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Energy distributions of Bianchi type-VI$_h$ Universe in general relativity and teleparallel gravity

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Abstract. In this paper, we have investigated the energy and momentum density distributions for the inhomogeneous generalizations of homogeneous Bianchi type-VI$_h$ metric with Einstein, Bergmann–Thomson, Landau–Lifshitz, Papapetrou, Tolman and Møller prescriptions in general relativity (GR) and teleparallel gravity (TG). We have found exactly the same results for Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum distributions in Bianchi type-VI$_h$ metric for different gravitation theories. The energy–momentum distributions of the Bianchi type-VI$_h$ metric are found to be zero for $h = -1$ in GR and TG. However, our results agree with Tripathy *et al*, Tryon, Rosen and Aygün *et al*.

Keywords. Bianchi VI$_h$ Universe; general relativity; teleparallel gravity; energy–momentum distribution.

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1. Introduction

The localization of energy–momentum distributions is still the most important and interesting problem in general relativity. The first energy–momentum study has been made by Einstein [1]. After this, many researchers including Papapetrou [2], Tolman [3], Bergmann–Thomson [4], Møller [5], Landau–Liftshitz [6] and Weinberg [7] have suggested different definitions for the energy–momentum distribution in GR and also in TG [8–10]. Vargas [8] has studied Einstein and Bergmann–Thomson energy–momentum complexes in teleparallel gravity and found that the total energy vanishes in FRW Universe. Except Møller [5] energy–momentum definition, other definitions are restricted to energy–momentum distribution in quasi-Cartesian coordinates [11]. Recently, Sinha *et al* [12] have studied Ayón-Beato and García (AG) metric with Møller energy and momentum definition in Schwarzschild and quasi-Cartesian coordinates. Also they have found exactly the same results for different coordinates and they proved that Møller energy and momentum definition gives coordinate-independent results. According to Vagenas [11] "The energy–momentum issue has been examined and criticized by many authors. Because its nature is nontensorial [13], different energy and momentum definitions give different results for the same frame [14] also these complexes are local objects and the suitable energy and momentum of gravitational field was only total, i.e. cannot be localized [15]". In the light of these developments, Virbhadra showed that Tolman and Landau–Lifshitz definitions give exactly the same result for Kerr–Newman black hole in GR [16]. In 1996, Aguirregabiria *et al* found that five different energy and momentum definitions give the same and acceptable results for Kerr–Schild Universe model [17]. Also, these studies provided important contributions to solve the problem. Virbhadra and his colleagues have done detailed studies on energy–momentum localization problem [18]. The only way to understand the nature of the space–time is to investigate the traces of energy–momentum distributions with probable astrophysical implications [19]. For this reason, we research energy–momentum distributions in various space–time models. Virbhadra and his co-authors have investigated distinctive gravitational lensing features of black holes and naked singularities by noticing that naked singularities have negative energy regions while black holes do not have [20–25]. Then, there have been many ventures to solve the energy–momentum problem in different gravitation theories [26–33]. However, many physicists have investigated the Bianchi models with different energy–momentum prescriptions in different gravitation theories. Kıy and Aygün have studied higher-dimensional energy–momentum problem for Bianchi types-V and I Universes using various gravitation theories [34]. Aygün and Tarhan have investigated energy–momentum localization problem for Bianchi type-IV Universe in GR and TG [35]. Banerjee–Sen [36] have studied Einstein prescription for Bianchi type-I space–time. Xulu [37] studied Bianchi type-I space–time using the Landau–Lifshitz, Papapetrou and Weinberg's energy–momentum complexes and found that the total energy vanishes. Aydoğdu and Saltı [38] have studied Bianchi type-V Universe in GR and TG and they found the same results. Radinschi [39,40] has investigated Møller and Tolman [40] complexes for Bianchi I space–time and she found that the total energy is zero. Nourinezhad and Mehdipour [41] have explored energy–momentum density in the context of Bianchi IX cosmological model. Radinschi has studied Bianchi type-VI_0 Universe using various energy–momentum definitions in GR [42,43]. Recently, Tripathy *et al* have investigated homogeneous Bianchi type-VI_h Universe in GR [44]. In this study, we shall consider energy–momentum localization problem with various energy–momentum definitions for inhomogeneous generalizations of homogeneous model of Bianchi type-VI_h Universe. The plan of the paper is as follows. In the next section, we present Bianchi type-VI_h Universe and carry out some necessary calculations for this model. In §3, we introduce energy–momentum definitions of Einstein, Bergmann–Thomson and Landau–Lifshitz, Møller, Papapetrou and Tolman in GR. In §4, we introduce energy–momentum definitions of Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller in TG. The last section is devoted to discussion and conclusion. Throughout this paper, all indices run from 0 to 3 and we use geometrized units where $G = 1$, $c = 1$.

2. The Bianchi type-VI_h Universe

Inhomogeneous generalizations of homogeneous Bianchi type-VI_h Universe is given by [45]

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha}(\mathrm{d}x^2 - \mathrm{d}t^2) + \mathrm{e}^{\beta+\gamma+2x}\mathrm{d}y^2 + \mathrm{e}^{\beta-\gamma+2hx}\mathrm{d}z^2, \tag{1}$$

where $\alpha = \alpha(x, t)$, $\beta = \beta(t)$ and $\gamma = \gamma(t)$. h is a chosen exponent to describe the model. The line element (1) is defined by

$$(g_{\mu\nu}) = \begin{pmatrix} \mathrm{e}^{2\alpha} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\beta+\gamma+2x} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\beta-\gamma+2hx} & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{2\alpha} \end{pmatrix}. \tag{2}$$

The inverse of $g_{\mu\nu}$ is evidently,

$$(g^{\mu\nu}) = \begin{pmatrix} \mathrm{e}^{-2\alpha} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-(\beta+\gamma+2x)} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-(\beta-\gamma+2hx)} & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-2\alpha} \end{pmatrix}. \tag{3}$$

Introducing the tetrad, ω^i $(i = 0, 1, 2, 3)$ given by

$$\omega^0 = \mathrm{e}^{\alpha}\mathrm{d}t;\quad \omega^1 = \mathrm{e}^{\alpha}\mathrm{d}x;\quad \omega^2 = \mathrm{e}^{((\beta+\gamma)/2)+x}\mathrm{d}y;$$
$$\omega^3 = \mathrm{e}^{((\beta-\gamma)/2)+hx}\mathrm{d}z \tag{4}$$

the metric (1) can be expressed in the simple form

$$\mathrm{d}s^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2. \tag{5}$$

Using this relation, we obtain the tetrad components

$$h^0{}_0 = \mathrm{e}^{\alpha},\quad h^1{}_1 = \mathrm{e}^{\alpha},\quad h^2{}_2 = \mathrm{e}^{((\beta+\gamma)/2)+x},$$
$$h^3{}_3 = \mathrm{e}^{((\beta-\gamma)/2)+hx} \tag{6}$$

and inverse of $h^a{}_\mu$ is

$$h_0{}^0 = \mathrm{e}^{-\alpha},\quad h_1{}^1 = \mathrm{e}^{-\alpha},\quad h_2{}^2 = \mathrm{e}^{-((\beta+\gamma)/2)+x}$$
$$h_3{}^3 = \mathrm{e}^{-((\beta-\gamma/2)+hx)}. \tag{7}$$

3. Energy–momentum definitions in general relativity

In this section, we introduce Einstein, Bergmann–Thomson, Landau–Lifshitz, Møller, Papapetrou and Tolman energy–momentum definitions, respectively.

3.1 *Einstein energy–momentum definition in GR*

The Einstein energy–momentum definition can be written as [1]

$$E^{\nu}_{\mu} = \frac{1}{16\pi}\Theta^{\nu\alpha}_{\mu,\alpha}, \tag{8}$$

where

$$\Theta^{\nu\alpha}_{\mu} = \frac{g_{\mu\beta}}{\sqrt{-g}}\Psi^{\nu\alpha\beta\rho}_{,\rho}. \tag{9}$$

Here, $\Psi^{\nu\alpha\beta\rho}$ is given by

$$\Psi^{\nu\alpha\beta\rho} = [-g(g^{\nu\beta}g^{\alpha\rho} - g^{\alpha\beta}g^{\nu\rho})], \quad (10)$$

where E_0^0 is the energy density, E_α^0 are the momentum density components and E_0^α are the components of energy current density [18]. In order to calculate the energy and momentum densities in Einstein's complex associated with the Bianchi type-VI$_h$ metric, we need to calculate the required components of $\Theta_\mu^{\nu\alpha}$:

$$\begin{aligned}
\Theta_1^{01} &= 2e^{\beta+x+hx}(h+1),\\
\Theta_2^{02} &= (\beta_t + 2\alpha_t)e^{\beta+x+hx},\\
\Theta_3^{03} &= (\beta_t + 2\alpha_t)e^{\beta+x+hx},\\
\Theta_0^{01} &= 2e^{\beta+x+hx}(h+1),
\end{aligned} \quad (11)$$

where t describes the derivative with respect to t. Substituting these results into eq. (8), we get Einstein energy–momentum densities in the form

$$\begin{aligned}
E_0^0 &= \frac{e^{\beta+x+hx}(h+1)^2}{8\pi},\\
E_1^0 &= \frac{e^{\beta+x+hx}(h+1)\beta_t}{8\pi},\\
E_2^0 &= 0,\\
E_3^0 &= 0.
\end{aligned} \quad (12)$$

3.2 *Bergmann–Thomson (B–T) energy–momentum definition in GR*

The Bergmann–Thomson energy–momentum definition can be written as [4]

$$B^{\beta\nu} = \frac{1}{16\pi}\Xi^{\beta\nu\alpha}_{,\alpha}, \quad (13)$$

where

$$\Xi^{\beta\nu\alpha} = g^{\beta\mu}X_\mu^{\nu\alpha} \quad (14)$$

with

$$X_\mu^{\nu\alpha} = -X_\mu^{\alpha\nu} = \frac{g_{\mu\beta}}{\sqrt{-g}}\Psi^{\nu\alpha\beta\rho}_{,\rho}. \quad (15)$$

Here B_0^0 is the energy density, B_μ^0 are the momentum density components and B_0^μ are the components of energy current density [18]. The required components of $\Xi^{\mu\nu\alpha}$ are:

$$\begin{aligned}
\Xi^{101} &= 2\beta_t e^{\beta+x+hx-2\alpha}\\
\Xi^{202} &= (\beta_t + 2\alpha_t)e^{hx-x-\gamma}\\
\Xi^{303} &= (\beta_t + 2\alpha_t)e^{-hx+x+\gamma}\\
\Xi^{001} &= -2e^{\beta+x+hx-2\alpha}(h+1).
\end{aligned} \quad (16)$$

If we substitute these results into eq. (13) we obtain Bergmann–Thomson energy–momentum densities in GR as follows:

$$\begin{aligned}
B_0^0 &= \frac{e^{hx+x+\beta}(h+1)(h-2\alpha_x+1)}{8\pi},\\
B_1^0 &= \frac{e^{hx+x+\beta}(h-2\alpha_x+1)\beta_t}{8\pi},\\
B_2^0 &= 0,\\
B_3^0 &= 0,
\end{aligned} \quad (17)$$

where x describes the derivative with respect to x.

3.3 *Landau–Lifshitz (LL) energy–momentum definition in GR*

The Landau–Lifshitz energy–momentum definition can be written as [6]

$$L^{\nu\beta} = \frac{1}{16\pi}\Psi^{\nu\alpha\beta\rho}_{,\alpha\rho}. \quad (18)$$

Here L_0^0 is the energy density, L_ν^0 are the momentum density components and L_0^ν are the components of energy current density [18]. The required non-vanishing components of $\Psi^{\nu\alpha\beta\rho}$ are

$$\begin{aligned}
\Psi^{1001} &= e^{2(\beta+x+hx)}\\
\Psi^{2002} &= e^{2\alpha+\beta-\gamma+2hx}\\
\Psi^{3003} &= e^{2\alpha+\beta+\gamma+2x}\\
\Psi^{0303} &= -e^{2\alpha+\beta+\gamma+2x}.
\end{aligned} \quad (19)$$

Using eqs (19) into eq. (18), we obtain LL energy–momentum densities in GR as follows:

$$\begin{aligned}
L_0^0 &= \frac{e^{2(\beta+x+hx+\alpha)}(h+1)^2}{4\pi}\\
L_1^0 &= \frac{e^{2(\beta+x+hx+\alpha)}(h+1)\beta_t}{4\pi}\\
L_2^0 &= 0\\
L_3^0 &= 0.
\end{aligned} \quad (20)$$

3.4 *Møller energy–momentum definition in GR*

The Møller energy–momentum complex can be written as [5]

$$M_\mu^\nu = \frac{1}{8\pi}\lambda^{\nu\alpha}_{\mu,\alpha}, \quad (21)$$

where the antisymmetric superpotential $\lambda_\mu^{\nu\alpha}$ is

$$\lambda_\mu^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \quad (22)$$

M_0^0 is the energy density and M_α^0 are the momentum density components [18]. Using the metric in eq. (1), the required non-vanishing components of $\lambda_\mu^{\nu\alpha}$ are:

$$\begin{aligned}
\lambda_1^{01} &= -2e^{hx+x+\beta}\alpha_t \\
\lambda_2^{02} &= -2e^{hx+x+\beta}\beta_t \\
\lambda_3^{03} &= -2e^{hx+x+\beta}\beta_t \\
\lambda_0^{01} &= -2e^{hx+x+\beta}\alpha_x.
\end{aligned} \tag{23}$$

Using these components in eq. (21), we obtain the Møller energy and momentum densities as follows:

$$\begin{aligned}
M_0^0 &= -\frac{e^{hx+x+\beta}(\alpha_x h + \alpha_{xx} + \alpha_x)}{4\pi} \\
M_1^0 &= -\frac{e^{hx+x+\beta}(\alpha_t h + \alpha_t + \alpha_{tx})}{4\pi} \\
M_2^0 &= 0 \\
M_3^0 &= 0.
\end{aligned} \tag{24}$$

3.5 *Papapetrou energy–momentum definition in GR*

The Papapetrou energy–momentum complex can be written as [2]

$$\Sigma^{\mu\nu} = \frac{1}{16\pi} N^{\mu\nu\alpha\beta}_{\ ,\alpha\beta} \tag{25}$$

where

$$N^{\mu\nu\alpha\beta} = \sqrt{-g}(g^{\mu\nu}\eta^{\alpha\beta} - g^{\mu\alpha}\eta^{\nu\beta} + g^{\alpha\beta}\eta^{\mu\nu} - g^{\nu\beta}\eta^{\mu\alpha}). \tag{26}$$

Σ_0^0 is the energy density, Σ_μ^0 are the momentum density components and Σ_0^μ are the components of energy current density [18]. The required non-vanishing components of $N^{\mu\nu\alpha\beta}$ are

$$\begin{aligned}
N^{1010} &= 2e^{hx+x+\beta} \\
N^{2020} &= (e^{2\alpha} + e^{\beta+\gamma+2x})e^{hx-x-\gamma} \\
N^{3030} &= (e^{2\alpha} + e^{\beta-\gamma+2hx})e^{x+\gamma-hx} \\
N^{0033} &= -(e^{2\alpha} + e^{\beta-\gamma+2hx})e^{x+\gamma-hx}.
\end{aligned} \tag{27}$$

Using these components in eq. (25) we get Papapetrou energy–momentum distribution as follows:

$$\begin{aligned}
\Sigma_0^0 &= \frac{e^{2\alpha+\beta+x+hx}(h+1)^2}{8\pi} \\
\Sigma_1^0 &= \frac{e^{2\alpha+\beta+x+hx}(h+1)\beta_t}{8\pi} \\
\Sigma_2^0 &= 0 \\
\Sigma_3^0 &= 0.
\end{aligned} \tag{28}$$

3.6 *Tolman energy–momentum definition in GR*

The Tolman energy–momentum complex can be written as [3]

$$\Upsilon_i^k = \frac{1}{8\pi}\Im_{i,l}^{kl}, \tag{29}$$

where Υ_0^0 and Υ_α^0 are energy and momentum components. Also

$$\Im_i^{kl} = \sqrt{-g}\left(-g^{pk}V_{ip}^l + \frac{1}{2}g_i^k g^{pm}V_{pm}^l\right) \tag{30}$$

with

$$V_{jk}^i = -\Gamma_{jk}^i + \frac{1}{2}g_j^i\Gamma_{mk}^m + \frac{1}{2}g_k^i\Gamma_{mj}^m. \tag{31}$$

Using eq. (30) the required non-vanishing components of $\Im_i^{kl}$ are

$$\begin{aligned}
\Im_1^{01} &= \frac{1}{2}e^{hx+x+\beta}\beta_t \\
\Im_1^{00} &= \frac{1}{2}e^{hx+x+\beta}(h+1) \\
\Im_2^{02} &= e^{hx+x+\beta}\alpha_t \\
\Im_3^{03} &= e^{hx+x+\beta}\alpha_t \\
\Im_0^{01} &= e^{hx+x+\beta}(h+1).
\end{aligned} \tag{32}$$

Using eq. (32) in eq. (29) we get Tolman energy–momentum distribution as follows:

$$\begin{aligned}
\Upsilon_0^0 &= \frac{e^{hx+x+\beta}(h+1)^2}{8\pi} \\
\Upsilon_1^0 &= \frac{e^{hx+x+\beta}(h+1)\beta_t}{8\pi} \\
\Upsilon_2^0 &= 0 \\
\Upsilon_3^0 &= 0.
\end{aligned} \tag{33}$$

4. Einstein, Bergmann–Thomson, Landau–Lifshitz, and Møller energy–momentum definitions in TG

The tetrad theory of gravitation or teleparallel gravity was developed by Einstein [46]. The first study about tetrad theory of gravitation was made by Weitzenböck [47]. In this section, we introduce Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller energy–momentum definitions in TG. The energy–momentum definitions of Einstein, Bergmann–Thomson, and Landau–Lifshitz in teleparallel gravity [8] are given by the following prescriptions:

$$hE_\nu^\mu = \frac{1}{4\pi}\partial_\lambda(U_\nu^{\mu\lambda}) \tag{34}$$

$$hB^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda\left(g^{\mu\beta}U_\beta{}^{\nu\lambda}\right) \tag{35}$$

$$hL^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda\left(hg^{\mu\beta}U_\beta{}^{\nu\lambda}\right). \tag{36}$$

Here $h = \mathtt{det}(h^a{}_\mu)$ and $U_\beta{}^{\nu\lambda}$ is the Freud's superpotential, which is given by

$$U_\beta{}^{\nu\lambda} = hF_\beta{}^{\nu\lambda}. \tag{37}$$

Here $F^{\mu\nu\lambda}$ is a tensor and

$$F^{\mu\nu\lambda} = e_1 T^{\mu\nu\lambda} + \frac{e_2}{2}\left(T^{\nu\mu\lambda} - T^{\lambda\mu\nu}\right) + \frac{e_3}{2}\left(g^{\mu\lambda}T^{\beta\nu}{}_\beta - g^{\nu\mu}T^{\beta\lambda}{}_\beta\right) \tag{38}$$

with e_1, e_2 and e_3 the three dimensionless coupling constants of teleparallel gravity [48]. For the teleparallel equivalent of general relativity these constants are given by [49,50]

$$e_1 = \frac{1}{4}, \qquad e_2 = \frac{1}{2}, \qquad e_3 = -1. \tag{39}$$

To calculate this tensor, we must calculate Weitzenböck connection

$$\Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \partial_\nu h^a{}_\mu \tag{40}$$

and after this calculation we obtain the torsion of the Weitzenböck connection

$$T^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda}. \tag{41}$$

Einstein, Bergmann–Thomson and Landau–Lifshitz energy and momentum distributions are given by the following equations, respectively:

$$P_\mu{}^E = \int_\Sigma hE^0{}_\mu \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{42}$$

$$P_\mu{}^B = \int_\Sigma hB^0{}_\mu \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{43}$$

$$P_\mu{}^L = \int_\Sigma hL^0{}_\mu \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{44}$$

where P_μ for $\mu = 1, 2, 3$ give the momentum components, while P_0 gives the energy and the integration hypersurface Σ is described by $x^0 = t =$ constant [8,49,50]. Using tetrad components and their inverse in eqs (6) and (7), we get the following non-vanishing Weitzenböck connection components:

$$\begin{aligned}
\Gamma^1{}_{11} &= \Gamma^0{}_{01} = \alpha_x\\
\Gamma^1{}_{10} &= \Gamma^0{}_{00} = \alpha_t\\
\Gamma^2{}_{20} &= \frac{1}{2}(\beta_t + \gamma_t)\\
\Gamma^2{}_{21} &= 1\\
\Gamma^3{}_{31} &= h\\
\Gamma^3{}_{30} &= \frac{1}{2}(\beta_t - \gamma_t)
\end{aligned} \tag{45}$$

and the corresponding non-vanishing torsion components are obtained as

$$\begin{aligned}
T^1_{10} &= -T^1_{01} = -\alpha_t\\
T^2_{12} &= -T^2_{21} = 1\\
T^2_{20} &= -T^2_{02} = -\frac{1}{2}(\beta_t + \gamma_t)\\
T^3_{13} &= -T^3_{31} = h\\
T^3_{30} &= -T^3_{03} = \frac{1}{2}(-\beta_t + \gamma_t)\\
T^0_{10} &= -T^0_{01} = \alpha_t.
\end{aligned} \tag{46}$$

Using these components in eq. (38), we get the required non-vanishing components of the tensor $F^{\mu\nu\lambda}$

$$\begin{aligned}
F^{110} &= -F^{101} = -\frac{1}{2}\mathrm{e}^{-4\alpha}\beta_t\\
F^{212} &= -F^{221} = -\frac{1}{2}\mathrm{e}^{-(\beta+\gamma+2x+2\alpha)}(h + \alpha_x)\\
F^{220} &= -F^{202} = -\frac{1}{4}\mathrm{e}^{-(\beta+\gamma+2x+2\alpha)}(\beta_t - \gamma_t + 2\alpha_t)\\
F^{313} &= -F^{331} = -\frac{1}{2}\mathrm{e}^{-(\beta+\gamma-2hx-2\alpha)}(1 + \alpha_x)\\
F^{330} &= -F^{303} = -\frac{1}{4}\mathrm{e}^{-\beta+\gamma-2hx-2\alpha)}(\beta_t + \gamma_t + 2\alpha_t)\\
F^{010} &= -F^{001} = \frac{1}{2}\mathrm{e}^{-4\alpha}(1 + h).
\end{aligned} \tag{47}$$

From eq. (37) we get non-vanishing components of Freud's superpotential as follows:

$$\begin{aligned}
U_1{}^{10} &= -U_1{}^{01} = -\frac{1}{2}\beta_t \mathrm{e}^{\beta+x+hx}\\
U_2{}^{12} &= -U_2{}^{21} = -\frac{1}{2}\mathrm{e}^{\beta+x+hx}\beta_t(h + \alpha_x)\\
U_2{}^{20} &= -U_2{}^{02} = -\frac{1}{4}\mathrm{e}^{\beta+x+hx}(\beta_t - \gamma_t + 2\alpha_t)\\
U_3{}^{13} &= -U_3{}^{31} = -\frac{1}{2}\mathrm{e}^{\beta+x+hx}(1 + \alpha_x)\\
U_3{}^{30} &= -U_3{}^{03} = -\frac{1}{4}\mathrm{e}^{\beta+x+hx}(\beta_t + \gamma_t + 2\alpha_t)\\
U_0{}^{10} &= -U_0{}^{01} = -\frac{1}{2}\mathrm{e}^{\beta+x+hx}(1 + h).
\end{aligned} \tag{48}$$

Substituting eq. (48) into eqs (34)–(36), we get Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum densities in teleparallel gravity, respectively.

$$hE_0^0 = \frac{(1+h)^2 e^{\beta+x+hx}}{8\pi}, \quad hE_1^0 = \frac{(1+h)e^{\beta+x+hx}\beta_t}{8\pi},$$
$$hE_2^0 = hE_3^0 = 0. \tag{49}$$

$$hB_0^0 = \frac{(1+h)(1+h-2\alpha_x)e^{\beta+x+hx}}{8\pi},$$
$$hB_1^0 = \frac{(1+h-2\alpha_x)e^{\beta+x+hx}\beta_t}{8\pi},$$
$$hB_2^0 = hB_3^0 = 0. \tag{50}$$

$$hL_0^0 = \frac{(1+h)^2 e^{2(\beta+x+hx+\alpha)}}{4\pi},$$
$$hL_1^0 = \frac{(1+h)e^{2(\beta+x+hx+\alpha)}\beta_t}{4\pi},$$
$$hL_2^0 = hL_3^0 = 0. \tag{51}$$

4.1 *Møller energy–momentum prescription in teleparallel gravity*

To elaborate energy–momentum localization problem, Møller used Saez's [51] theory and transformed GR [5]. Also, Mikhail *et al* [52] have designated Møller's superpotential in TG as follows [53]:

$$\Lambda_\mu^{va} = \frac{(-g)^{1/2}}{2\kappa}\Re_{\chi\rho\sigma}^{\tau va}\left[\Phi^\rho g^{\sigma\chi}g_{\mu\tau} - \zeta g_{\tau\mu}\xi^{\chi\rho\sigma} - (1-2\zeta)g_{\tau\mu}\xi^{\sigma\rho\chi}\right], \tag{52}$$

where

$$\Re_{\chi\rho\sigma}^{\tau va} = \delta_\chi^\tau g_{\rho\sigma}^{va} + \delta_\rho^\tau g_{\sigma\chi}^{va} - \delta_\sigma^\tau g_{\chi\rho}^{va} \tag{53}$$

with $g_{\rho\sigma}^{va}$ being a tensor defined by

$$g_{\rho\sigma}^{va} = \delta_\rho^v\delta_\sigma^a - \delta_\sigma^v\delta_\rho^a \tag{54}$$

and $\gamma_{\mu va}$ is the con-torsion tensor given by

$$\xi_{\mu va} = h_{i\mu}h_{v;a}^i, \tag{55}$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols, g is the determinant of $g_{\mu v}$ and Φ_μ is the basic vector field defined by

$$\Phi_\mu = \xi_{\mu\rho}^\rho. \tag{56}$$

κ is the Einstein constant and ζ is the free dimensionless parameter. The energy may be expressed by the surface integral

$$E_{\mathrm{TG}}^{\mathrm{Moller}} = \lim_{r\to\infty}\int_{r=\mathrm{constant}} \Lambda_0^{0b} n_b \mathrm{d}S, \tag{57}$$

where n_b is the unit three-vector normal to the surface element dS [52,53]. Taking the results which are given by (6) and (7) into eq. (55) we get the non-vanishing components of $\xi_{\mu v\beta}$ as

$$\xi_{010} = -\xi_{100} = \alpha_x e^{2\alpha}$$
$$\xi_{011} = -\xi_{101} = \alpha_t e^{2\alpha}$$
$$\xi_{022} = -\xi_{202} = \frac{1}{2}(\beta_t + \gamma_t)e^{\beta+\gamma+2x}$$
$$\xi_{033} = -\xi_{303} = \frac{1}{2} = (\beta_t - \gamma_t)e^{\beta-\gamma+2hx}$$
$$\xi_{122} = -\xi_{212} = e^{\beta+\gamma+2x}$$
$$\xi_{133} = -\xi_{313} = he^{\beta-\gamma+2hx}. \tag{58}$$

From eqs (56) and (58), we get the following non-vanishing component of the basic vector field:

$$\Phi^0 = (\alpha_t + \beta_t)e^{-2\alpha}$$
$$\Phi^1 = -e^{-2\alpha}(1 + \alpha_x + h). \tag{59}$$

Using eqs (57)–(59) we get Møller's energy–momentum components as follows:

$$M_0^{0\mathrm{TG}} = \frac{(1+h)^2 e^{\beta+x+hx}}{\kappa}$$
$$M_1^{0\mathrm{TG}} = \frac{(1+h)e^{\beta+x+hx}(\beta_t - 1 - h)}{2\kappa}$$
$$M_2^{0\mathrm{TG}} = \frac{(2\alpha_x - 3 - h)e^{-2\alpha+2\beta+3x+hx+\gamma}(1+h)}{2\kappa}$$
$$M_3^{0\mathrm{TG}} = \frac{(2\alpha_x - 1 - 3h)e^{-2\alpha+2\beta+x+3hx-\gamma}(1+h)}{2\kappa}. \tag{60}$$

5. Summary and discussions

To overcome the problem of energy–momentum many researchers calculated energy–momentum complexes for various Universe models in GR and also in TG. Some of these researchers showed that these energy–momentum complexes can provide meaningful results ([18,54,55] and reference therein). Also Virbhadra [56], Sharif [57], Yang and Radinschi [58] showed that two or more energy–momentum complexes do not give the same result. In this study, we have showed that (except Einstein and Tolman energy–momentum distributions) Bergmann–Thomson, Landau–Lifshitz, Møller and Papapetrou energy–momentum complexes do not give the same result, because these energy–momentum complexes are pseudotensors and not covariant objects. This is in accordance with the equivalence principle [59] which implies that the gravitational field cannot be detected at a point [31]. We also get the following results:

(i) Einstein and Tolman energy–momentum distributions are exactly the same in GR.

$$E_0^0 = \Upsilon_0^0 \quad \text{and} \quad E_1^0 = \Upsilon_1^0.$$

also

$$E_2^0 = E_3^0 = \Upsilon_2^0 = \Upsilon_3^0 = 0.$$

(ii) There is constant ratio $((h+1)/\beta_t)$ between Einstein, Bergmann–Thomson, Landau–Lifshitz, Papapetrou and Tolman's energy and momentum distributions in GR as follows:

$$E_0^0 = \frac{E_1^0(h+1)}{\beta_t}, \quad L_0^0 = \frac{L_1^0(h+1)}{\beta_t},$$
$$B_0^0 = \frac{B_1^0(h+1)}{\beta_t}, \quad \Sigma_0^0 = \frac{\Sigma_1^0(h+1)}{\beta_t},$$
$$\Upsilon_0^0 = \frac{\Upsilon_1^0(h+1)}{\beta_t}. \tag{61}$$

(iii) All energy and first momentum components of energy–momentum densities are well defined and do not vanish but other momentum densities vanish in GR. The results of this paper also support the Cooperstock's hypothesis [60]. We can see these results in table 1.

(iv) There is constant ratio $((h+1)/\beta_t)$ between Einstein, Bergmann–Thomson and Landau–Lifshitz's energy and first momentum distributions in TG as follows:

$$hE_0^0 = \frac{hE_1^0(h+1)}{\beta_t}, \quad hL_0^0 = \frac{hL_1^0(h+1)}{\beta_t},$$
$$hB_0^0 = \frac{hB_1^0(h+1)}{\beta_t}. \tag{62}$$

(v) All energy and first momentum components of energy complexes are well defined and also do not vanish but other momentum densities vanish in TG. We can see these results in table 2.

(vi) Tripathy *et al* have investigated Einstein, Bergmann–Thomson, Landau–Lifshitz and Papapetrou energy–momentum prescription for homogeneous Bianchi type-VI$_h$ Universe in GR and they have found that energy–momentum densities vanish for $h = -1$. Except for Møller energy–momentum and Bergmann–Thomson's first momentum components, the energy and momentum densities of inhomogeneous generalizations of homogeneous Bianchi type-VI$_h$ Universe are zero for $h = -1$ in our model. These results agree with the study of Tripathy *et al* [44]. We can say that $h = -1$ case is so important and special in GR and also in TG. According to Roy and Prasad "each of the $h = -1$ and $h \neq -1$ gives rise to families of Universes. Some of the models start with a big-bang and ultimately stop expanding or collapse into a singularity" [45]. Tryon [61] presented a big bang model and showed that the total energy–momentum of a homogeneous and isotropic closed Universe may

Table 1. Non-zero energy–momentum distributions of Bianchi type-VI$_h$ Universe in GR.

General relativity	Energy density	Momentum density
Einstein	$E_0^0 = \frac{e^{\beta+x+hx}(h+1)^2}{8\pi}$	$E_1^0 = \frac{e^{\beta+x+hx}(h+1)\beta_t}{8\pi}$
Bergmann–Thomson	$B_0^0 = \frac{e^{hx+x+\beta}(h+1)(h-2\alpha_x+1)}{8\pi}$	$B_1^0 = \frac{e^{hx+x+\beta}(h-2\alpha_x+1)\beta_t}{8\pi}$
Landau–Lifshitz	$L_0^0 = \frac{e^{2(\beta+x+hx+\alpha)}(h+1)^2}{4\pi}$	$L_1^0 = \frac{e^{2(\beta+x+hx+\alpha)}(h+1)\beta_t}{4\pi}$
Møller	$M_0^0 = -\frac{e^{hx+x+\beta}(\alpha_x h+\alpha_{xx}+\alpha_x)}{4\pi}$	$M_1^0 = -\frac{e^{hx+x+\beta}(\alpha_t h+\alpha_t+\alpha_{tx})}{4\pi}$
Papapetrou	$\Sigma_0^0 = \frac{e^{2\alpha+\beta+x+hx}(h+1)^2}{8\pi}$	$\Sigma_1^0 = \frac{e^{2\alpha+\beta+x+hx}(h+1)\beta_t}{8\pi}$
Tolman	$\Upsilon_0^0 = \frac{e^{hx+x+\beta}(h+1)^2}{8\pi}$	$\Upsilon_1^0 = \frac{e^{hx+x+\beta}(h+1)\beta_t}{8\pi}$

Table 2. Non-zero energy–momentum distributions of Bianchi type-VI$_h$ Universe in TG.

Teleparallel gravity	Energy density	Momentum density
Einstein	$hE_0^0 = \frac{e^{\beta+x+hx}(h+1)^2}{8\pi}$	$hE_1^0 = \frac{e^{\beta+x+hx}(h+1)\beta_t}{8\pi}$
Bergmann–Thomson	$hB_0^0 = \frac{e^{hx+x+\beta}(h+1)(h-2\alpha_x+1)}{8\pi}$	$hB_1^0 = \frac{e^{hx+x+\beta}(h-2\alpha_x+1)\beta_t}{8\pi}$
Landau–Lifshitz	$hL_0^0 = \frac{e^{2(\beta+x+hx+\alpha)}(h+1)^2}{4\pi}$	$hL_1^0 = \frac{e^{2(\beta+x+hx+\alpha)}(h+1)\beta_t}{4\pi}$
Møller	$M_0^{0\mathrm{TG}} = \frac{(1+h)^2 e^{\beta+x+hx}}{\kappa}$	$M_1^{0\mathrm{TG}} = \frac{(1+h)e^{\beta+x+hx}(\beta_t-1-h)}{2\kappa}$ $M_2^{0\mathrm{TG}} = \frac{(2\alpha_x-3-h)e^{-2\alpha+2\beta+3x+hx+\gamma}(1+h)}{2\kappa}$ $M_3^{0\mathrm{TG}} = \frac{(2\alpha_x-1-3h)e^{-2\alpha+2\beta+x+3hx-\gamma}(1+h)}{2\kappa}$

Table 3. Energy–momentum distributions of Bianchi type-VI$_h$ Universe in GR and TG for $h = -1$.

For $h=-1$ in GR and TG	Energy density	Momentum density
Einstein	$E_0^0 = hE_0^0 = 0$	$E_1^0 = hE_0^0 = 0$
Bergmann–Thomson	$B_0^0 = hB_0^0 = 0$	$B_1^0 = hB_1^0 - (\beta_t e^{\beta}\alpha_x/4\pi)$
Landau–Lifshitz	$L_0^0 = hL_0^0 = 0$	$L_1^0 = hL_1^0 = 0$
Møller GR	$M_0^0 = -(\alpha_{xx}e^{\beta}/4\pi)$	$M_1^0 = -(\alpha_{tx}e^{\beta}/4\pi)$
Møller TG	$M_0^{0\mathrm{TG}} = 0$	$M_1^{0\mathrm{TG}} = M_2^{0\mathrm{TG}} = M_3^{0\mathrm{TG}} = 0$
Papapetrou	$\Sigma_0^0 = 0$	$\Sigma_1^0 = 0$
Tolman	$\Upsilon_0^0 = 0$	$\Upsilon_1^0 = 0$

be indeed zero. Later, Rosen [62] considered FRW metric and obtained that the total energy of the FRW Universe is zero. In the present study, we showed that the energy and momentum vanish for $h = -1$ in Bianchi VI$_h$ Universe model. For $h = -1$, our results agree with Tryon, Rosen, Roy–Prasad and Aygün *et al* [63] in GR and TG. We can see these results in table 3.

(vii) Virbhadra [10,18] and his colleagues have demonstrated that different energy–momentum complexes can give meaningful results. Based on this idea, we have researched the energy–momentum distributions of the inhomogeneous generalizations of homogeneous Bianchi type-VI$_h$ Universe. We have found that Einstein, Bergmann–Thomson and Landau–Lifshitz energy (E) and momentum (M) distributions are exactly the same in GR and TG.

$$E_{\mathrm{GR}} = E_{\mathrm{TG}}, \qquad M_{\mathrm{GR}} = M_{\mathrm{TG}}.$$

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References

[1] A Einstein, *Preuss. Akad. Wiss. Berlin* **47**, 778 (1915)
[2] A Papapetrou, *Proc. R. Irish. Acad. A* **52**, 11 (1948)
[3] R C Tolman, *Relativity, thermodynamics and cosmology* (Oxford Univ. Press, London, 1934) p. 227
[4] P G Bergmann and R Thomson, *Phys. Rev.* **89**, 400 (1953)
[5] C Møller, *Ann. Phys. (NY)* **4**, 347 (1958); *Ann. Phys. (NY)* **12**, 118 (1961)
[6] L D Landau and E M Lifshitz, *The classical theory of fields* 4th edn (Pergamon Press, Oxford, 1987)
[7] S Weinberg, *Gravitation and cosmology: Principle and applications of general theory of relativity* (John Wiley and Sons, Inc, New York, 1972)
[8] T Vargas, *Gen. Relativ. Gravit.* **36**, 1255 (2004)

[9] F I Mikhail, M I Wanas, A Hindawi and E I Lashin, *Int. J. Theor. Phys.* **32**, 1627 (1993)
[10] K S Virbhadra, *Phys. Rev. D* **41**, 1086 (1990)
[11] E C Vagenas, *Int. J. Mod. Phys. A* **18**, 5949 (2003); *Int. J. Mod. Phys. D* **14**, 573 (2005); *Mod. Phys. Lett. A* **21**, 1947 (2006)
[12] G K A K Sinha, A K Pandey, B C Bhaskar, A K Rai, S Jha, S Kumar and S Xulu, *Mod. Phys. Lett.* **30**, 1550120 (2015)
[13] S Chandrasekhar and V Ferrari, *Proc. R. Soc. London A* **435**, 645 (1991)
[14] G Bergqvist, *Class. Quantum Gravit.* **9**, 1917 (1992); *Class. Quantum Gravit.* **9**, 1753 (1992)
[15] C M Chen and J Nester, *Class. Quantum Gravit.* **16**, 1279 (1999)
[16] K S Virbhadra, *Phys. Rev. D* **41**, 1088 (1990)
[17] J M Aguirregabiria, A Chamorro and K S Virbhadra, *Gen. Relativ. Gravit.* **28**, 1393 (1996)
[18] K S Virbhadra, *Phys. Rev. D* **42**, 1066 (1990); *D* **42**, 2919 (1990); *Pramana – J. Phys.* **38**, 31 (1992)
N Rosen and K S Virbhadra, *Gen. Relativ. Gravit.* **25**, 429 (1993)
K S Virbhadra and J C Parikh, *Phys. Lett. B* **317**, 312 (1993)
K S Virbhadra and J C Parikh, *Phys. Lett. B* **331**, 302 (1994)
[19] I Radinschi, F Rahaman, T Grammenos, A Spanou and S Islam, *Adv. Math. Phys.* **2015**, 530281 (2015)
[20] K S Virbhadra, D Narasimha and S M Chitre, *Astron. Astrophys.* **337**, 1 (1998)
[21] K S Virbhadra and G F R Ellis, *Phys. Rev. D* **62**, 084003 (2000)
[22] K S Virbhadra and G F R Ellis, *Phys. Rev. D* **65**, 103004 (2002)
[23] K S Virbhadra and C R Keeton, *Phys. Rev. D* **77**, 124014 (2008)
[24] K S Virbhadra, *Phys. Rev. D* **79**, 083004 (2009)
[25] C Clarissia-Marie, K S Virbhadra and G F R Ellis, *J. Math. Phys.* **42**, 818 (2001)
[26] S Aygün, *Int. J. Theor. Phys.* **49**, 2288 (2010)
[27] S Aygün, İ Tarhan and H Baysal, *Astrophys. Space Sci.* **314**, 323 (2008)
[28] P K Sahoo *et al*, *Chin. Phys. Lett.* **32**, 020402 (2015)
[29] S S Xulu, *Int. J. Theor. Phys.* **39**, 1153 (2000); *Mod. Phys. Lett. A* **15**, 1511 (2000); *Chin. Phys. Lett.* **44**, 348 (2006); *Int. J. Theor. Phys.* **37**, 1773 (1998); *Int. J. Mod. Phys. D* **7**, 773 (1998)
[30] I Radinschi, *Acta Phys. Slov.* **49**, 789 (1999)
I-Ching Yang and I Radinschi, *Chin. J. Phys.* **41**, 326 (2003)
I Radinschi, *Mod. Phys. Lett. A* **15**, 2171 (2000)
[31] M Sharif and Tasnim Fatima, *Int. J. Mod. Phys. A* **20**, 4309 (2005)
[32] R Gad, *Astrophys. Space Sci.* **295**, 451 (2005); *Astrophys. Space Sci.* **293**, 453 (1994)
[33] M U Doğru *et al*, *Int. J. Mod. Phys. D* **21**, 1250078 (2012)
M U Doğru *et al*, *Int. J. Theor. Phys.* **51**, 1545 (2012)
M Aygün, İ Yılmaz, *Chin. Phys. Lett.* **24**, 874 (2007)
[34] G Kıy and S Aygün, *Int. J. Geo. Meth. Mod. Phys.* **12**, 1550045 (2015)
[35] S Aygün and İ Tarhan, *Pramana – J. Phys.* **78**, 531 (2012)
[36] N Banerjee and S Sen, *Pramana – J. Phys.* **49**, 609 (1997)
[37] S S Xulu, *Int. J. Mod. Phys. A* **15**, 4849 (2000)
[38] O Aydoğdu and M Saltı, *Czech. J. Phys.* **56**, 7898 (2006)
[39] I Radinschi, *Fizika B* **9**, 203 (2000)
[40] I Radinschi, *Acta Phys. Slov.* **50**, 609 (2000)
[41] Z Nourinezhad and S H Mehdipour, *Indian J. Phys.* **86**, 919 (2012)
[42] I Radinschi, *Chin. J. Phys.* **5**, 393 (2001)
[43] I Radinschi, *Chin. J. Phys.* **39**, 231 (2001)
[44] S K Tripathy, B Mishra, G K Pandey, A K Singh, T Kumar and S S Xulu, *Adv. High Energy Phys.* **2015**, 705262 (2015)
[45] S R Roy and A Prasad, *Astrophys. Space Sci.* **225**, 151 (1995)
[46] A Einstein, *Math. Ann.* **102**, 658 (1930)
[47] R Weitzenböck, *Invariantentheorie* (Noordhoff, Groningen, 1923)
[48] K Hayashi and T Shirafuji, *Prog. Theor. Phys.* **64**, 866 (1980) **65**, 525 (1980)
[49] G G L Nashed, *Ann. Phys.* **523**, 450 (2011)
[50] M Saltı, *Astrophys. Space Sci.* **299**, 159 (2005)
[51] D Saez, *Phys. Rev. D* **27**, 2839 (1983)
[52] F I Mikhail, M I Wanas, A Hindawi and E I Lashin, *Int. J. Theor. Phys.* **32**, 1627 (1993)
[53] G G L Nashed, *Astrophys. Space Sci.* **344**, 87 (2013)
[54] S S Xulu, *Int. J. Mod. Phys. A* **19**, 2979 (2000); *Phys. Lett. A* **15**, 1511 (2000)
[55] M Sharif, *Int. J. Mod. Phys. A* **17**, 1175 (2002)
[56] K S Virbhadra, *Phys. Rev. D* **60**, 104041 (1999)
[57] M Sharif, *Int. J. Mod. Phys. A* **18**, 4361 (2003); Erratum, *Int. J. Mod. Phys. A* **19**, 1495 (2004)
[58] I Ching Yang and I Radinschi, *AIP. Conf. Proc.* **895**, 325 (2007)
[59] C W Misner, K S Thorne and J A Wheeler, *Gravitation* (W.H. Freeman, New York, 1973) p. 603
[60] F I Cooperstock and R S Sarracino, *J. Phys. A* **11**, 877 (1978)
[61] E P Tryon, *Nature* **246**, 396 (1973)
[62] N Rosen, *Gen. Relativ. Gravit.* **26**, 319 (1994)
[63] S Aygün, H Baysal and İ Tarhan, *Int. J. Theor. Phys.* **46**, 2607 (2007)