1 we discuss simple application of particles in one-dimensional infinite potential well. In §8.2 we depict graphical representation of fractional wave function. Section 8.3 depicts the graphical representation of probability density. In §8.4 we discuss the energy calculations. In Appendix, we have defined fractional quantities that are used in the paper.

2. Some definitions of fractional calculus

There are several definitions of fractional derivative. The leading definitions are Riemann–Liouville (RL) definition [5], Caputo definition [1] and modified RL definition [9].

2.1 *Riemann–Liouville (RL) definition of fractional derivative*

Riemann–Liouville (RL) fractional derivative of a function $f(x)$ is defined as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha+m+1)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} \times \int_a^x (x-\tau)^{m-\alpha} f(\tau)\mathrm{d}\tau, $$

where $m \le \alpha < m+1$, m is a positive integer. This definition says that the fractional derivative of a non-zero constant function is not zero [6]; which is contrary to the classical calculus.

2.2 *Jumarie-modified RL definition of fractional derivative*

To get rid of the above-mentioned problem of RL fractional derivative, Jumarie modified [9,13,15] this definition, for a continuous (but not necessarily differentiable) function $f(x)$, with start point of function $a=0$ such that $f(a)=f(0)$ is finite at the start point of the function, as follows:

$$ f^{(\alpha)}(x) = {}_0^JD_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x (x-\xi)^{-\alpha-1} f(\xi)\mathrm{d}\xi, & \alpha<0, \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^x (x-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\mathrm{d}\xi, & \\ \quad 0<\alpha<1, & \\ \left(f^{(\alpha-n)}(x)\right)^{(n)}, \quad n\le\alpha<n+1, & n\ge 1. \end{cases} $$

In Leibniz's classical sense, the Jumarie fractional derivative is defined via fractional difference. Let $f:\mathbb{R}\to\mathbb{R}$ denotes a continuous (but not necessarily differentiable) function such that $x\to f(x)$ for all $x\in\mathbb{R}$. Let $h>0$ denotes a constant infinitesimal step. Define a forward operator $E_h[f(x)] = f(x+h)$; then the right-hand fractional difference of order α $(0<\alpha<1)$ is,

$$ \Delta_+^{(\alpha)} f(x) = (E_h-1)^\alpha f(x) = \sum_{k=0}^{\infty}(-1)^k({}^\alpha C_k) f\left(x+(\alpha-k)h\right), $$

where ${}^{\alpha}C_k = (\alpha!/k!(\alpha-k)!)$ are the generalized binomial coefficient. Then the Jumarie fractional derivative is

$$f_{+}^{(\alpha)}(x) = \lim_{h\downarrow 0}\frac{\Delta_{+}^{(\alpha)}[f(x)-f(0)]}{h^{\alpha}} = \frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}.$$

Similarly, one can have left Jumarie derivative by defining backward shift operator. In this Jumarie definition we subtract the value of the function at the start point, from the function itself and then the fractional derivative is taken (in Riemann–Liouvelli sense). The offsetting of the function by subtracting the start point value makes the fractional derivative of constant function as zero. This also gives conjugation with classical integer-order calculus, especially regarding chain rule for fractional derivatives, fractional derivative of the product of two functions etc. [13]. In the rest of the paper, the derivative operator D^{α} will be regarded as the modified Riemann–Liouville (Jumarie) derivative.

2.3 *Some techniques of Jumarrie derivative*

Consider a function $f[u(x)]$ which is not differentiable in the classical sense but fractionally differentiable. Jumarie suggested [13] three different ways depending upon the characteristics of the function, depicted by formulas (i), (ii) and (iii), as follows:

(i) $D^{\alpha}(f[u(x)]) = f_u^{(\alpha)}(u)(u'_x)^{\alpha}$,

(ii) $D^{\alpha}(f[u(x)]) = (f/u)^{1-\alpha}(f'_u(u))^{\alpha}u^{\alpha}(x)$,

(iii) $D^{\alpha}(f[u(x)]) = (1-\alpha)!u^{\alpha-1}f_u^{(\alpha)}(u)u^{\alpha}(x)$.

2.4 *Mittag-Leffler function and fractional trigonometric functions*

The one-parameter Mittag-Leffler function [10] is defined as infinite series in the following form:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}, \quad z\in\mathbb{C}, \quad \mathrm{Re}(\alpha)>0.$$

For $\alpha = 1$, it is a simple exponential function $E_1(z) = \mathrm{e}^z$. The fractional sine and cosine functions [4] defined by Mittag-Leffler function are as follows:

$$\cos_{\alpha}(t^{\alpha}) = \frac{E_{\alpha}(it^{\alpha})+E_{\alpha}(-it^{\alpha})}{2} = \sum_{k=1}^{\infty}(-1)^k\frac{t^{2k\alpha}}{(2k\alpha)!} = \sum_{k=1}^{\infty}(-1)^k\frac{t^{2k\alpha}}{\Gamma(2k\alpha+1)}$$

$$\sin_{\alpha}(t^{\alpha}) = \frac{E_{\alpha}(it^{\alpha})-E_{\alpha}(-it^{\alpha})}{2i} = \sum_{k=1}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{(2k\alpha+\alpha)!} \times\sum_{k=1}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(2k\alpha+\alpha+1)}.$$

One of the most important properties of Mittag-Leffler function [11] is ${}^{J}D_x^{\alpha}E_{\alpha}(ax^{\alpha}) = aE_{\alpha}(ax^{\alpha})$. This means that the Jumarie fractional derivative of order α of the Mittag-Leffler function of order α (in scaled variable x^{α}) is just returning the function itself. Thus, the function $E_{\alpha}(ax^{\alpha})$ is the eigenfunction for the Jumarie derivative operator. We mention here that this function $E_{\alpha}(ax^{\alpha})$ is also eigenfunction of Caputo derivative operator. This function $E_{\alpha}(ax^{\alpha})$ is also termed as α-exponential function $\tilde{\mathrm{e}}_{\alpha}(x) = E_{\alpha}(x^{\alpha})$. This is in conjugation with classical calculus similar to exponential function and very useful in solving fractional differential equations composed of Jumarie fractional derivative.

3. Fractional-order wave equation and its solution

In this section we consider a plane progressive wave propagating in the positive x direction with a constant velocity v. The general form is $f(x,t) = f(x-vt)$ [21]. The fractional plane progressive wave propagating in fractional space–time can be considered in the following form:

$$f(x,t) = f(x^{\alpha}-v_{\alpha}t^{\alpha}), \quad 0<\alpha\le 1. \tag{1}$$

Here v_{α} is the fractional velocity (we shall elaborate this in Appendix). When α tends to one, this fractional plane progressive wave turns to one-dimensional plane progressive wave. Thus, eq. (1) represents an αth-order fractional plane progressive wave, where the space and time axes are transformed to x^{α} and t^{α} respectively (with $0<\alpha\le 1$). Thus, the wave we consider here is a fractional plane wave moving in the x direction. Jumarie-type fractional derivative [9,13] is used to find various physical quantities and physical properties of the corresponding wave. Let us define the operators as ${}^{J}D_x^{\alpha} \equiv \partial^{\alpha}/\partial x^{\alpha}$, ${}^{J}D_x^{2\alpha} \equiv \partial^{2\alpha}/\partial x^{2\alpha}$ and ${}^{J}D_t^{\alpha} \equiv \partial^{\alpha}/\partial t^{\alpha}$, ${}^{J}D_t^{2\alpha} \equiv \partial^{2\alpha}/\partial t^{2\alpha}$. These are Jumarie derivative operators. Now consider the following with the condition, i.e. $0<\alpha\le 1$, with $u(x,t) = x^{\alpha}-v_{\alpha}t^{\alpha}$ and write function of $u(x,t)$ as

$$f\,(u(x,t)) = f(x^{\alpha}-v_{\alpha}t^{\alpha}).$$

We now choose the differential trick that is $D^\alpha(f[u(x)]) = (1-\alpha)!u^{\alpha-1} f_u^{(\alpha)}(u) u^\alpha(x)$; the formula (iii), of Jumarie [22]. We know the following expression:

$$u_x^{(\alpha)} = D_x^\alpha u(x,t) = D_x^\alpha [x^\alpha - v_\alpha t^\alpha] = D_x^\alpha [x^\alpha] = \alpha! = \Gamma(\alpha+1).$$

Therefore, applying Jumarie fractional derivative with respect to x; with formula (iii) we write (2)

$$D_x^\alpha [f\,(u(x,t))] = \alpha!(1-\alpha)!u^{\alpha-1} f_u^{(\alpha)}(u). \tag{2}$$

Doing similar operation with respect to t, we write the following equation:

$$D_t^\alpha [f\,(u(x,t))] = v_\alpha \alpha!(1-\alpha)!u^{\alpha-1} f_u^{(\alpha)}(u). \tag{2a}$$

From eqs (2) and (2a) we get the following equations:

$$D_t^\alpha [f\,(u(x))] = v_\alpha (D_x^\alpha f[u(x)]).$$

On operating D_x^α on both sides of the above expression we get

$$D_x^\alpha D_t^\alpha f[u(x)] = v_\alpha (D_x^\alpha D_x^\alpha f[u(x)]) = v_\alpha (D_x^{2\alpha} f[u(x)]). \tag{2b}$$

Now on operating D_t^α on both sides of (2b) we get

$$D_t^\alpha D_t^\alpha f[u(x)] = D_t^{2\alpha} f[u(x)] = v_\alpha (D_t^\alpha D_x^\alpha f[u(x)]). \tag{2c}$$

Using Theorem A.8 and combining eqs (2b) and (2c) we get the following expression:

$$D_x^{2\alpha} f[u(x)] = \frac{1}{v_\alpha^2}(D_t^{2\alpha} f[u(x)])$$

$${}^J D_t^{2\alpha}[f(x^\alpha - v_\alpha t^\alpha)] = v_\alpha^2 ({}^J D_x^{2\alpha}[f(x^\alpha - v_\alpha t^\alpha)]). \tag{3}$$

Equation (3) represents the fractional wave equation of α order. If $\alpha = 1$ the equation changes to one-dimensional classical wave equation for the plane progressive wave.

4. Solution of the fractional wave equation

To find solution of the fractional wave equation (3) by separation of the variable method, we consider its solution of the form $f(x,t) = g(x^\alpha)h(t^\alpha)$. This reduces eq. (3) to

$$\frac{1}{g(x^\alpha)} D_x^{2\alpha} g(x^\alpha) = \frac{1}{v_\alpha^2}\frac{1}{h(t^\alpha)} D_t^{2\alpha} h(t^\alpha)$$

which implies the following expression:

$$h(t^\alpha) D_x^{2\alpha} g(x^\alpha) = \frac{1}{v_\alpha^2} g(x^\alpha) D_t^{2\alpha} h(t^\alpha),$$

where left-hand side is space-dependent and right-hand side is time-dependent. This is possible if and only if both sides of this equation is constant. Let the constant be k_α^2.

The space part of the equation is now

$$\frac{1}{g(x^\alpha)} D_x^{2\alpha} g(x^\alpha) = k_\alpha^2 \quad \text{or} \quad D_x^{2\alpha} g(x^\alpha) = k_\alpha^2 g(x^\alpha) \tag{3a}$$

Solution of this equation [11,23] is

$$g(x^\alpha) = bE_\alpha(\pm i k_\alpha x^\alpha),$$

where b is a constant. Similarly, the time part of the equation is $D_t^{2\alpha} h(t^\alpha) = k_\alpha^2 v_\alpha^2 h(t^\alpha)$. The solution to this is

$$h(t^\alpha) = BE_\alpha(\pm i\omega_\alpha t^\alpha).$$

Here we put $\omega_\alpha = k_\alpha v_\alpha$ and assume B as a constant. Thus, the general solution of (3a) is of the following form:

$$f(x,t) = AE_\alpha(\pm i k_\alpha x^\alpha) E_\alpha(\pm i\omega_\alpha t^\alpha), \tag{4}$$

where $A = bB$ is a constant.

5. Fractional Schrödinger equation: Derivation and solution

Consider a particle of mass m moving with velocity v. According to de Broglie hypothesis [24], there is a wave associated with every moving particle. The mathematical form of de Broglie hypothesis is $p = \hbar k$. Here p is the momentum of the particle and k is the wave vector in one dimension; $\hbar$ is the reduced Planck's constant. Now Planck's hypothesis [24] shows that the energy ε of a particle in a particular quantum level is proportional to the angular frequency, that is, $\varepsilon = \hbar\omega$. In this context, it is assumed that de Broglie hypothesis and Planck's hypothesis are also valid in fractional αth order with a modified form

$$p_\alpha = \hbar_\alpha k_\alpha \tag{5}$$

$$\varepsilon_\alpha = \hbar_\alpha \omega_\alpha. \tag{6}$$

It is clear that if $\alpha = 1$, eqs (5) and (6) reduce to the original form of de Broglie and Planck's hypotheses. Here $\hbar_\alpha$ is the reduced Planck's constant of α order; $\hbar = h/2\pi$, and h is the Planck's constant. Here, ω_α is the fractional-order angular frequency and k_α is the fractional-order wave vector (which is described in Appendix).

The general solution for eq. (3) is $u = Af(x^\alpha - v_\alpha t^\alpha)$, $0 < \alpha \le 1$ where A is a constant. To find the explicit form of this solution, Mittag-Leffler [10] function is taken as a trial solution [11] similar to the classical differential equation where we consider $\exp(x)$ as the trial solution [25]. Thus, we can take

$$f(x^\alpha, t^\alpha) = AE_\alpha(ik_\alpha x^\alpha - i\omega_\alpha t^\alpha) = AE_\alpha(ik_\alpha x^\alpha)E_\alpha(-i\omega_\alpha t^\alpha), \quad (7)$$

where $E_\alpha(ik_\alpha x^\alpha)$ and $E_\alpha(-i\omega_\alpha t^\alpha)$ are one-parameter Mittag-Leffler functions in complex variable. Thus, (7) is a trial solution of eq. (3). Now fractional velocity v_α can be defined as $v_\alpha = \omega_\alpha/k_\alpha$, which is the fractional velocity of the particle as well as the fractional phase velocity of the wave. This fractional velocity is assumed to be constant.

Consider a particle that possesses constant momentum p_α and constant energy ε_α, i.e. its energy and momentum do not vary with the propagation of the wave in space and time. Using conditions (5) and (6), solution (7) can be written as

$$f(x^\alpha, t^\alpha) = AE_\alpha\left(\frac{i}{\hbar_\alpha} p_\alpha x^\alpha\right)E_\alpha\left(-\frac{i}{\hbar_\alpha}\varepsilon_\alpha t^\alpha\right). \quad (8)$$

This particle has some hidden physical properties. To investigate these properties we derive the space and time evolution of the function $f(x^\alpha, t^\alpha)$.

5.1 *Derivation of fractional Schrödinger equation*

To derive α-order fractional Schrödinger equation we first find the α-order partial derivative of (8) with respect to space coordinate in the form

$${}^J D_x^\alpha f(x^\alpha, t^\alpha) = A\left(\frac{ip_\alpha}{\hbar_\alpha}\right)E_\alpha\left(\frac{i}{\hbar_\alpha} p_\alpha x^\alpha\right) \times E_\alpha\left(-\frac{i}{\hbar_\alpha}\varepsilon_\alpha t^\alpha\right)$$

$${}^J D_x^\alpha f(x^\alpha, t^\alpha) = \frac{i}{\hbar_\alpha}(p_\alpha f(x^\alpha, t^\alpha)). \quad (9)$$

Repeating the previous operation once again to eq. (9) we get the following expression:

$${}^J D_x^{2\alpha} f(x^\alpha, t^\alpha) = -\frac{1}{\hbar_\alpha^2} p_\alpha^2 f(x^\alpha, t^\alpha), \quad (10)$$

where

${}_0^J D_x^\alpha \left[E_\alpha(ax^\alpha)\right] = aE_\alpha(ax^\alpha)$ [11].

Let us define $p_\alpha^2 = 2^\alpha m_\alpha \varepsilon_{\alpha K}$, where m_α is the mass (in fractional sense) and $\varepsilon_{\alpha K}$ is the kinetic energy of fractional order α. Then expression (10) can be written as

$${}^J D_x^{2\alpha} f(x^\alpha, t^\alpha) = -\frac{1}{\hbar_\alpha^2}(2^\alpha m_\alpha \varepsilon_{\alpha K}) f(x^\alpha, t^\alpha).$$

This gives the following expression:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}\left({}^J D_x^{2\alpha}\left[f(x^\alpha, t^\alpha)\right]\right) = \varepsilon_{\alpha K} f(x^\alpha, t^\alpha). \quad (11)$$

The variation of the function (8) with time is studied. Repeating the above steps now by taking Jumarie fractional derivative of order α with respect to time (t), we have the following equation:

$${}^J D_t^\alpha f(x^\alpha, t^\alpha) = -\frac{i}{\hbar_\alpha}\varepsilon_\alpha f(x^\alpha, t^\alpha). \quad (12)$$

Here ε_α is the total energy of the system. From the conservation of energy in fractal space we write

Total energy (ε_α) = (kinetic energy $\varepsilon_{\alpha k}$) + (potential energy $V(x^\alpha, t^\alpha)$).

That is,

$$\varepsilon_\alpha = \varepsilon_{\alpha K} + V(x^\alpha, t^\alpha). \quad (13)$$

Substituting (13) in eq. (12) and combining them with eq. (11) we get the following expression:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}({}^J D_x^{2\alpha}[f(x^\alpha, t^\alpha)]) = ((\varepsilon_\alpha - V(x^\alpha, t^\alpha))f(x^\alpha, t^\alpha).$$

Now by rearranging the above expression we obtain the following form:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}({}^J D_x^{2\alpha} f(x^\alpha, t^\alpha)) + V(x^\alpha, t^\alpha) f(x^\alpha, t^\alpha) = i\hbar_\alpha({}^J D_t^\alpha f(x^\alpha, t^\alpha)), \quad (14)$$

where

$$f(x^\alpha, t^\alpha) = AE_\alpha\left(\frac{i}{\hbar_\alpha} p_\alpha x^\alpha\right)E_\alpha\left(-\frac{i}{\hbar_\alpha}\varepsilon_\alpha t^\alpha\right).$$

This is the α-order fractional Schrödinger equation. In the limit $\alpha = 1$ this equation reduces to the 'classical' Schrödinger equation in one-dimensional space and time. Equation (14) has the solution which will lead to certain interesting physical properties.

5.2 *Solution of fractional Schrödinger equation*

The basic method of the solution of eq. (14) is the method of separation of variables. In this method the solution is identified as the product of two different functions $\Phi(x^\alpha)$ and $T(t^\alpha)$. Here, $\Phi(x^\alpha)$ depends on the transformed space variable x^α and $T(t^\alpha)$ depends on transformed time variable t^α. Thus, the function $f(x^\alpha, t^\alpha) = \Phi(x^\alpha)T(t^\alpha)$. Substituting $f(x^\alpha, t^\alpha) =$

$\Phi(x^\alpha)T(t^\alpha)$ in eq. (14), we obtain the following expression:

$$-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha}\frac{1}{\Phi(x^\alpha)}\frac{\mathrm{d}^{2\alpha}[\Phi(x^\alpha)]}{\mathrm{d}x^{2\alpha}}+V(x^\alpha)\Phi(x^\alpha) = \frac{i\hbar_\alpha}{T(t^\alpha)}\frac{\mathrm{d}^\alpha[T(t^\alpha)]}{\mathrm{d}t^\alpha}. \quad (15)$$

Left-hand side of eq. (15) is only space-dependent and right-hand side is only time-dependent. Thus, to satisfy eq. (15) both sides must be equal to some constant. On the left side of eq. (15) we have a fractional potential term. This has the dimension of fractional energy, that is

$$[ML^2T^{-2}]^\alpha = [M^\alpha L^{2\alpha}T^{-2\alpha}].$$

Clearly the constant must have the dimension of fractional energy due to homogeneity of dimension. Observing the right-hand side of the equation, the dimension analysis allows us to choose the unit of the constant ε_α as $(\text{Joule})^\alpha$ for fractional values of α which is also supported by eq. (12). Equation (15) can be written as the following two different equations, of which one is solely time-dependent and other is solution space-dependent, as described below:

$$\frac{i\hbar_\alpha}{T(t^\alpha)}\frac{\mathrm{d}^\alpha[T(t^\alpha)]}{\mathrm{d}t^\alpha}=\varepsilon_\alpha, \quad (16)$$

$$-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha}\frac{1}{\Phi(x^\alpha)}\frac{\mathrm{d}^{2\alpha}[\Phi(x^\alpha)]}{\mathrm{d}x^{2\alpha}}+V(x^\alpha)=\varepsilon_\alpha. \quad (17)$$

Solution of equation of type (16) was found by Ghosh *et al* [11] using the Mittag-Leffler functions in the form

$$T(t^\alpha)\approx E_\alpha\left(-\frac{i}{\hbar_\alpha}\varepsilon_\alpha t^\alpha\right).$$

Thus, the solution of eq. (15) is (by omitting the integral constant) the following:

$$f(x^\alpha,t^\alpha)=\Psi_\alpha=\Phi(x^\alpha)E_\alpha\left(-\frac{i}{\hbar_\alpha}\varepsilon_\alpha t^\alpha\right). \quad (18)$$

For $\alpha = 1$, i.e. in limiting case, the solution (18) turns to the solution of one-dimensional classical Schrödinger wave equation.

5.3 *Time-independent fractional Schrödinger equation and fractional Hamiltonian*

Equation (17) has no time-dependent part. This can be rearranged as follows:

$$-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha}\frac{\mathrm{d}^{2\alpha}[\Phi(x^\alpha)]}{\mathrm{d}x^{2\alpha}}-\left(\varepsilon_\alpha-V(x^\alpha)\right)\Phi(x^\alpha)=0. \quad (19)$$

This equation is called the time-independent fractional Schrödinger equation. It is potential-dependent. So it is not possible to solve eq. (19) without knowing the character of the potential function. However, it can be confirmed that solution of (19) has only space dependency. A Hamiltonian can be constructed with the analogy of classical Schrödinger's one-dimensional quantum wave equation. The Hamiltonian in terms of non-integer order derivative is thus defined as follows:

$$\hat{H}_\alpha=-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha}\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}x^{2\alpha}}-V(x^\alpha). \quad (20)$$

Therefore, eq. (19) can be written in terms of Hamiltonian as depicted below:

$$\hat{H}_\alpha\Phi=\varepsilon_\alpha\Phi. \quad (21)$$

Equation (21) is an eigenequation with the eigenvalue ε_α. The eigenfunction of the equation is Φ. This eigenfunction is the 'information centre' of a particle. One can operate it in various ways to find the corresponding physical property. The Hamiltonian (21) gives the correct information about the energy of the particle.

6. Continuity equation: Conservation of probability current density

Consider the Schrödinger equation previously derived in eq. (14)

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}({}^J D_x^{2\alpha}[f(x^\alpha,t^\alpha)])+V(x^\alpha,t^\alpha)f(x^\alpha,t^\alpha) = i\hbar_\alpha({}^J D_t^\alpha[f(x^\alpha,t^\alpha)]).$$

Multiply the above equation with the complex conjugate of the solution, say $f^*(x^\alpha,t^\alpha)$, to write the following expression:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}f^*(x^\alpha,t^\alpha)({}^J D_x^{2\alpha}[f(x^\alpha,t^\alpha)]) + V(x^\alpha,t^\alpha)f^*(x^\alpha,t^\alpha)f(x^\alpha,t^\alpha) = i\hbar_\alpha f^*(x^\alpha,t^\alpha)({}^J D_t^\alpha[f(x^\alpha,t^\alpha)]). \quad (22)$$

Let us take complex conjugate of eq. (14) and multiply with the function $f(x^\alpha,t^\alpha)$ and write the following equation:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}f(x^\alpha,t^\alpha)({}^J D_x^{2\alpha}[f^*(x^\alpha,t^\alpha)]) + V(x^\alpha,t^\alpha)f(x^\alpha,t^\alpha)f^*(x^\alpha,t^\alpha) = -i\hbar_\alpha f(x^\alpha,t^\alpha)({}^J D_t^\alpha[f^*(x^\alpha,t^\alpha)]). \quad (23)$$

Subtracting eq. (22) from eq. (23) we have following (by dropping x^α, t^α) equation:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}(f^*({}^J D_x^{2\alpha}[f]) - f({}^J D_x^{2\alpha}[f^*])) \\ = i\hbar_\alpha(f^*({}^J D_t^\alpha[f]) + f({}^J D_t^\alpha f^*)). \quad (24)$$

Equation (24) can be rewritten in the following form:

$$-\frac{\hbar_\alpha^2}{2^\alpha m_\alpha}{}^J D_x^\alpha[f^*({}^J D_x^\alpha[f]) - f({}^J D_x^\alpha[f^*])] \\ = i\hbar_\alpha(f^*({}^J D_t^\alpha[f]) + f({}^J D_t^\alpha[f^*])). \quad (25)$$

Let us define

$$\left(\frac{i\hbar_\alpha}{2^\alpha m_\alpha}\right)(f^{*J} D_x^\alpha f - f\,{}^J D_x^\alpha f^*) = j_\alpha$$

as the probability current density of α order. Define $f^* f = \rho_\alpha$ as the probability density of α order. For $\alpha = 1$ the fractal probability density $f^* f = \rho_\alpha$ turns to the one-dimensional probability density. Thus, eq. (25) reduces to

$${}^J D_x^\alpha[j_\alpha] = {}^J D_t^\alpha[\rho_\alpha]. \quad (26)$$

This is the equation of continuity of α order in one dimension. If probability mass density $f^* f = \rho_\alpha$ is independent of time, right-hand side of (26) is zero. Thus, the left-hand side is also equal to zero. This implies that the one-dimensional variation of current density with space is zero. Physical significance of the fact is that there is no source or sink of probability current density. This is the condition of stationary state. To satisfy the above condition of ρ_α the solution must be of type

$$f(x^\alpha, t^\alpha) = \Psi_\alpha = \Phi(x^\alpha)E_\alpha(-i(\varepsilon_\alpha t^\alpha/\hbar_\alpha)).$$

This is the stationary state solution of α order. For $\alpha = 1$ the state is the same as that of the one-dimensional stationary state solution.

6.1 *Some properties of fractional wave function*

For further investigation it is needed to characterize the basic properties of the solution of fractional Schrödinger equation. We list them as follows:

(a) The fractional wave function must be continuous and should be single valued. As the particle has physical existence, the fractional wave function of the particle must be continuous at every position of space and time. If the fractional wave function is not continuous for some position or time, then the particle will vanish in the middle of its trajectory which is not possible at all. Fractional wave function must be single valued, i.e. for every position of space–time the property of the particle is unique.

(b) The fractional wave function must be square integrable in fractional sense in the region $a \le x \le b$, i.e.

$$\int_a^b \Psi_\alpha \Psi_\alpha^* \mathrm{d}x^\alpha < \infty.$$

The notation $\int f(x)\mathrm{d}x^\alpha$ implies fractional integration that is given as follows:

$$\int_{-\infty}^x f(x)\mathrm{d}x^\alpha = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi$$

with $\alpha > 0$.

(c) Linear combination of solutions of the fractional Schrödinger wave equation itself is a solution. Thus, linear combination of the wave function is another wave function.

(d) The fractional wave function must vanish at the boundary. If it is does not vanish at the boundary, then the boundary itself loses its significance. The boundaries have property to seize the motion of particle to go further away from it. As a result, the particle has to stop at the boundary and consequently the fractional wave function vanishes. Here boundary means perfectly rigid boundary. If we have an analogy with a vibrating string bounded by two points, then we cannot get amplitude of vibrating string at the two end points. Mathematically, the condition for the wave function in the region $a \le x \le b$ is thus described as follows:

$$\Psi_\alpha(a) = \Psi_\alpha(b) = 0.$$

(e) The fractional Schrödinger equation suggests that the α-order fractional derivative ${}^J D_x^\alpha \equiv \partial^\alpha/\partial x^\alpha$ of the wave function Ψ_α is continuous and single valued.

(f) The α-order fractional derivative of the wave function must vanish at the boundary. If not, condition of stationary state will violate as suggested in the equation of continuity.

(g) The wave function must be normalized. This signifies the existence of the particle with certainty within the considered boundary.

6.2 *Further study on fractional wave function*

The general solution of fractional wave equation is

$$f(x^\alpha, t^\alpha) = \Psi_\alpha = \Phi(x^\alpha)E_\alpha(-i\varepsilon_\alpha t^\alpha/\hbar_\alpha).$$

Its complex conjugate is

$\Psi_\alpha^* = \Phi^*(x^\alpha)E_\alpha\left(i\varepsilon_\alpha t^\alpha/\hbar_\alpha\right).$

Multiplying Ψ_α with Ψ_α^* we get

$\Psi_\alpha\Psi_\alpha^* = \Phi(x^\alpha)\Phi^*(x^\alpha).$

This quantity is independent of time. We define this quantity as 'existence intensity' and Ψ as existence amplitude. In a certain considered boundary the particle exists with certainty. Let Ψ be defined in the boundary $-\infty \le x \le +\infty$. Then we have

$$\int_{-\infty}^{+\infty}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha = \text{constant}.$$

Note that the notation $\int f(x)\mathrm{d}x^\alpha$ implies fractional integration, that is,

$$\int_{-\infty}^{x} f(x)\mathrm{d}x^\alpha = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi \quad \text{with } \alpha > 0.$$

If the particle does not exist at the boundary, the integration vanishes. We can now define $\int_{-\infty}^{+\infty}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha = 1$ if the particle exists with certainty and $\int_{-\infty}^{+\infty}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha = 0$ if the particle does not exist anywhere. Clearly, the existence parameter $\int_{-\infty}^{+\infty}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha$ is such that the condition $0 \le \int_{-\infty}^{+\infty}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha \le 1$, gets satisfied.

Suppose we have to find the information about the existence over a certain region considered inside the boundary, and then the quantity $\int_{-a}^{+b}\Psi_\alpha\Psi_\alpha^*\mathrm{d}x^\alpha = l < 1$ should be less than 1. This defines that particle is not localized and behaves as waves that are not localized. If all these existence parameters (or probability) are added up, the total probability is unity. From eqs (9), (10), (12), (17) we find that wave function is an eigenfunction of various operators.

6.3 *Orthogonal and orthonormal conditions of wave functions*

Two functions $F(x)$ and $G(x)$ defined in the region $a \le x \le b$ are orthogonal if their inner product is zero [25]. From the analogy of this orthogonal condition in $\{x\}$ space we can define the orthogonal condition for $\{x^\alpha\}$ space with the following fractional integration operation such that

$$\langle F|G\rangle = \int_a^b F^*(x^\alpha)G(x^\alpha)\mathrm{d}x^\alpha = 0. \tag{27}$$

Here

$$\langle F|G\rangle = \int_a^b F^*(x^\alpha)G(x^\alpha)\mathrm{d}x^\alpha$$

is defined as the inner product of order α where $F^*(x^\alpha)$ is the complex conjugate of $F(x^\alpha)$. In the same way, the orthonormal condition is defined by the following fractional integration:

$$\langle F|G\rangle = \int_a^b F^*(x^\alpha)G(x^\alpha)\mathrm{d}x^\alpha = 1. \tag{28}$$

The general solution of wave function is

$$\Psi_\alpha = \sum_n^\infty c_n\psi_n,$$

where c_n is some constant. Here n is a dummy index. The complex conjugate of the solution is

$$\Psi_\alpha^* = \sum_m^\infty c_m^*\psi_m^*$$

and the inner product using Dirac bracket notation is

$$\langle\Psi_\alpha|\Psi_\alpha\rangle = \sum_m^\infty\sum_n^\infty c_m^*c_n\langle\psi_{m\alpha}^*\psi_{n\alpha}\rangle. \tag{29}$$

From orthogonal and orthonormal conditions we have the following equation:

$$\langle\Psi_\alpha|\Psi_\alpha\rangle = \sum_m^\infty\sum_n^\infty c_m^*c_n\langle\psi_{m\alpha}^*\psi_{n\alpha}\rangle = 0, \quad \text{if } n \ne m$$

and

$$\langle\Psi_\alpha|\Psi_\alpha\rangle = \sum_m^\infty\sum_n^\infty c_m^*c_n\langle\psi_{m\alpha}^*\psi_{n\alpha}\rangle = 1, \quad \text{if } n = m.$$

In a similar way we write

$\langle\psi_{m\alpha}|\psi_{n\alpha}\rangle = 1, \quad \text{if } n = m.$

Clearly

$$\langle\Psi_\alpha|\Psi_\alpha\rangle = \sum_n^\infty c_nc_n^* = 1.$$

More precisely, it can be written as

$$\langle\Psi_\alpha|\Psi_\alpha\rangle = \sum_n^\infty |c_n|^2 = 1.$$

We can therefore define c_n as the existence coefficient or probability coefficient.

7. Operators and expectation values

In quantum mechanics all the measureable quantities which cannot be measured directly are measured by their expectation values [25]. So in the case of α-order quantum mechanics we need to define operators for every measurable quantity. For this purpose, there must be some rules of choosing operators which we list as follows:

(i) Every operator must be an eigenoperator of the wave function.
(ii) Eigenvalue of the operator defines measurable quantity.
(iii) Expectation value of an operator is the measure of the corresponding operator.

Consider an operator $\hat{A}_\alpha$ that operates on a certain function $\Psi_{n\alpha}$ such that

$$\hat{A}_\alpha \Psi_{n\alpha} = \lambda_n \Psi_{n\alpha}. \tag{30}$$

From the general form of Ψ_α, eq. (30) turns to be

$$\hat{A}_\alpha[\psi_\alpha] = \sum_n^\infty c_n \hat{A}_\alpha \psi_{n\alpha} = \sum_n^\infty \lambda_n c_n \psi_{n\alpha}.$$

Thus, λ_n cannot be determined directly. For the correct information of the system we have to find the mean value (or expectation) of the system. Expectation value of an operator is defined as follows:

$$\langle A \rangle = \frac{\int_{-\infty}^{+\infty} \psi_{n\alpha} \hat{A}_\alpha \psi_{n\alpha}^* \mathrm{d}x^\alpha}{\int_{-\infty}^{+\infty} \psi_{n\alpha} \psi_{n\alpha}^* \mathrm{d}x^\alpha}. \tag{31}$$

For every physical measurable quantity there is a corresponding expectation value.

8. Simple application: Particles in one-dimensional infinite potential well

Consider a particle bounded by a one-dimensional infinite potential well with length from $x = 0$ to $x = a$ for $0 \le x \le a$. The potential defined here is of the type of $V = 0$ if $0 \le x \le a$ and $V = \infty$ otherwise. Thus, the particle is strictly bounded by the potential well in the transformed scale, i.e. x^α too. The wave function is zero outside the well. For continuity, the wave function must vanish at the boundaries also, i.e. $\Phi(0) = \Phi(a^\alpha) = 0$. The fractional Schrödinger equation as suggested in eq. (19) is as follows:

$$-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha} \frac{\mathrm{d}^{2\alpha}\Phi(x^\alpha)}{\mathrm{d}x^{2\alpha}} - (\varepsilon_\alpha - V(x^\alpha))\Phi(x^\alpha) = 0.$$

With $V(x^\alpha) = 0$, this equation takes the form

$$-\frac{\hbar_\alpha^2}{(2)^\alpha m_\alpha} \frac{\mathrm{d}^{2\alpha}\Phi(x^\alpha)}{\mathrm{d}x^{2\alpha}} - \varepsilon_\alpha \Phi(x^\alpha) = 0.$$

Rearranging the above expression we write the following equation:

$$\frac{\mathrm{d}^{2\alpha}\Phi(x^\alpha)}{\mathrm{d}x^{2\alpha}} + \frac{(2)^\alpha m_\alpha \varepsilon_\alpha}{\hbar_\alpha^2} \Phi(x^\alpha) = 0.$$

Let us take the following constant:

$$\frac{(2)^\alpha m_\alpha \varepsilon_\alpha}{\hbar_\alpha^2} = k_\alpha^2. \tag{31a}$$

With this, the equation now is as follows:

$$\frac{\mathrm{d}^{2\alpha}\Phi(x^\alpha)}{\mathrm{d}x^{2\alpha}} + k_\alpha^2 \Phi(x^\alpha) = 0. \tag{32}$$

This equation has solution as suggested by Ghosh *et al* [11]

$$\Phi(x^\alpha) = AE_\alpha(-ik_\alpha x^\alpha) + BE_\alpha(ik_\alpha x^\alpha). \tag{33}$$

Using boundary condition $\Phi(0) = \Phi(a^\alpha) = 0$, we get $A + B = 0$. Thus, the solution of (32) is

$$\Phi(x^\alpha) = B(E_\alpha(-ik_\alpha x^\alpha) - E_\alpha(ik_\alpha x^\alpha)).$$

Using the definition of fractional sine function [9] we write the following equation:

$$\Phi(x^\alpha) = C \sin_\alpha(k_\alpha x^\alpha). \tag{34}$$

Using boundary condition on eq. (34) we get

$$\Phi(a^\alpha) = C \sin_\alpha(k_\alpha a^\alpha) = \Phi(0) = 0. \tag{34a}$$

As defined by Jumarie [9]

$$\sin_\alpha(x^\alpha) = \sin_\alpha((x + M_\alpha)^\alpha).$$

Here we defined M_α as first-order zero or first zero crossing for the fractional sine function [26]. As $\sin_\alpha(0) = 0$, we have

$$\sin_\alpha((M_\alpha)^\alpha) = 0. \tag{34b}$$

Comparing eqs (34a) and (34b), $\sin_\alpha(k_\alpha a^\alpha) = \sin_\alpha((M_\alpha)^\alpha)$, we get $ka^\alpha = (M_\alpha)^\alpha$ which gives the following expression:

$$k_\alpha = \left(\frac{M_\alpha}{a}\right)^\alpha. \tag{34c}$$

Using $k_\alpha = (M_\alpha/a)^\alpha$ in eq. (34) the solution is

$$\Phi(x^\alpha) = C \sin_\alpha\left(\left(\frac{M_\alpha}{a}\right)^\alpha x^\alpha\right).$$

8.1 *Normalization of wave function*

The normalization condition for wave function of α order is as follows:

$$\int_0^a \Phi_\alpha \Phi_\alpha^* \mathrm{d}x^\alpha = 1. \tag{35}$$

As $\Phi_\alpha(x^\alpha) = \Phi(x^\alpha) = C \sin_\alpha(k_\alpha x^\alpha)$ is a real function, $\Phi_\alpha \Phi_\alpha^* = |\Phi_\alpha|^2$. Then,

$$\int_0^a |\Phi_\alpha|^2 \, \mathrm{d}x^\alpha = 1.$$

Here $\Phi_\alpha \Phi_\alpha^* = |\Phi_\alpha|^2 = |\Phi(x^\alpha)|^2 = C^2 \sin_\alpha^2(k_\alpha x^\alpha)$. Thus, the integration is

$$C^2 \int_0^a \sin_\alpha^2(k_\alpha x^\alpha) \mathrm{d}x^\alpha = 1.$$

To integrate the above equation an identity must be developed, which is described next. By definition we have following expression:

$$\cos_\alpha(2x^\alpha) = \frac{E_\alpha(2ix^\alpha) + E_\alpha(-2ix^\alpha)}{2}.$$

We also have the following identities [11]:

$$\cos_\alpha(2x^\alpha) - 1 = \frac{E_\alpha(2ix^\alpha) + E_\alpha(-2ix^\alpha)}{2} - 1$$

$$\cos_\alpha(2x^\alpha) - 1 = \frac{E_\alpha(2ix^\alpha) + E_\alpha(-2ix^\alpha) - 2}{2}.$$

The Mittag-Leffler function $E_\alpha(ax^\alpha)$ satisfies the relation

$$E_\alpha(ax^\alpha)E_\alpha(bx^\alpha) = E_\alpha((a+b)x^\alpha).$$

Putting $a = b$ we get $E_\alpha(2x^\alpha) = (E_\alpha(x^\alpha))^2$ (Theorem 1 of [11]). With this, we manipulate the above expression to get the following equation:

$$\cos_\alpha(2x^\alpha) - 1 = \frac{(E_\alpha(ix^\alpha))^2 + (E_\alpha(-ix^\alpha))^2 - 2E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha)}{2}$$

$$\cos_\alpha(2x^\alpha) - 1 = \frac{(E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha))^2}{2}.$$

Again, by definition [11] we have the following expression:

$$\sin_\alpha(x^\alpha) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i}. \quad (36)$$

So, we get the following identity:

$$1 - \cos_\alpha(2x^\alpha) = 2\sin_\alpha^2(x^\alpha). \quad (37)$$

Using the identity of eq. (37), and by using the formula of fractional integration, i.e.

$$\int_0^x \cos_\alpha(\lambda y^\alpha)\, (\mathrm{d}y)^\alpha = \lambda^{-1} \sin_\alpha(\lambda x^\alpha)$$

and

$$\int_0^x (\mathrm{d}y)^\alpha = (\alpha!)^{-1}\, x^\alpha$$

we write following steps:

$$C^2 \int_0^a \sin_\alpha^2(k_\alpha x^\alpha) \mathrm{d}x^\alpha = 1$$

$$C^2 \left(\frac{1}{2}\right) \left(\int_0^a (1 - \cos_\alpha 2k_\alpha x^\alpha) \mathrm{d}x^\alpha \right) = 1$$

$$C^2 \left(\frac{1}{2}\right) \int_0^a \mathrm{d}x^\alpha - C^2 \left(\frac{1}{2}\right) \int_0^a \cos_\alpha(2k_\alpha x^\alpha) \mathrm{d}x^\alpha = 1$$

$$\frac{C^2}{2} \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\sin_\alpha(2k_\alpha x^\alpha)}{2k_\alpha} \right)_{x=0}^{x=a} = 1$$

$$\frac{C^2}{2} \left(\frac{a^\alpha}{\Gamma(1+\alpha)} - \frac{\sin_\alpha(2k_\alpha a^\alpha)}{2k_\alpha} \right) = 1.$$

The term $\sin_\alpha(2k_\alpha a^\alpha)/2k_\alpha$ is zero as suggested by boundary condition. Thus, $(C^2/2)(a^\alpha/\Gamma(1+\alpha)) = 1$ or $C = \sqrt{2\Gamma(1+\alpha)/a^\alpha}$. With all these derivations we get the solution as follows:

$$\Phi(x^\alpha) = \sqrt{\frac{2\Gamma(1+\alpha)}{a^\alpha}} \sin_\alpha\left(\left(\frac{M_\alpha}{a}\right)^\alpha x^\alpha \right).$$

For $\alpha = 1$ the solution is converted to one-dimensional solution for one-dimensional Schrödinger equation of infinite potential well.

8.2 *Graphical representation of wave function*

Graphical presentation of

$$\Phi(x^\alpha) = \sqrt{\frac{2\Gamma(1+\alpha)}{a^\alpha}} \sin_\alpha(k_\alpha x^\alpha)$$

for different values of fractional order for $a = 10$ unit is shown in figure 1. We need to know the values of M_α for various α before plotting the graph. We found using Wolfram Mathematica-9 the various approximate values of (M_α) for 10,000 terms of Mittag-Leffler function and $\sin_\alpha(x^\alpha)$ function. Numerically, it is observed that $\sin_\alpha(x^\alpha)$ losses periodicity for $\alpha < 1$. They are listed in table 1.

The plot is drawn for $\Phi(x^\alpha) = \sqrt{2\Gamma(1+\alpha)/a^\alpha}$ $\sin_\alpha(k_\alpha x^\alpha)$ against x. Here the box width is taken as 10 units, that is, $a = 10$. From numerical analysis we found that the quantum boundary conditions are satisfied up to $\alpha \approx 0.736$ from $\alpha = 1$. The plot suggests that the maxima of the wave function shift to the right with the increase in value of order α. The nature of wave function also changes with value of α. When $\alpha = 1$, the plot is the same as suggested by one-dimensional Schrödinger potential box problem. When $\alpha \approx 0.736$, the curve is not symmetrical and more area gets covered at the left side of the graph. Less the value of α, more asymmetrical is the plot. These plots have one zero crossing [26]. This means that the quantum number of the system is 1. The system is in the ground state. To compare the wave functions for various α values we have another plot (see figure 2).

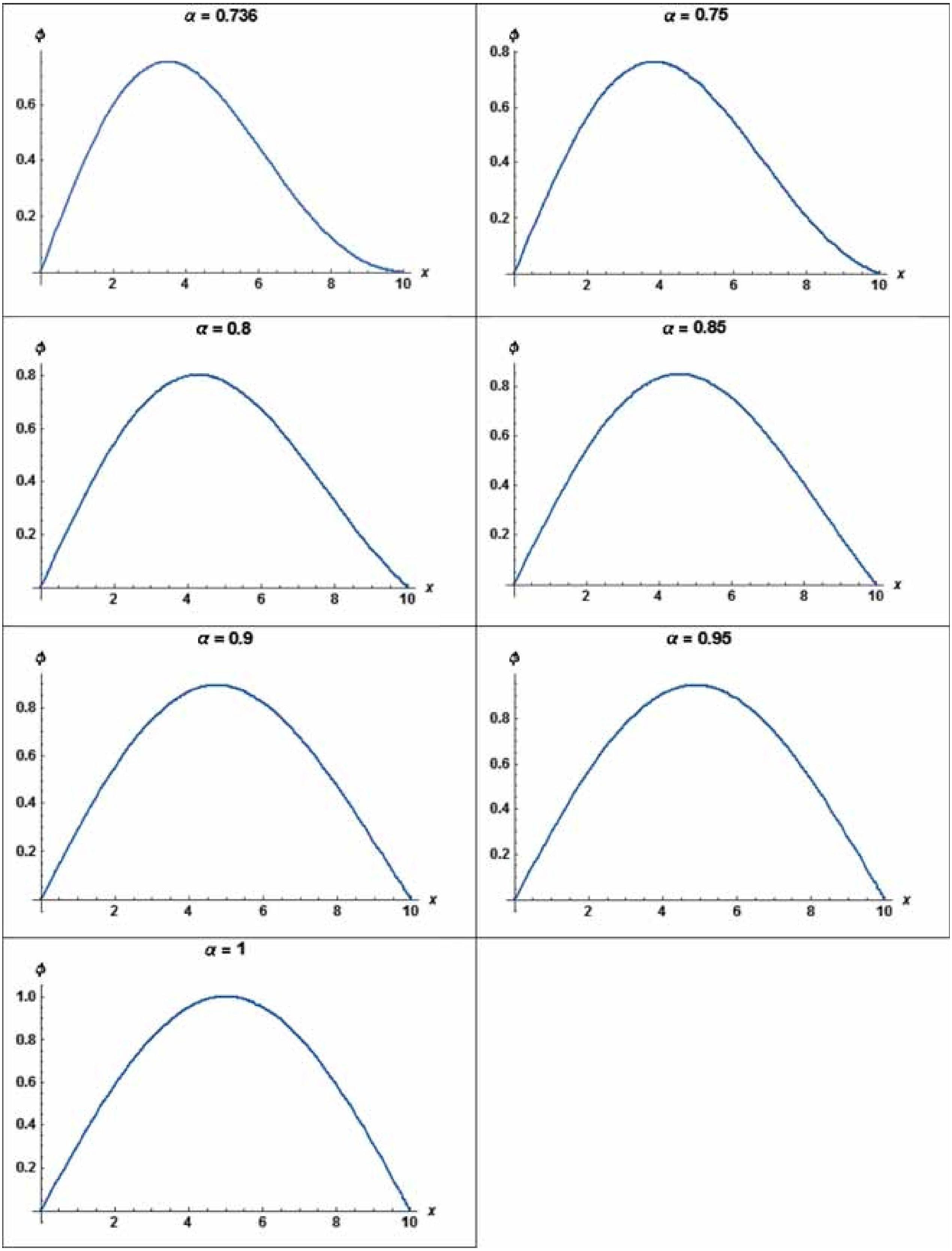

Figure 1. Graphical presentation of one-dimensional Schrödinger potential box problem. $\Phi(x^{\alpha}) = \sqrt{2\Gamma(1+\alpha)/a^{\alpha}}\,\sin_{\alpha}(k_{\alpha}x^{\alpha})$ for $\alpha = 0.736,\ 0.75,\ 0.8,\ 0.85,\ 0.9,\ 0.95$ and 1.0.

Table 1. First zeros of function $\sin_\alpha(x^\alpha)$ after $x = 0$ for different α.

α	(M_α)
0.736	3.19590
0.75	2.96354
0.80	2.80104
0.85	2.80556
0.90	2.87596
0.95	2.99051
1.0	3.14159

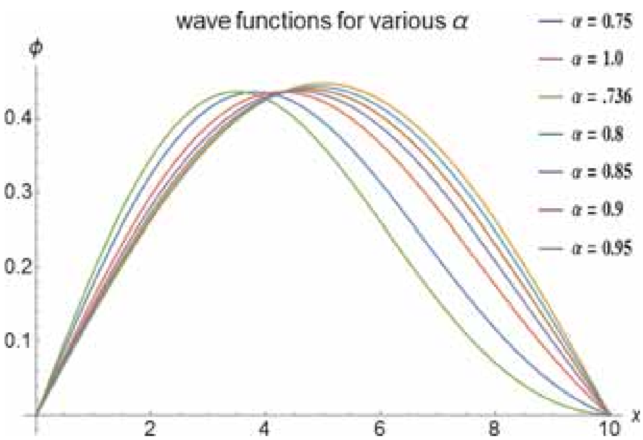

Figure 2. Graphical presentation of $\Phi(x^\alpha)$ for different values of α for $a = 10$.

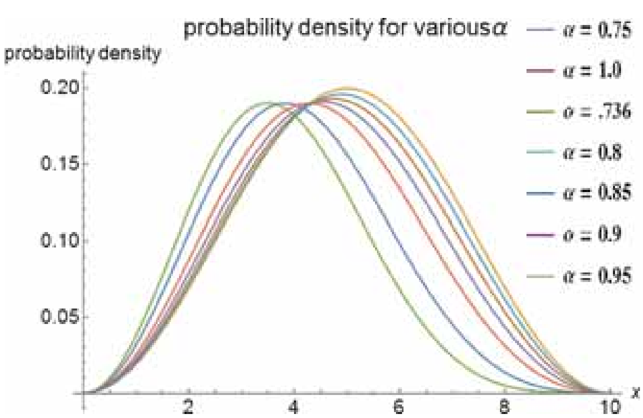

Figure 3. Graphical presentation of $\psi_\alpha^* \psi_\alpha$ for different values of α for $a = 10$.

8.3 *Probability density*

As we got wave functions for various values of α, we can also get probability density, i.e. $\rho_\alpha = \psi_\alpha^* \psi_\alpha$. Probability density plot for various α is given in figure 3.

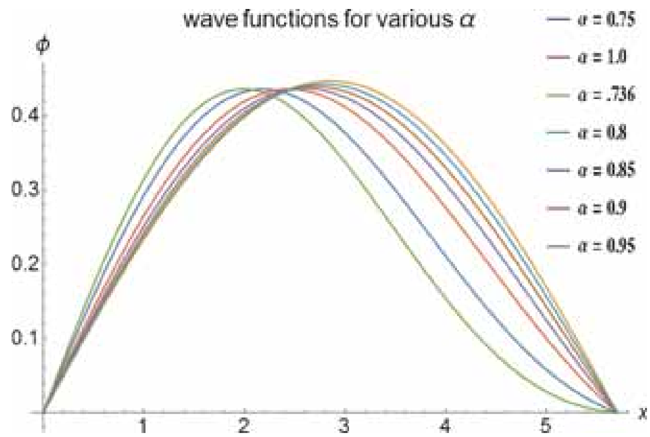

Figure 4. Graphical presentation of $\Phi(x^\alpha)$ for different values of α for $a = 5.7$.

8.4 *Energy calculation*

Now we shall calculate the fractional energy ε_α of the particle. Use of (31a) and (34c) gives

$$(2^\alpha m_\alpha \varepsilon_\alpha)/\hbar_\alpha^2 = (M_\alpha/a)^{2\alpha}$$

implying

$$\varepsilon_\alpha = (\hbar_\alpha^2/2^\alpha m_\alpha)\,(M_\alpha/a)^{2\alpha}\,.$$

For $\alpha = 1$ the energy is $\varepsilon_1 = (\hbar^2/2m)\,(\pi/a)^2$. This is the energy for the first quantum state as described in quantum mechanics.

If we choose the box length as 5.7 units, the plot is as in figure 4.

9. Conclusions

Using fractional derivative of Jumarie-type in the fractional Schrödinger equation, we found that in the range $0.736 < \alpha \leq 1$ of order of fractional derivative, the quantum behaviour of the particle in a one-dimensional box changes dramatically. In this range of fractional derivative, the equation of continuity is successfully maintained and the stationary condition also holds. For $\alpha = 1$ all the equations are reduced to the classical Schrödinger equation. Here, we studied particle in a box problem and found that the wave equation in fractional sense also meets the condition described in the classical sense. Further, the wave function is not symmetric till $\alpha = 1$. When $0.736 < \alpha < 1$ the peak of the wave function is left sided, i.e. the peak is on the left side of the middle point of the box. The existence amplitude, i.e. wave function has higher amplitude for higher α and it is maximum when $\alpha = 1$. Thus, the wave function or existence amplitude is α-dependent. We need to further study the fractional quantum mechanics of $\alpha < 0.736$ to understand the internal behaviour of the quantum states.

Acknowledgement

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Appendix

A.1 *Fractional mass*

Fractional mass m_α is defined as $m_\alpha = \int \rho \mathrm{d}x^\alpha$, where ρ is the fractional linear mass density in one dimension; when ρ is constant, $m_\alpha = \rho \int \mathrm{d}x^\alpha$. We have considered that the density is the same as the case $\alpha = 1$.

A.2 *Fractional velocity*

The fractional change of displacement, i.e. $\mathrm{d}^\alpha x = \Gamma(1+\alpha)\mathrm{d}x$ per unit change in fractional time differential $(\mathrm{d}t)^\alpha$ is the fractional velocity, i.e. with $0 < \alpha < 1$

$$\vec{v}_\alpha = \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \alpha! \frac{\mathrm{d}x}{(\mathrm{d}t)^\alpha}.$$

Then we can write the following equation:

$$\frac{\mathrm{d}^{1-\alpha}}{\mathrm{d}t^{1-\alpha}}(\vec{v}_\alpha) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{v}.$$

A.3 *Fractional wavelength*

Fractional wavelength is demonstrated in figure A1 of a fractional wave of the order $\alpha = 0.8$.

The wavelength is the distance AB. That is the distance covered by a fractional wave in a full fractional cycle. Fractional wavelength is not a fixed quantity. It changes with the evolution of fractional time like a damped oscillating wave.

A.4 *Fractional time period*

The time taken N_α for a wave to cover the distance AB (figure A1) is the fractional time period. We should note that this is the first-order time period. As wavelength changes, the time period also changes with the wave propagation. But we assume that $\lambda_\alpha = v_\alpha N_\alpha$.

A.5 *Fractional angular frequency*

Figure A2 is the polar plot of the fractional wave of the order $\alpha = 0.8$. In this polar plot we can easily see that the wave is returned to the same point after completing a fractional cycle, i.e. to its origin. From the polar plot we can say that the angle traversed in a full fractional cycle is 2π. Thus, the fractional angular frequency can be assigned as

$$\omega_\alpha = \frac{2\pi}{N_\alpha}.$$

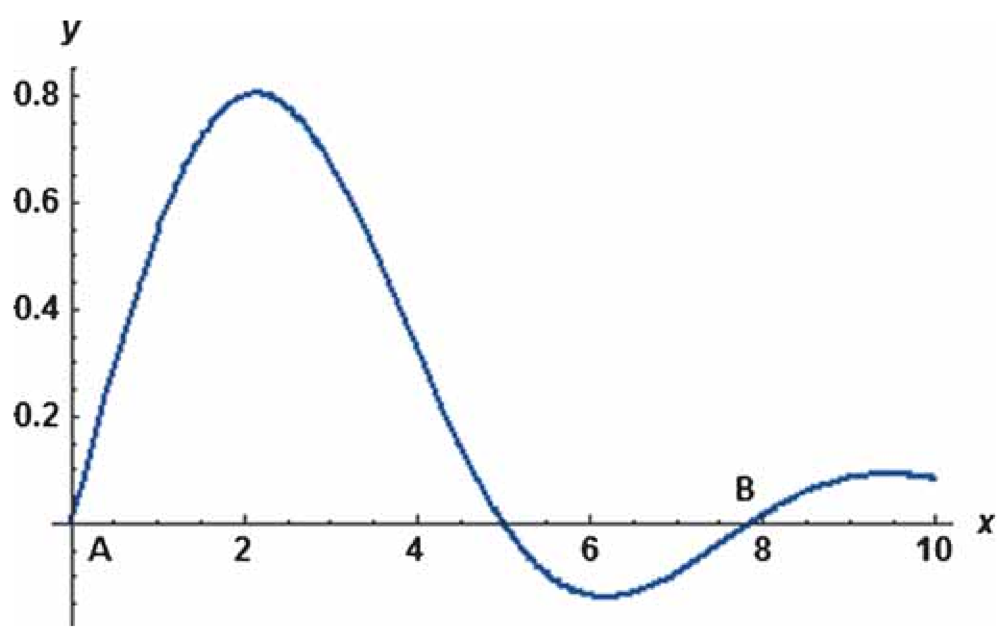

Figure A1. Fractional wavelength of a fractional wave of the order $\alpha = 0.8$.

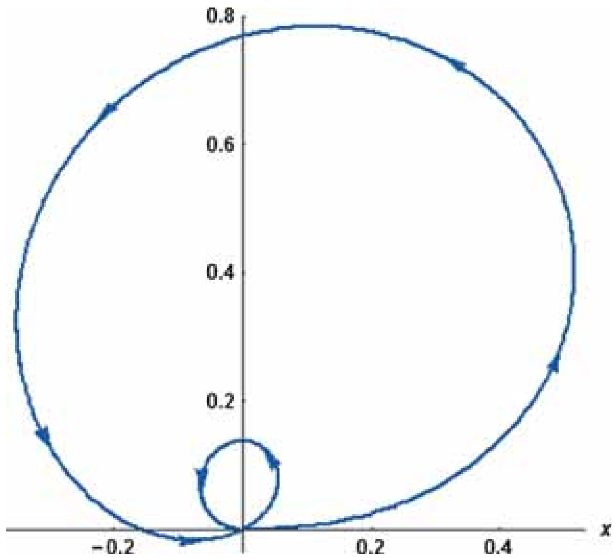

Figure A2. The plot showing the concept of fractional angular frequency.

In figure A2 the initial line is along the horizontal line. From this, we can see that the product of fractional angular momentum and fractional time period N_α is always 2π though both are varying. In limiting condition we have

$$\lim_{\alpha\to 1}\omega_\alpha=\omega \quad \text{or} \quad \lim_{\alpha\to 1}\frac{\omega_\alpha}{\omega}=1.$$

A.6 *Fractional wave constant (or vector in three dimensions)*

From the analysis of fractional wave of eq. (8) if $k_\alpha x^\alpha-\omega_\alpha t^\alpha=0$, that is the phase part of the wave is zero, we get $k_\alpha=\omega_\alpha\,(t^\alpha/x^\alpha)=\omega_\alpha/\upsilon_\alpha$ as $x^\alpha/t^\alpha=\upsilon_\alpha$ is the fractional velocity. We have $k_\alpha=(\omega_\alpha/\upsilon_\alpha)=2\pi/(\upsilon_\alpha N_\alpha)$, using Appendix (A.5). Now by using Appendix (A.4) we have $k_\alpha=2\pi/\lambda_\alpha$.

A.7 *Fractional reduced Planck's constant*

In this paper we have introduced fractional Planck's constant $\hbar_\alpha$ as a basic constant. For the limiting condition of α this constant is of the form of reduced Planck's constant $\hbar$. Consider integer (i.e. classical) and fractional energies, which are $E=\hbar\omega=h\upsilon$ and $\varepsilon_\alpha=\hbar_\alpha\omega_\alpha=h_\alpha\upsilon_\alpha$ respectively. Now we can write for the limiting condition, i.e. with $\alpha\to 1$,

$$E=h\upsilon=\lim_{\alpha\to 1}(h_\alpha\upsilon_\alpha).$$

Here, υ is the integer-order frequency and υ_α is the fractional-order frequency. We write the following expression with the above description:

$$h=\lim_{\alpha\to 1}\left(\frac{\upsilon_\alpha}{\upsilon}\right)h_\alpha, \quad \hbar=\lim_{\alpha\to 1}\frac{\omega_\alpha}{\omega}\hbar_\alpha.$$

Using Appendix A.5, we have, $\hbar=\lim_{\alpha\to 1}\hbar_\alpha$. We assume that the energy is proportional to angular frequency and this assumption does not depend upon the value of α. Therefore, we have, $E\propto\omega$ and $\varepsilon_\alpha\propto\omega_\alpha$. If we remove the proportionality, we can have a constant, such that the following condition is satisfied:

$$\frac{\varepsilon_{0.73}}{\omega_{0.73}}=\frac{\varepsilon_{0.79}}{\omega_{0.79}}=\frac{\varepsilon_{0.8}}{\omega_{0.8}}=\cdots=\frac{E}{\omega}=\hbar.$$

Therefore, we have $\hbar_\alpha=\hbar$ and hence $h_\alpha=h$. We can conclude that the fractional Planck's constant is nothing but Planck's constant. The reduced fractional Planck's constant is also the reduced Planck's constant.

A.8 Theorem

If a function, i.e. $f(x,y)$ is fractionally differentiable, with order α with respect to both the variables x and y, then $D_y^\alpha D_x^\alpha f(x,y)=D_x^\alpha D_y^\alpha f(x,y)$ or $f_{xy}^{2\alpha}(x,y)=f_{yx}^{2\alpha}(x,y)$ are equivalent. where $0\le\alpha\le 1$ and D^α is the Jumarie derivative operator.

Proof. Consider a function $\phi(x)=f(x,y+k)-f(x,y), k>0$. The fractional mean value theorem states, for $0<\theta<1$ and $0\le\alpha\le 1$, the following [14] equation:

$$\phi(x+h)-\phi(x)=\frac{h^\alpha}{\Gamma(1+\alpha)}\phi_x^\alpha(x+\theta h),$$

$$\phi(x+h)-\phi(x)=\frac{h^\alpha}{\Gamma(1+\alpha)}[f_x^\alpha(x+\theta h,y+k)-f_x^\alpha(x+\theta h,y)].$$

Let $F(y)=f_x^\alpha(x+\theta h,y)$. Then using fractional mean value theorem, we have the following expression:

$$\begin{aligned}\phi(x+h)-\phi(x)&=\frac{h^\alpha}{\Gamma(1+\alpha)}[F(y+k)-F(y)]\\&=\frac{h^\alpha}{\Gamma(1+\alpha)}\frac{k^\alpha}{\Gamma(1+\alpha)}[F_y^\alpha(y+\theta_1 k)-F_y^\alpha(y)]\\&=\frac{h^\alpha}{\Gamma(1+\alpha)}\frac{k^\alpha}{\Gamma(1+\alpha)}[f_{yx}^\alpha(x+\theta h,y+\theta_1 k)]\end{aligned}$$

where $0<\theta,\theta_1<1, 0\le\alpha\le 1$.

On the other hand, we have

$$\phi(x+h) = f(x+h, y+k) - f(x+h, y).$$

Therefore,

$$\phi(x+h) - \phi(x) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)$$

$$f_y^{\alpha}(x, y) = \Gamma(1+\alpha) \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k^{\alpha}}$$

or

$$\begin{aligned} f_{xy}^{2\alpha}(x, y) &= \Gamma(1+\alpha) \lim_{h\to 0} \frac{f_y^{\alpha}(x, y+h) - f_y^{\alpha}(x, y)}{h^{\alpha}} \\ &= (\Gamma(1+\alpha))^2 \lim_{h\to 0} \lim_{k\to 0} \frac{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)}{k^{\alpha} h^{\alpha}} \\ &= (\Gamma(1+\alpha))^2 \lim_{h\to 0} \lim_{k\to 0} \frac{\phi(x+h) - \phi(x)}{k^{\alpha} h^{\alpha}} \\ &= (\Gamma(1+\alpha))^2 \lim_{h\to 0} \lim_{k\to 0} f_{yx}^{2\alpha}(x+\theta h, y+\theta_1 k) \end{aligned}$$

or

$$f_{xy}^{2\alpha}(x, y) = f_{yx}^{2\alpha}(x, y).$$

Hence the theorem is proved. □

We have used this theorem in our derivation.

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