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Multiwave solutions of time-fractional (2 + 1)-dimensional Nizhnik–Novikov–Veselov equations

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Abstract. In this paper, we present a generalized unified method for finding multiwave solutions of the time-fractional (2+1)-dimensional Nizhnik–Novikov–Veselov equations. The fractional derivatives are described in the modified Riemann–Liouville sense. The fractional complex transform has been suggested to convert fractional-order differential equations with modified Riemann–Liouville derivatives into integer-order differential equations, and the reduced equations can be solved by symbolic computation. Multiauxiliary equations have been introduced in this method to obtain not only multisoliton solutions but also multiperiodic or multielliptic solutions. It is shown that the considered method is very effective and convenient for solving wide classes of nonlinear partial differential equations of fractional order.

Keywords. The generalized unified method; the modified Riemann–Liouville derivative; time-fractional (2+1)-dimensional Nizhnik–Novikov–Veselov equations; multiwave solutions.

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1. Introduction

Nonlinear evolution equations have been used to study some nonlinear wave phenomena. They are encountered in a variety of scientific and engineering fields [1–8]. Among these equations; fractional differential equations (FDEs) which are essential tools to describe many phenomena and dynamic processes in physics, mechanics, signal processing, control theory, systems identification, chemistry, biology and other areas [9–12]. We think that finding exact solutions of FDEs is the most suitable way for the better understanding of the dynamics and the mechanism of related physical models. The explicit solutions of these equations, if available, facilitate the verification of numerical solvers and aid in the stability analysis of solutions.

Recently, a large amount of literature has been provided to construct the solutions of FDEs. Among these methods; the fractional subequation method [13–15], the (G'/G)-expansion method [16] and the exp-function method [17,18].

In [19], Jumarie proposed a modified Riemann–Liouville derivative. With this kind of fractional derivative and some useful formulas, we can convert FDEs into integer-order differential equations by fractional complex transformation [20]. Also, Li and He [21,22] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODEs). So all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus.

Our objective in this study is to introduce the generalized unified method which is a new method. In fact, this method is a general method to the unified method [23–25]. By using this method in the sense of the modified Riemann–Liouville derivative in tackling nonlinear time-fractional differential equations, we construct multiwave solutions for time-fractional differential equations such as multisolitary wave solutions, multiperiodic wave solutions, multielliptic wave solutions and combined formal multiwave solutions.

Here, we use the generalized unified method to find multiwave solutions of the time-fractional (2+1)-dimensional Nizhnik–Novikov–Veselov equations (FNNVEs) which is a variation of (2+1)-dimensional Nizhnik–Novikov–Veselov (NNV) system [26–28]. FNNVEs are obtained from the NNV system by replacing the first time derivative term by a fractional derivative of order α, $0 < \alpha \le 1$.

The reason for using fractional-order differential equations is that they are naturally related to systems with memory, closely related to fractals which are abundant in an incompressible fluid. Fractional derivatives are used to model anomalous diffusion, where a particle plume spreads at a rate inconsistent with the classical model, and the plume may be asymmetric.

The results derived from the fractional model are of a more general nature. A fractal time derivative leads to subdiffusion, where a cloud of particle spreads slower than the classical rate. Also, a fractal time derivative models particle sticking and trapping, a subdiffusive effect [29,30].

The remainder of this paper is organized as follows. In §2, a brief review of the theory of fractional calculus and some basic properties of the modified Riemann–Liouville derivative have been provided. A description of the generalized unified method and the fractional complex transform are presented in §3. Section 4 is devoted to the application of the generalized unified method to FNNVEs. Conclusions are given in §5.

2. The modified Riemann–Liouville derivative

There are different definitions for the concept of a fractional derivative. Some of these are: Riemann–Liouville, Grunwald–Letnikow, Caputo and modified Riemann–Liouville derivative [9,10]. The most commonly used definitions are the Riemann–Liouville and Caputo derivatives.

Jumarie proposed a modified Riemann–Liouville derivative. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper [19].

Assume that $f{:}R \to R$, $x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $k > 0$ in the interval [0, 1].

Through the fractional Riemann–Liouville integral

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} f(t_1)\mathrm{d}t_1, \quad \alpha > 0, \tag{1}$$

and the Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression [19,31]:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-t_1)^{-\alpha}(f(t_1)-f(0))\mathrm{d}t_1, \\ 0 < \alpha < 1 \\ (f^{(n)}(t))^{(\alpha-n)}, \qquad n \le \alpha < n+1, n \ge 1. \end{cases} \tag{2}$$

Some useful formulas and results of the fractional modified Riemann–Liouville derivative are summarized [20–22].

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad \beta > 0 \tag{3}$$

$$D_t^\alpha (f(t)\,g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t) \tag{4}$$

$$D_t^\alpha h[f(t)] = h'_f[f(t)]\, D_t^\alpha f(t) = D_t^\alpha h[f(t)](f'(t))^\alpha, \tag{5}$$

where $f(t)$ and $g(t)$ satisfy the definition of the modified Riemann–Liouville derivative and $h(t)$ is an α-order differentiable function.

3. The fractional complex transform and the mathematical formulation of the generalized unified method

In this section, we present the outline of the generalized unified method to find multiwave solutions to the time-fractional $(q+1)$-dimensional FDEs in the sense of the modified Riemann–Liouville derivative.

We consider the following general nonlinear time-fractional $(q+1)$-dimensional FDEs of the type

$$F_i(u_j,\ D_t^\alpha(u_j),\ D_t^\alpha D_t^\alpha(u_j), ..., (u_j)_{x_1}, ..., (u_j)_{x_q}, (u_j)_{x_1 x_2}, (u_j)_{x_1 x_3}, ...) = 0, \quad 0 < \alpha < 1, \tag{6}$$

where $u_j = u_j(t, x_1, ..., x_q)$, $i,\ j = 1, 2, ..., l$.

By the fractional complex transform in [19–22] and by using eqs (1)–(5), each physical observable u_j possesses $(q+1)$ basic travelling wave solutions that satisfy the equation

$$H_i(U_j, (U_j)_{z_1}, ..., (U_j)_{z_q}, (U_j)_{z_1 z_2}, (U_j)_{z_1 z_3}, ...) = 0,$$
$$z_j = \frac{\beta_j t^\alpha}{\Gamma(1+\alpha)} + \sum_{s=1}^{q} \beta_{j,s} x_s, \tag{7}$$

where β_j, $\beta_{j,s}$ are arbitrary constants and $U_j = U_j(z_1, ..., z_q)$.

The fundamental rules and objectives of the unified method are also used here (for details, see [23–25]). The only distinction is that: the main aim in [23–25] is to search for a single travelling wave solution, namely

$$U_j = U_j(z), \quad z = \beta_0 t + \sum_{j=1}^{q} \alpha_j x_j.$$

For N-soliton (periodic or elliptic) wave solutions of (7), we have to construct solutions of (6) in the form

$$u(x_1, ..., x_q, t) = U(z_1, ..., z_N). \tag{8}$$

Similarly, as we did in the unified method [23–25], we search for:

(i) polynomial function solutions,
(ii) rational function solutions.

In this paper, we confine ourselves to find multiwave solutions as polynomial function solutions when $N = 2$. In a future work, we shall find multiwave solutions as rational function solutions when $N = 2$ and $N = 3$.

3.1 *The polynomial function solutions*

In this section, we introduce the steps of computations of N-wave polynomial solutions as follows:

Step 1. The generalized unified method asserts that, the N-wave solutions of degree n to (7)

$$U(z_1, z_2, ..., z_N) = a_0 + \sum_{I_N=1}^{n} a_{i_1,i_2,...,i_N} \times \phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2), ..., \phi_N^{i_N}(z_N),$$
$$I_N = \sum_{j=1}^{N} i_j, \quad N \geq 2 \tag{9}$$

where a_0 and $a_{i_1,i_2,...,i_N}$ are arbitrary constants to be determined later and $\phi_j(z_j)$, $j = 1, 2, ..., N$ satisfy the auxiliary equations

$$(\phi_j'(z_j))^p = \sum_{r=0}^{pk} b_{j,r}\,\phi_j^r(z_j),$$
$$z_j = \frac{\beta_{j,0}\,t^\alpha}{\Gamma(1+\alpha)} + \sum_{s=1}^{q} \beta_{j,s}\,x_s, \quad p = 1, 2, \tag{10}$$

where $b_{j,r}$, $\beta_{j,s}$ and $\beta_{j,0}$ are constants. It is worth noting that, n and k are determined from the balance equation by the criteria given in [23–25].

Also, a second condition (the consistency condition), which asserts that the constants in (9) can be consistently determined, is used.

When p=1, (10) solves elementary solutions (explicit or implicit) and when $p = 2$, it solves elliptic solutions.

Step 2. By inserting (9) together with (10) into (7), we get an equation which is splitting to a set of nonlinear algebraic equations namely "the principle equations". They are solved by any computer algebra system.

Step 3. Solving the auxiliary equations in (10).

Step 4. Finding the formal exact solutions which is given in (9).

4. Multiwave solutions of FNNVEs using the generalized unified method

In this section, we apply the generalized unified method described in §3 to find multiwave solutions of the time-fractional (2+1)-dimensional Nizhnik–Novikov–Veselov equations (FNNVEs) which read as

$$D_t^\alpha u - u_{xxx} + \lambda\,(u\,v)_x = 0, \quad 0 < \alpha \leq 1,$$
$$u_x + \gamma\,v_y = 0, \tag{11}$$

where λ and γ are arbitrary constants. The FNNVE system is a variation of (2+1)-dimensional Nizhnik–Novikov–Veselov (NNV) system. When $\alpha \to 1$, FNNVE system may be considered as a model for an incompressible fluid (NNV system) where u and v are components of the (dimensionless) velocity [32]. Boiti *et al* solved NNV system of equations via the inverse scattering transformation [33]. It is well known that, the NNV system of equations is an isotropic Lax integrable extension of the well-known (1+1)-dimensional KdV equation and has physical significance [34]. Also, NNV system of equations can be obtained from the inner parameter-dependent symmetry constraint of the KP equation [35].

4.1 *The polynomial function solutions (when $N = 2$)*

From eqs (9) and (10), we have

$$u(x,t) = U(z_1, z_2) = p_0 + \sum_{i_1+i_2=1}^{n_1} p_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2),$$
$$v(x,t) = V(z_1, z_2) = q_0 + \sum_{i_1+i_2=1}^{n_2} q_{i_1,i_2}(t)\,\phi_1^{i_1}(z_1) \times \phi_2^{i_2}(z_2),$$
$$(\phi_1'(z_1))^p = \sum_{r=0}^{pk} b_r \phi_1^r(z_1),$$
$$(\phi_2'(z_2))^p = \sum_{r=0}^{pk} c_r \phi_2^r(z_2), \qquad p = 1,2, \tag{12}$$

where

$$z_1 = \frac{\alpha_0\,t^\alpha}{\Gamma(1+\alpha)} + \alpha_1\,x + \alpha_2\,y,$$
$$z_2 = \frac{\beta_0\,t^\alpha}{\Gamma(1+\alpha)} + \beta_1 x + \beta_2 y$$

and p_0, p_{i_1,i_2}, q_0, q_{i_1,i_2} are arbitrary constants.

Case 1. When $p = 1$

When $p = 1$, the balance condition yields $n_1 = n_2 = 2(k-1)$, $k > 1$ and the consistency condition gives rise to $k \leq 3$ [23–25]. Thus, in this case, the solutions exist when $k = 2, 3$.

(I_1) When $k = 2$, $n_1 = n_2 = 2$.

In this case, we have

$$u(x,y,t)=U(z_1,z_2)=p_0+\sum_{i_1+i_2=1}^{2} p_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2),$$

$$v(x,y,t)=V(z_1,z_2)=q_0+\sum_{i_1+i_2=1}^{2} q_{i_1,i_2}(t)\phi_1^{i_1}(z_1)\phi_2^{i_2}(z_2),$$

$$\phi_1'(z_1)=\sum_{r=0}^{2} b_r\,\phi_1^r(z_1),\quad \phi_2'(z_2)=\sum_{r=0}^{2} c_r\,\phi_2^r(z_2). \tag{13}$$

By substituting (13) into (11) and by using any package in symbolic computations (here, we use the elimination method, or other suitable solvable method, with the aid of MATHEMATICA or MAPLE), we get

$$\begin{aligned}
p_0 &= \frac{\beta_2\,\nu(-\beta_1\,\alpha_0+\alpha_1\,(b_1^2\,\alpha_1^2\,\beta_1+8\,b_2\,b_0\,\alpha_1^2\,\beta_1-c_1^2\,\beta_1^3-8\,c_2\,c_0\,\beta_1^3+\beta_0))}{2\,\alpha_1\,\beta_1^2\,\lambda},\\
p_{1,0} &= \frac{6\,b_1\,b_2\,\alpha_1^2\,\beta_2\,\nu}{\beta_1\,\lambda},\quad p_{0,1}=-\frac{6\,c_1\,c_2\,\beta_1\,\beta_2\,\nu}{\lambda},\quad p_{1,1}=0,\\
p_{0,2} &= -\frac{6\,c_2^2\,\beta_1\,\beta_2\,\nu}{\lambda},\quad p_{2,0}=\frac{6\,b_2^2\,\alpha_1^2\,\beta_2\,\nu}{\beta_1\,\lambda},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
q_0 &= \frac{b_1^2\,\alpha_1^3\,\beta_1+8\,b_2\,b_0\,\alpha_1^3\,\beta_1+c_1^2\,\alpha_1\,\beta_1^3+8\,c_2\,c_0\,\alpha_1\,\beta_1^3-\beta_1\,\alpha_0-\alpha_1\,\beta_0}{2\,\alpha_1\,\beta_1\,\lambda},\\
q_{1,0} &= \frac{6\,b_1\,b_2\,\alpha_1^2}{\lambda},\quad q_{0,1}=\frac{6\,c_1\,c_2\,\beta_1^2}{\lambda},\quad q_{1,1}=0,\\
q_{0,2} &= \frac{6\,c_2^2\,\beta_1^2}{\lambda},\quad q_{2,0}=\frac{6\,b_2^2\,\alpha_1^2}{\lambda},\quad \alpha_2=-\frac{\alpha_1\,\beta_2}{\beta_1}.
\end{aligned} \tag{15}$$

It remains to solve the auxiliary equations in $(13)_3$. By a direct calculation, we get

$$\begin{aligned}
\phi_1(z_1) &= -\frac{b_1+R_1\,\tanh((1/2)\,R_1\,z_1)}{2\,b_2},\\
\phi_2(z_2) &= -\frac{c_1+R_2\,\tanh((1/2)\,R_2\,z_2)}{2\,c_2},
\end{aligned} \tag{16}$$

where $R_1^2=b_1^2-4\,b_2\,b_0>0$ and $R_2^2=c_1^2-4\,c_2\,c_0>0$.

Substituting (14), (15) and (16) into (13) we get the solution of (11), namely

$$\begin{aligned}
U(z_1,z_2) &= \frac{\beta_2\,\nu}{2\,\alpha_1\,\beta_1^2\,\lambda}\Bigg(-\beta_1\,\alpha_0+\alpha_1\beta_1\Bigg(-2\,b_1^2\,\alpha_1^2\\
&\quad+8\,b_2\,b_0\,\alpha_1^2+2\,c_1^2\,\beta_1^2-8\,c_2\,c_0\,\beta_1^2\\
&\quad+3\,R_1^2\,\alpha_1^2\,\tanh^2\left(\frac{1}{2}\,R_1\,z_1\right)\\
&\quad-3\,R_2^2\,\beta_1^2\,\tanh^2\left(\frac{1}{2}\,R_2\,z_2\right)\Bigg)+\alpha_1\,\beta_0\Bigg)
\end{aligned}$$

$$\begin{aligned}
V(z_1,z_2) &= \frac{1}{2\,\alpha_1\,\beta_1\,\lambda}\Bigg(-\beta_1\,\alpha_0+\alpha_1\,\beta_1\Bigg(-2\,b_1^2\,\alpha_1^2\\
&\quad+8\,b_2\,b_0\,\alpha_1^2-2\,c_1^2\,\beta_1^2+8\,c_2\,c_0\,\beta_1^2\\
&\quad+3R_1^2\,\alpha_1^2\,\tanh^2\left(\frac{1}{2}\,R_1\,z_1\right)\\
&\quad+3R_2^2\,\beta_1^2\,\tanh^2\left(\frac{1}{2}\,R_2\,z_2\right)-\alpha_1\,\beta_0\Bigg)\Bigg)
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
z_1 &= \alpha_1\,x-\frac{\alpha_1\,\beta_2}{\beta_1}\,y+\frac{\alpha_0\,t^\alpha}{\Gamma(1+\alpha)},\\
z_2 &= \beta_1\,x+\beta_2\,y+\frac{\beta_0\,t^\alpha}{\Gamma(1+\alpha)},\\
&\quad \beta_0,\ \beta_1,\ \beta_2,\alpha_0,\ \alpha_1,\lambda
\end{aligned}$$

and ν are arbitrary constants and $0<\alpha<1$.

The solution (17) is shown in figures 1 and 2 for different values of α.

Figure 1 shows the overlapping of rogue and antirogue waves. They envelope soliton and antisoliton waves (kink waves).

Figure 2 shows the overlapping of two rogue waves in the same directions. In both figures, the overlapping region between these waves depends on the values of α, $0<\alpha<1$.

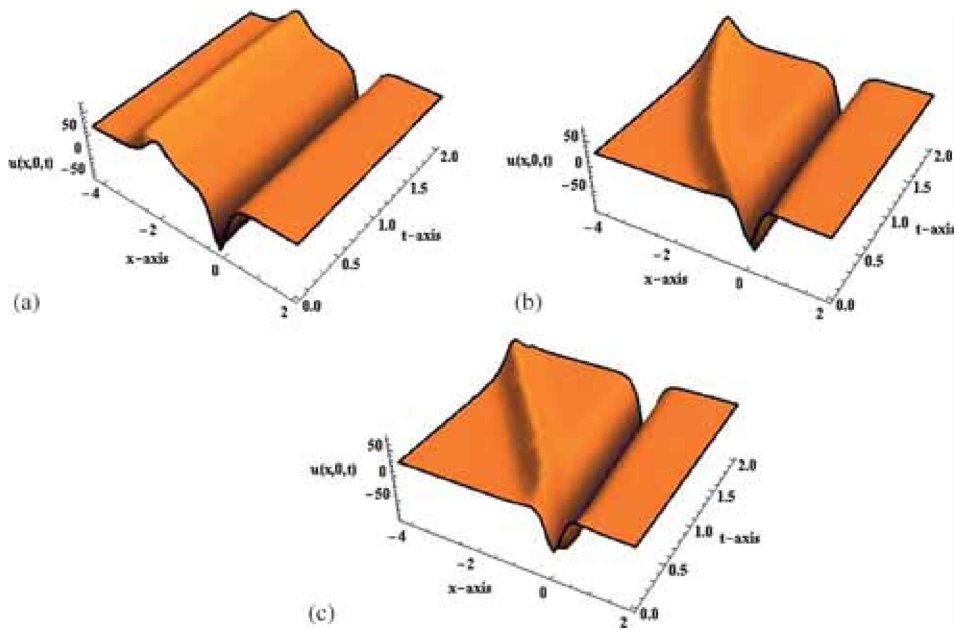

Figure 1. 3D plot for $u(x, y, t)$ when $y = 0$. (**a**) $\alpha = 0.02$, (**b**) $\alpha = 0.6$, (**c**) $\alpha = 0.99$. $\alpha_1 = 2,\ \alpha_3 = 5,\ \beta_1 = -11,\ \beta_2 = 7,\ \beta_3 = -3$ and $b_0 = 1,\ c_0 = 0.1,\ b_1 = 1,\ c_1 = 0,\ b_2 = c_2 = -1$. $\lambda = 0.2,\ \nu = 0.5$.

The time evolution of solution (17) is depicted in figures 3 and 4.

Figure 3 shows two soliton and antisoliton waves with no interference region moving along the x-axis from right to left with the time evolution.

In figure 4 there is an interaction of two solitons where a rogue wave is generated. They are symmetric about the region in the xy-plane and moving along the x-axis from right to left with the time evolution.

(I_2) When $k = 3, n_1 = n_2 = 4$.

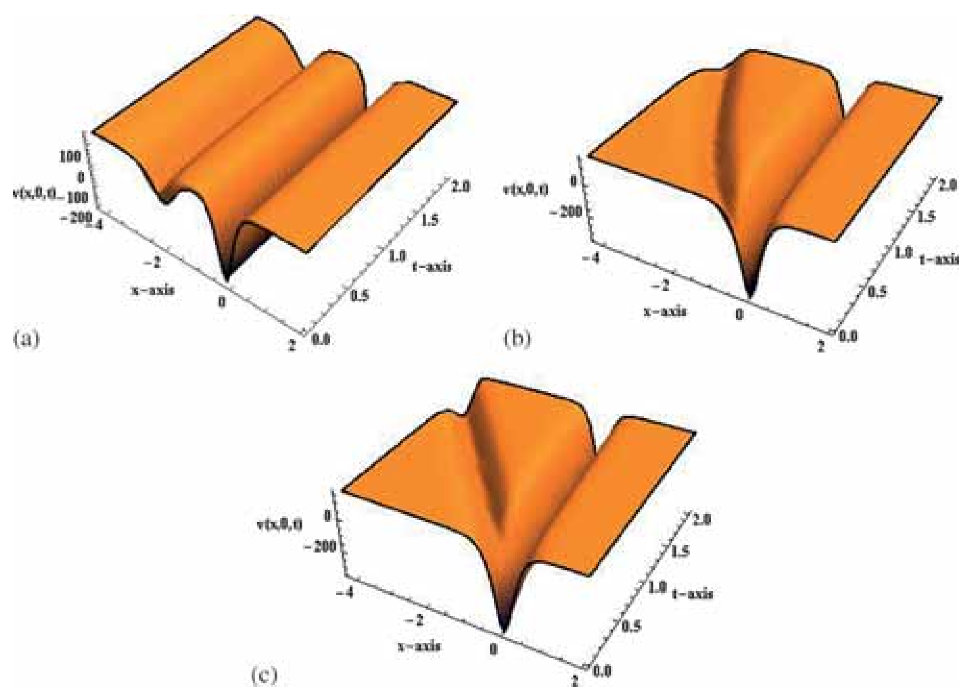

Figure 2. 3D plot for $v(x, y, t)$ when $y = 0$. (**a**) $\alpha = 0.02$, (**b**) $\alpha = 0.6$, (**c**) $\alpha = 0.99$ with the same caption as in figure 1.

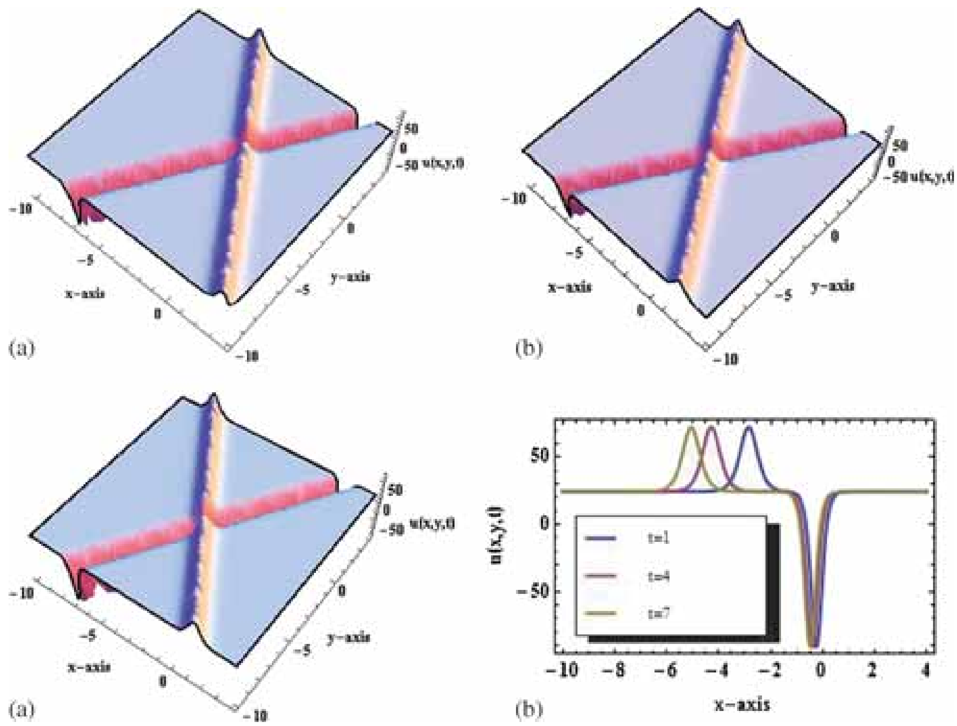

Figure 3. (**a**)–(**c**) 3D plot for $u(x, y, t)$ when $\alpha = 0.3$. (**a**) $t = 1$, (**b**) $t = 4$, (**c**) $t = 7$. (**d**) 2D plot for $u(x, y, t)$ when $\alpha = 0.3$, $y = 0$. $\alpha_1 = 2$, $\alpha_3 = 5$, $\beta_1 = -11$, $\beta_2 = 7$, $\beta_3 = -3$ and $b_0 = 1$, $c_0 = 0.1$, $b_1 = 1$, $c_1 = 0$, $b_2 = c_2 = -1$. $\lambda = 0.2$, $\nu = 0.5$.

In this case, we have

$$\begin{aligned} u(x,y,t) &= U(z_1,z_2) \\ &= p_0 + \sum_{i_1+i_2=1}^{4} p_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2), \\ v(x,y,t) &= V(z_1,z_2) \\ &= q_0 + \sum_{i_1+i_2=1}^{4} q_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2), \\ \phi_1'(z_1) &= \sum_{r=0}^{3} b_r\,\phi_1^r(z_1), \\ \phi_2'(z_2) &= \sum_{r=0}^{3} c_r\,\phi_2^r(z_2). \end{aligned} \tag{18}$$

By a similar way as we did in part I_1, we get the solution of (11) in the form

$$\begin{aligned} U(z_1,z_2) = \frac{\beta_2\,\nu}{2\,\beta_1\,\lambda}\Bigg(& 4\,b_1^2\,\alpha_1^2 + \frac{48\,b_1^2\,b_3\,\alpha_1^2\,\exp(2\,b_1\,z_1)}{1-b_3\,\exp(2\,b_1\,z_1)} \\ & + \frac{48\,b_1^2\,b_3^2\,\alpha_1^2\,\exp(4\,b_1\,z_1)}{(1-b_3\,\exp(2\,b_1\,z_1))^2} \\ & - 4\,c_1^2\,\beta_1^2 - \frac{48\,c_1^2\,c_3^2\,\beta_1^2\,\exp(4\,c_1\,z_2)}{(-1+c_3\,\exp(2\,c_1\,z_2))^2} \\ & + \frac{48\,c_1^2\,c_3\,\beta_1^2\,\exp(2\,c_1\,z_2)}{-1+c_3\,\exp(2\,c_1\,z_2)} - \frac{\alpha_0}{\alpha_1} + \frac{\beta_0}{\beta_1}\Bigg), \end{aligned}$$

$$\begin{aligned} V(z_1,z_2) = \frac{1}{2\,\lambda}\Bigg(& 4\,b_1^2\,\alpha_1^2 + \frac{48\,b_1^2\,b_3\,\alpha_1^2\,\exp(2\,b_1\,z_1)}{1-b_3\,\exp(2\,b_1\,z_1)} \\ & + \frac{48\,b_1^2\,b_3^2\,\alpha_1^2\,\exp(4\,b_1\,z_1)}{(1-b_3\,\exp(2\,b_1\,z_1))^2} \\ & + 4\,c_1^2\,\beta_1^2 + \frac{48\,c_1^2\,c_3^2\,\beta_1^2\,\exp(4\,c_1\,z_2)}{(-1+c_3\,\exp(2\,c_1\,z_2))^2} \\ & + \frac{48\,c_1^2\,c_3\,\beta_1^2\,\exp(2\,c_1\,z_2)}{1-c_3\,\exp(2\,c_1\,z_2)} - \frac{\alpha_0}{\alpha_1} - \frac{\beta_0(t)}{\beta_1}\Bigg). \end{aligned} \tag{19}$$

Case 2. When $p = 2$

In this case, we find the polynomial function solutions for (11) in elliptic function forms. To this end, we put $n_1 = n_2 = 2$, $k = 2$ in (12).

(I_3) When $k = 2$, $n_1 = n_2 = 2$ (where $p = 2$).

In this case, we assume that

$$\begin{aligned} u(x,y,t) &= U(z_1,z_2) \\ &= p_0 + \sum_{i_1+i_2=1}^{2} p_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2), \\ v(x,y,t) &= V(z_1,z_2) \\ &= q_0 + \sum_{i_1+i_2=1}^{2} q_{i_1,i_2}\,\phi_1^{i_1}(z_1)\,\phi_2^{i_2}(z_2), \end{aligned}$$

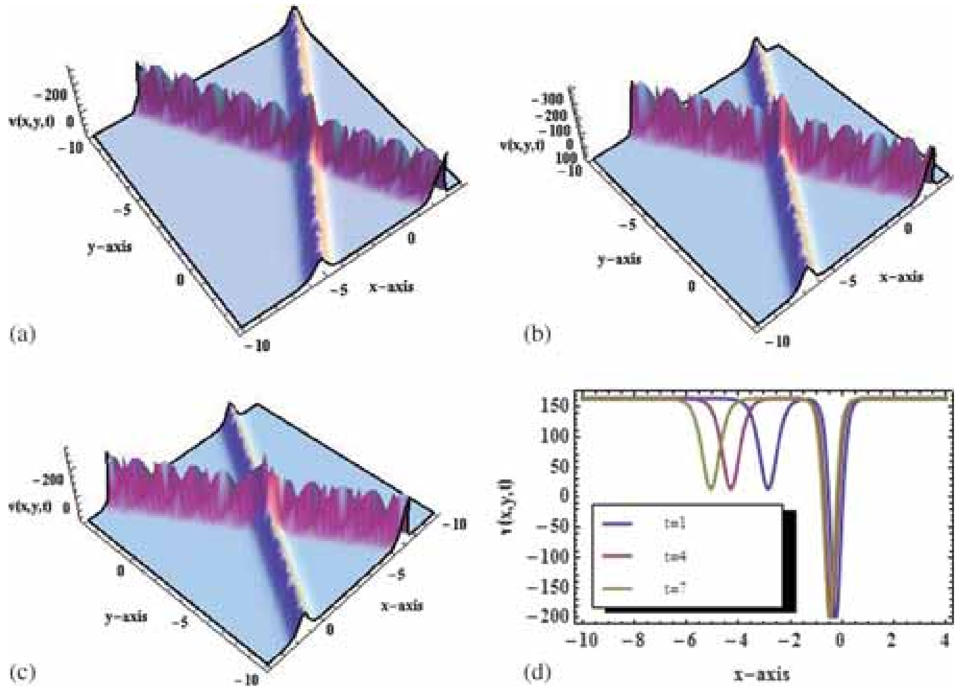

Figure 4. **(a)–(c)** 3D plot for $v(x, y, t)$ when $\alpha = 0.3$. (**a**) $t = 1$, (**b**) $t = 4$, (**c**) $t = 7$. (**d**) 2D plot for $v(x, y, t)$ when $\alpha = 0.3$, $y = 0$ with the same caption as in figure 3.

$$\phi_1'(z_1) = \sqrt{b_0 + b_2\,\phi_1^2(z_1) + b_4\,\phi_1^4(z_1)},$$

$$\phi_2'(z_2) = \sqrt{c_0 + c_2\,\phi_2^2(z_2) + c_4\,\phi_2^4(z_2)}. \qquad (20)$$

By substituting (20) into (11) and by using any package in symbolic computations, we get

$$p_0 = \frac{\beta_2\,\nu\,(-\beta_1\,\alpha_0 + \alpha_1(4\,b_2\,\alpha_1^2\,\beta_1 - 4\,c_2\,\beta_1^3 + \beta_0))}{2\,\alpha_1\,\beta_1^2\,\lambda},$$

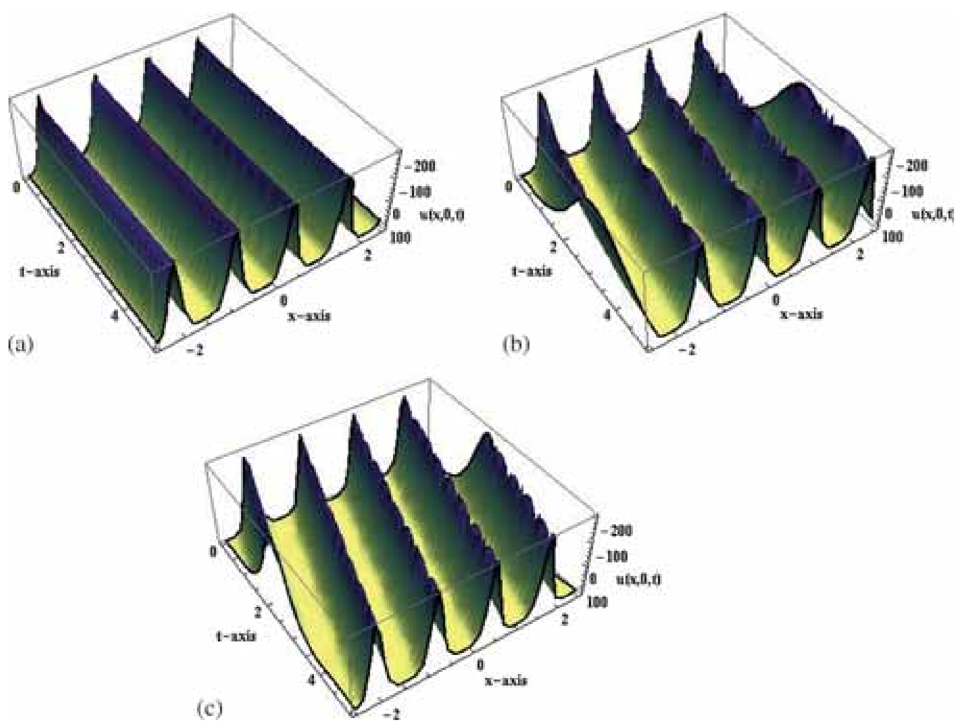

Figure 5. 3D plot for $u(x, y, t)$ when $y = 0$. (**a**) $\alpha = 0.02$, (**b**) $\alpha = 0.6$, (**c**) $\alpha = 0.99$. $\alpha_1 = -2$, $\alpha_3 = -5$, $\beta_1 = 7$, $\beta_2 = 5$, $\beta_3 = 2$ and $b_0 = -(1 - m_1^2)^2/4$, $b_2 = (1 + m_1^2)^2/2$, $b_4 = -1/4$, $c_0 = -(1 - k_1^2)^2/4$, $c_2 = (1 + k_1^2)^2/2, c_4 = -1/4$. $\lambda = 0.1$, $\nu = -0.2$.

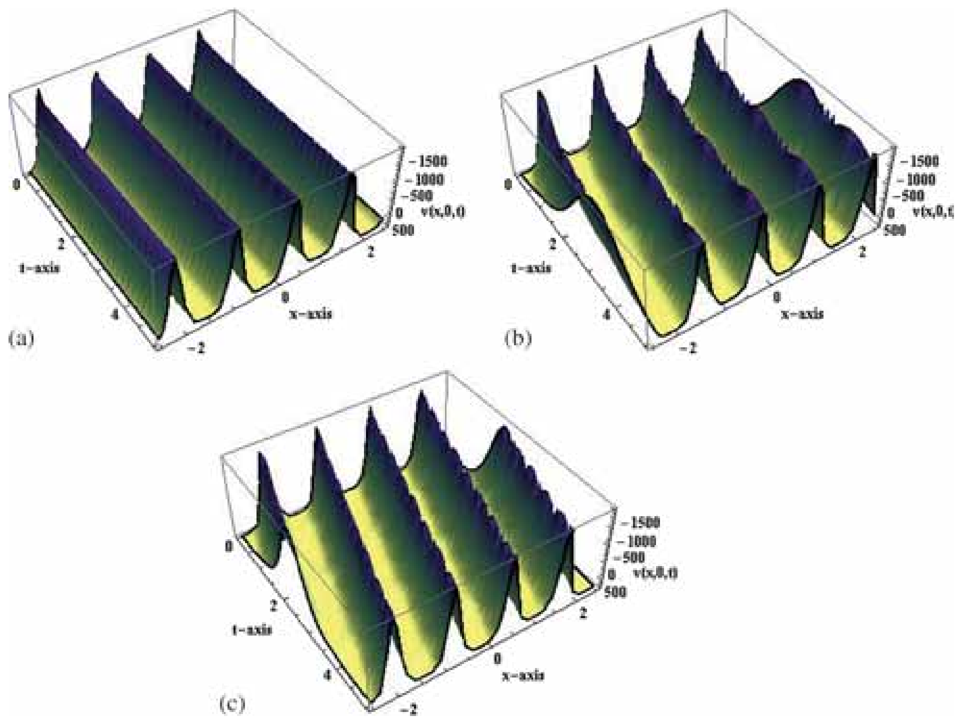

Figure 6. 3D plot for $v(x, y, t)$ when $y = 0$. (**a**) $\alpha = 0.02$, (**b**) $\alpha = 0.6$, (**c**) $\alpha = 0.99$ with the same caption as in figure 3.

$$p_{1,0} = p_{0,1} = p_{1,1} = 0,$$
$$p_{0,2} = -\frac{6\,c_4\,\beta_1\,\beta_2\,\nu}{\lambda}, \quad p_{2,0} = \frac{6\,b_4\,\alpha_1^2\,\beta_2\,\nu}{\beta_1\,\lambda} \tag{21}$$

and

$$q_0 = -\frac{\beta_1\,\alpha_0 + \alpha_1\,(-4\,(b_2\,\alpha_1^2\,\beta_1 + c_2\,\beta_1^3) + \beta_0)}{2\,\alpha_1\,\beta_1\,\lambda},$$
$$q_{1,0} = q_{0,1} = q_{1,1} = 0,$$
$$q_{0,2} = \frac{6\,c_4\,\beta_1^2}{\lambda}, \quad q_{2,0}(t) = \frac{6\,b_4\,\alpha_1^2}{\lambda},$$
$$\alpha_2 = -\frac{\alpha_1\,\beta_2}{\beta_1}. \tag{22}$$

It remains to solve the auxiliary equations in $(20)_3$. For particular values of b_r and c_r, $r = 0, 2, 4$ we get different solutions in Jacobi elliptic functions.

Here, if we take (according to the classification in [36])

$$b_0 = -\frac{(1 - m_1^2)^2}{4}, \quad b_2 = \frac{1 + m_1^2}{2}, \quad b_4 = -\frac{1}{4},$$
$$c_0 = -\frac{(1 - k_1^2)^2}{4}, \quad c_2 = \frac{1 + k_1^2}{2}, \quad b_4 = -\frac{1}{4}, \tag{23}$$

and substituting into $(20)_3$, we get

$$\phi_1(z_1) = m_1\,\mathrm{cn}(z_1, m_1) + \mathrm{dn}(z_1, m_1),$$
$$\phi_2(z_2) = k_1\,\mathrm{cn}(z_2, k_1) - \mathrm{dn}(z_2, k_1), \tag{24}$$

where $0 < m_1 < 1$ and $0 < k_1 < 1$ are called the modulus of the Jacobi elliptic functions. When $m_1 \to 0$ (or $k_1 \to 0$), $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$ degenerate to $\sin(z)$, $\cos(z)$ and 1, respectively; and when $m_1 \to 1$ (or $k_1 \to 1$), $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$ degenerate to $\tanh(z)$, sech(z) and sech(z) respectively.

Finally, the general solution of (11) in terms of the Jacobi elliptic functions is given by

$$U(z_1, z_2) = \frac{\beta_2\,\nu}{2\,\beta_1\,\lambda}\Bigg(- 3\alpha_1^2(m_1\mathrm{cn}(z_1, m_1) + \mathrm{dn}(z_1, m_1))^2 + 3\,\beta_1^2(-k_1\,\mathrm{cn}(z_2, k_1) + \mathrm{dn}(z_2, k_1))^2 + \frac{-\beta_1\,\alpha_0 + \alpha_1(2\,\beta_1\,((1+m_1^2)\alpha_1^2 - (1+k_1^2)\beta_1^2) + \beta_0)}{\alpha_1\,\beta_1}\Bigg),$$

$$V(z_1, z_2) = \frac{1}{2\,\lambda}\Bigg(- 3\,\alpha_1^2\,(m_1\mathrm{cn}(z_1, m_1) + \mathrm{dn}(z_1, m_1))^2 - 3\,\beta_1^2(-k_1\,\mathrm{cn}(z_2, k_1) + \mathrm{dn}(z_2, k_1))^2 + \frac{2\,\alpha_1^3\,\beta_1\,(1+m_1^2) + 2\,\alpha_1\,\beta_1^3(1+k_1^2) - \alpha_1\,\beta_0 - \beta_1\,\alpha_0}{\alpha_1\,\beta_1}\Bigg), \tag{25}$$

where

$$z_1 = \alpha_1\,x - \frac{\alpha_1\,\beta_2}{\beta_1}\,y + \frac{\alpha_0\,t^\alpha}{\Gamma(1 + \alpha)},$$
$$z_2 = \beta_1\,x + \beta_2\,y + \frac{\beta_0\,t^\alpha}{\Gamma(1 + \alpha)},$$

β_0, β_1, α_0, α_1, λ and ν are arbitrary constants.

The solution (25) is shown in figures 5 and 6 for different values of α when $m_1 = 1/4$, $k_1 = 3/4$.

5. Conclusion

Here, we have investigated the multiwave solutions for FNNVEs. Our analytical descriptions are obtained by using the generalized unified method which is described in the modified Riemann–Liouville sense. We use this method to find multiwave solutions as polynomial function solutions. Multisoliton and multi-elliptic solutions were found. Moreover, the obtained results show that the proposed methods are quite effective, promising and convenient for solving nonlinear fractional differential equations. For small fractional time derivative, wave progresses are changed significantly.

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