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A new approach for cluster structures in ^{16}O and ^{20}Ne

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Abstract. In this article, the cluster–cluster interaction between α-clusters in ^{16}O and ^{20}Ne is studied theoretically. Using the generalized Nikiforov–Uvarov (NU) technique, the clusterization energy for these nuclei is calculated. Based on the obtained results, one can find out that the clustering phenomenon does not take place neither at the ground state, nor at the excited states of these nuclei and it is more probable at energies among excited levels. It is shown that the formulation presented for the clustering phenomenon reproduces the results obtained in previous experimental and theoretical attempts. It is worth mentioning that the consistency of the results with the previous experimental and theoretical predictions for clustering phenomenon in ^{16}O and ^{20}Ne indicates the reliability of this formulation for various types of light α-conjugate nuclei, such as ^{8}Be, ^{12}C, ^{24}Mg and so on.

Keywords. Cluster–cluster interaction; clusterization energy; α-conjugate nuclei, Nikiforov–Uvarov (NU) technique.

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1. Introduction

Although shell model theory is an acceptable theory in nuclear physics, but using this theory, to study the structure of even–even nuclei, it is a challenge [1]. In the cluster model, some basic assumptions can be considered to simplify the problem, which provide a good dynamical description for the quantum system of structures composed of α-particles, called α-conjugate nuclei, which are nuclei with equal number of protons and neutrons and $N = Z = 2k(k = 2, 3, ...)$ [2,3]. In general, the energy of the quantum system is the basic and the most important parameter which can affect the clustering phenomenon in nuclei. In other words, in each light α-conjugate nucleus, depending on the amount of energy, one can expect a variety of structural changes from a cluster perspective, all of which lead to structures resembling molecular structures in these nuclei. These changes start from a structure consisting of an α-particle and a smaller nucleus and eventually end with a structure consisting of several α-clusters [3,4].

There are two models to describe clustering phenomenon in light α-conjugate nuclei, namely the microscopic [5] and the non-microscopic [1,5,6] models. The microscopic cluster model has been used extensively so far and no phenomenological cluster–cluster potential is utilized in it. But in the non-microscopic multicluster model, the α-clusters are treated as structureless particles and the phenomenological interaction potential between α-clusters includes the attractive nuclear, repulsive Coulomb and centrifugal terms [5,7]. The main purpose of this research is to attain an appropriate model using a non-microscopic approach to study the clustering phenomenon in α-conjugate nuclei.

Following Ikeda's achievement in schematically showing various possible molecule-like structures, numerous studies have been conducted and a molecular viewpoint of nuclear systems has been provided that shed light on different dynamical aspects of light α-conjugate nuclei [6]. In the Ikeda's diagram, the required amount of energy for the formation of each $n\alpha$-cluster ($n = 2, 3, \ldots, 7$) is approximately mentioned [4].

Nevertheless, clustering phenomenon is challenging with much of its physical processes and measurements remaining unknown so far. Nuclear clustering will be one of the most fruitful subjects of nuclear physics, and physicists will face some of the greatest challenges and opportunities in the coming years. More experimental and theoretical works are needed to clarify the nature of

clustering phenomenon. To the best of our knowledge, despite the importance of the subject, no comprehensive analytical model has been proposed to investigate the nature of clustering phenomenon in α-conjugate nuclei with a non-microscopic model [8–12].

In this work, the non-microscopic α-cluster model [4,13] is used for computational analysis of clustering phenomenon in ^{16}O and ^{20}Ne nuclei as four and five α-cluster structures, respectively. Then, a suitable potential for the interaction between α-clusters is suggested and, for the first time, a numerical method is proposed for calculating the clusterization energy of these nuclei as non-relativistic multi-α-particle quantum systems. As mentioned before, the energy level of the system is a strategic parameter to investigate the clustering phenomenon. Therefore, this article focusses on calculating this main parameter. Based on the calculations, it is revealed that the α-cluster phase is situated between excited levels. Furthermore, the predicted energies of the system in the α-cluster phase are 112.78 MeV for ^{16}O and 140.97 MeV for ^{20}Ne. The proposed mathematical model is comprehensively discussed in the following sections and it will be demonstrated that the obtained results are consistent with the experimental and theoretical values reported earlier.

2. Proposed mathematical model

As mentioned earlier, the phenomenological interaction potential between α-clusters should be the sum of nuclear, Coulomb and centrifugal terms [7]. Accordingly, the nuclear and Coulomb part of cluster–cluster potential is suggested as eq. (1), which is suitable for studying the light α-conjugate nuclei in their α-cluster phase:

$$V(r) = \frac{b\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-ar})^2} - \frac{c\mathrm{e}^{-ar}}{(1-\mathrm{e}^{-ar})} + \frac{f}{r}, \tag{1}$$

where b, c and f are repulsive core strength (at near distances), attraction strength (at intermediate distances) and the strength of Coulomb repulsion (at far distances) respectively and a is related to the range of potential. It should be noted that, the first two terms in the proposed potential are of Manning–Rosen-type and the last term is of Coulomb-type. The Manning–Rosen potential is one of the most useful, applicable and reasonable short-range potentials in physics which is used not only as a mathematical model for describing diatomic molecular vibrations but also as a potential model for studying bound states and scattering parameters in nuclear physics [14–16].

To calculate the energy of a multicluster system from a non-relativistic perspective in the presence of such a local potential, which is only a function of the interparticle distance (r), one should solve the time-independent Schrödinger equation. In a method that is provided to solve Schrödinger equation for a system of N-identical particles with a hyperspherical formalism, the part of Schrödinger equation which is related to the hyper-radius x is expressed as [17]

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2} + \frac{D-1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x} + \frac{2\mu}{\hbar^2}\left[E - V(x) - \frac{\hbar^2 l(l+D-2)}{2\mu x^2}\right]R(x) = 0, \tag{2}$$

where $R(x)$ and $V(x)$ are the radial parts of the N-body wave function and the hyperpotential, respectively. In eq. (2), $D = 3N - 3$ and μ is the reduced mass. E represents the energy of the system and h is the Planck's constant represented by $\hbar = h/2\pi$. l indicates the angular momentum quantum number. Thus, in the hyperspherical formalism, the radial part of the time-independent Schrödinger equation in the presence of the proposed potential takes the following form:

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2} + \frac{D-1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x} + \frac{2\mu}{\hbar^2}\left[E - \frac{b\mathrm{e}^{-2ax}}{(1-\mathrm{e}^{-ax})^2} + \frac{c\mathrm{e}^{-ax}}{(1-\mathrm{e}^{-ax})} - \frac{f}{x} - \frac{\hbar^2 l(l+D-2)}{2\mu x^2}\right]R(x) = 0. \tag{3}$$

It is worth mentioning that in eq. (3) the potential equation (1) is added to a centrifugal term and the effective potential of the system is actually the sum of Manning–Rosen, Coulomb and centrifugal terms [18].

It is known that eq. (3) cannot be solved by the usual methods. Thus, like other researchers, we use a proper approximation for the centrifugal term [18–23]:

$$\frac{1}{x^2} \approx \frac{a^2\mathrm{e}^{-2ax}}{(1-\mathrm{e}^{-ax})^2} \tag{4}$$

and equivalently, we can consider

$$\frac{1}{x} \approx \frac{a\mathrm{e}^{-ax}}{1-\mathrm{e}^{-ax}}. \tag{5}$$

In the last two equations, a ensures the integrity of dimensions. In figure 1, it is demonstrated that for short potential ranges, eq. (4) is a proper approximation for $y(x) = \dfrac{1}{x^2}$.

As a result, eq. (3) becomes

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{(D-1)a\mathrm{e}^{-ax}}{(1-\mathrm{e}^{-ax})}\frac{\mathrm{d}R(x)}{\mathrm{d}x}+\frac{\left(a\mathrm{e}^{-ax}\right)^2}{(1-\mathrm{e}^{-ax})^2}\times\left[\left(\frac{2\mu E}{\hbar^2}\right)\left(\frac{(1-\mathrm{e}^{-ax})^2}{\left(a\mathrm{e}^{-ax}\right)^2}\right)-\frac{2\mu b}{\hbar^2a^2}+\left(\frac{2\mu c}{\hbar^2}\right)\left(\frac{(1-\mathrm{e}^{-ax})}{a^2\mathrm{e}^{-ax}}\right)-\left(\frac{2\mu f}{\hbar^2}\right)\left(\frac{(1-\mathrm{e}^{-ax})}{a\mathrm{e}^{-ax}}\right)-l(l+D-2)\right]R(x)=0. \tag{6}$$

By introducing a new variable as

$$s(x)^{-1}=\frac{a\mathrm{e}^{-ax}}{(1-\mathrm{e}^{-ax})}, \tag{7}$$

eq. (6) can be further transformed into the following form:

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}s(x)^2}+\frac{(D-1)}{s(x)}\frac{\mathrm{d}R(x)}{\mathrm{d}s(x)}+\frac{1}{s(x)^2}\times\left[\frac{2\mu E}{\hbar^2}s(x)^2-\frac{2\mu b}{\hbar^2a^2}+\left(\frac{2\mu c}{\hbar^2a}-\frac{2\mu f}{\hbar^2}\right)s(x)-l(l+D-2)\right]R(x)=0. \tag{8}$$

Then, with further analysis and simplification, eq. (8) becomes

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}s(x)^2}+\frac{(D-1)}{s(x)}\frac{\mathrm{d}R(x)}{\mathrm{d}s(x)}+\frac{1}{s(x)^2}\times[-\varepsilon^2s(x)^2+M_1s(x)+M_2]R(x)=0, \tag{9}$$

where

$$\begin{cases}\varepsilon^2=\dfrac{-2\mu E}{\hbar^2}, & \varepsilon>0,\\ M_1=\dfrac{2\mu c}{\hbar^2a}-\dfrac{2\mu f}{\hbar^2},\\ M_2=\dfrac{-2\mu b}{\hbar^2a^2}-l(l+D-2).\end{cases} \tag{10}$$

As it can be realized, the Schrödinger equation with the considered potential for the interaction between α-clusters has transformed to a second-order differential equation that agrees with the general form of the equation in Nikiforov–Uvarov method. This technique can be used to solve eq. (9). In the next section, the parametric NU method is briefly described.

3. The general framework of the Nikiforov–Uvarov (NU) technique

The Nikiforov–Uvarov method offers a powerful mathematical model to solve second-order differential equations. For a given potential, the Schrödinger equation is reduced to a generalized equation of hypergeometric-type with an appropriate coordinate transformation $s = s(r)$. In this method, the differential equations can be written as follows [24,25]:

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{11}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials that can be at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find a particular solution for eq. (11) by

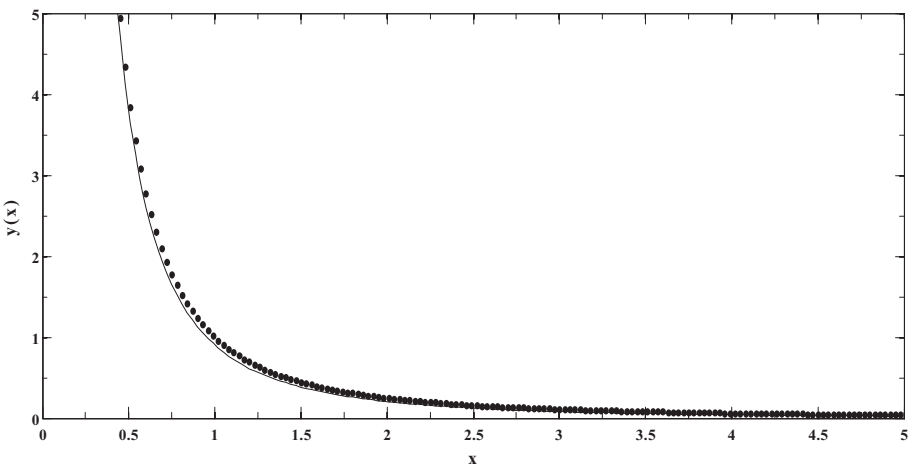

Figure 1. The centrifugal-type term (solid curve) and its approximation in eq. (4) (dotted curve).

separation of variables, the following transformation is applied:

$$\psi_n(s) = \varphi_n(s) y_n(s) \tag{12}$$

which reduces eq. (11) to a hypergeometric-type function:

$$\sigma(s) y_n''(s) + \tau(s) y_n'(s) + \lambda y_n(s) = 0 \tag{13}$$

where $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ and $\tau'(s) < 0$ which means $\tau(s)$ has a negative derivative. Additionally, λ is a parameter with the following definitions:

$$\begin{cases} \lambda_n = -n\tau'(s) - \dfrac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, ... \\ \lambda = k + \pi'(s) \end{cases} \tag{14}$$

and equality of the two parts in eq. (14) yields the energy eigenvalues of the intended multiparticle system.

Furthermore, the function $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{15}$$

$\pi(s)$ should be a first-degree polynomial and the expression under the square root must be square of a polynomial. In this way, the value of k will be determined [26,27].

4. Mathematical calculation and results

The NU method has been applied to present the results based on the discussed model. To this end, eqs (9) and (11) have been compared and the following expressions have been obtained:

$$\begin{cases} \tilde{\tau} = (D-1), \\ \sigma(s) = s(x), \\ \tilde{\sigma}(s) = [-\varepsilon^2 s(x)^2 + M_1 s(x) + M_2]. \end{cases} \tag{16}$$

Substituting the above expressions into eq. (15) and considering the NU method condition for $\pi(s)$, with some analysis and simplification, the following equation can be achieved:

$$\pi(s) = \frac{(2-D)}{2} \pm \left[\varepsilon s(x) + \left(-M_2 + \left[\frac{(2-D)}{2}\right]^2\right)^{1/2}\right]. \tag{17}$$

Since we have the polynomial $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ with a negative derivative, the suitable form has to be established for this parameter. Therefore,

$$\tau(s) = 1 - 2\left[\varepsilon s(x) + \left(-M_2 + \left[\frac{(2-D)}{2}\right]^2\right)^{1/2}\right]. \tag{18}$$

Finally, considering the notations of (10) and eqs (14), the equation of energy for such a system of α-clusters can be obtained:

$$E = -\left(\frac{2\mu}{\hbar^2}\right)\left(\frac{(c/a) - f}{(1+2n) - \left(\frac{8\mu b}{\hbar^2 a^2} + 4l(l+D-2) + (2-D)^2\right)^{1/2}}\right)^2. \tag{19}$$

Having achieved this important equation, we can calculate the clusterization energy for ^{16}O and ^{20}Ne nuclei in their α-cluster phase by assigning appropriate values to coefficients of the potential.

It is worth mentioning that in our non-microscopic model, the α-clusters are assumed to be structureless particles. Thus, we are dealing with four clusters in ^{16}O and five clusters in ^{20}Ne interacting via eq. (1). In table 1, the obtained results are compared with the previous theoretical and experimental values [3,12,28].

In this paper, binding energy is the energy of a bound system consisting of A nucleons, but clusterization energy is the energy of a bound system consisting of k α-clusters. (A is the number of nucleons and k is the number of α-clusters). It can be realized that the energy of the nucleonic system is different from the energy of the clustered system (and hence the equivalent mass of energy in these two situations is different). This reminder corresponds to Ikeda's achievements in 1968 [4]. Additionally, it should be noted that, for each of the nucleus listed in table 1, the potential coefficients are determined by fitting the parameters so that the clusterization energies mentioned in the Ikeda's diagram [4] are reproduced.

As shown in table 1, the clusterization energies, calculated with the proposed potential in this paper, are in good agreement with the experimental and theoretical results for the two intended nuclei, which are

Table 1. Binding energy (at ground state), energy of the system in ^{16}O and ^{20}Ne as four and five α-clusters respectively, and the comparison of results with previous experimental and theoretical attempts [3,4,12,28,29].

Nucleus	Coefficients of the potential	Binding energy (MeV)	Energy of the clustered system in our work (MeV)	Other experimental attempts (MeV)	Other theoretical attempts (MeV)
^{16}O (as a 4α-structure)	$a = 0.6$ fm^{-1}, $b = 2$ MeV, $c = 96.1$ MeV, $f = 7$ MeV·fm	-127.621	-112.78	-113.181	-110
^{20}Ne (as a 5α-structure)	$a = 0.7$ fm^{-1}, $b = 2$ MeV, $c = 179.3$ MeV, $f = 7$ MeV·fm	-160.6471	-140.97	-140.687	-137.5

mentioned as preliminary examples. Therefore, the proposed mathematical model can be reliable for other light α-conjugate nuclei such as ^{8}Be, ^{12}C, ^{24}Mg and so on, with $N = 2, 3, ...$ in eq. (2) and fitting the potential parameters for each specific nucleus. These investigations will be reported in a separate work later.

5. Conclusions

The main purpose of this research is to attain a unified method to utilize the α-cluster model for all light α-conjugate nuclei. Thus, using a proper non-microscopic approach, the molecule-like picture of ^{16}O and ^{20}Ne nuclei as four and five α-cluster structures respectively is considered. The suitable potential which is assumed for the interaction between α-clusters in this paper, is a composition of nuclear, Coulomb and centrifugal terms. The radial part of the time-independent Schrödinger equation with this potential is analytically solved. The energy equation for a system of N-α clusters is obtained and the clusterization energies for these two nuclei are calculated. It is revealed that the results are consistent with the previous experimental and theoretical attempts.

Finally, it is concluded that in the non-microscopic approach for the clustering phenomenon in ^{16}O and ^{20}Ne, the proposed potential and mathematical model can be appropriate for all other light α-conjugate nuclei. Under these interpretations, more theoretical works are required to clarify the application of cluster models to various properties of light α-conjugate nuclei.

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