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Hidden attractors without equilibrium and adaptive reduced-order function projective synchronization from hyperchaotic Rikitake system

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Abstract. By introducing an additional state feedback into classic Rikitake system, a new hyperchaotic system without equilibrium is derived. The proposed system is investigated through numerical simulations and analyses including time phase portraits, Lyapunov exponents, and Poincaré maps. Based on adaptive control and Lyapunov stability theory, we design a reduced-order projective synchronization scheme for synchronizing the hyperchaotic Rikitake system coexisting without equilibria and the original classic Rikitake system coexisting with two non-hyperbolic equilibria. Finally, numerical simulations are given to illustrate the effectiveness of the proposed synchronization scheme.

Keywords. Hyperchaos; hidden attractors; reduced order; projective synchronization; Lyapunov stability theorem.

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1. Introduction

In the past five decades, it has been found that chaos is useful in many application fields such as engineering, medicine, secure communications, and so on. The equilibrium is very important for showing chaotic attractors, especially for showing multiple wings or scrolls. In the theory of nonlinear systems, the equilibrium plays important roles. One of the most important methods to analyse chaotic system is Silnikov method, which has been used to check whether a system is chaotic or not [1].

For a generic 3D quadratic autonomous system, does there exist chaotic system without equilibrium, with one equilibrium or with any number of equilibria? More generally, through exhaustive computer searching, more than twenty simple 3D quadratic autonomous chaotic systems with no more than three equilibrium points have been found [2–5]. Thus, the main focus of the present report is to further find out new dynamical behaviours of this system. There has recently been interest in systems with coexisting attractors, some of which are hidden [6–9]. According to this new classification, there are two types of attractors: self-excited attractor and hidden attractor. A self-excited attractor has a basin of attraction that is excited from unstable equilibria. In contrast, a hidden attractor has a basin of attraction which does not contain neighbourhoods of equilibria.

Therefore, there has been increasing attention to some unusual examples. In 2010, Yang *et al* [10] introduced and analysed a new 3D chaotic system in a form very similar to the Lorenz, Chen, Lü and Yang–Chen systems, but with only two stable node-foci. In 2012, Wang and Chen [11] discovered a simple 3D autonomous quadratic system that has only one stable equilibrium, showing some new mysterious features of chaos. Later, Wei made the Sprott D system to preserve its chaotic dynamics by a tiny perturbation, and demonstrated that the perturbed Sprott D system without equilibrium produces a cascade of doubling period bifurcations that lead to some chaotic attractors [12]. Subsequently, from 2012 to 2014, chaotic systems with stable equilibrium (equilibria) [13–16], no equilibrium [17–20], any number of equilibria [21] have been presented. From a computational point of view, these hidden attractors cannot be found easily by numerical methods. Furthermore, knowledge about equilibria does not help in the localization of hidden attractors [22–31].

In 2015, Feng and Wei [32] investigated the effect of delayed feedback on the generalized Sprott B system with hidden attractors and local Hopf bifurcation. In addition, Feng and Wei studied the adaptive switched generalized function projective synchronization between four-dimensional hyperchaotic diffusionless Lorenz equations and modified four-dimensional Lorenz–Stenflo system when the two system parameters are unknown [33]. Investigation on no-equilibrium hyperchaotic system with hidden attractors is an attractive topic [34]. Hyperchaos characterized by more than one positive Lyapunov exponent has attracted increasing attention from various scientific and engineering communities. Generating a hidden hyperchaotic attractor, in particular designing a hyperchaotic system purposefully from an originally chaotic system, will be an extension of three-dimensional hidden attractors. It has important theoretical and practical meanings to carry out this research further. To our knowledge, there are not many reports about the coexistence of a hyperchaotic attractor with another attractor in the 4D autonomous system without equilibrium [35,36].

On the other hand, chaos synchronization problems in many physical system have been developed by many methods and techniques over the last few decades, such as feedback approach, adaptive method and different kinds of synchronization, including some complete synchronizations: phase synchronization, lag synchronization, anticipatory synchronization, multiplexing synchronization etc. [37–39]. Nowadays, the complex synchronization phenomena of chaotic systems with different orders are more important. Therefore, understanding the local and the global behaviours and all applications of chaotic systems or hyperchaotic systems with hidden attractors is of great importance. Motivated by the above research, a new four-dimensional hyperchaotic system without equilibrium from the Rikitake system is proposed and analysed in this paper. We also will design asynchronization scheme to realize adaptive reduced-order function projective synchronization between the hyperchaotic Rikitake system coexisting without equilibrium and the 3D chaotic Rikitake system coexisting with two non-hyperbolic equilibria. Numerical simulations are presented to demonstrate the effectiveness of the proposed adaptive controllers.

2. The hyperchaotic Rikitake system without equilibrium

Stimulated by the works of Leonov and Kuznetsov [9], we perform a systematic search for chaos in three-dimensional autonomous systems with quadratic nonlinearities. A simple mechanical model used to study the reversal of the magnetic field of the Earth is due to Rikitake [40], and consists of two connected identical single Faraday-disk dynamos of the Bullard type. The dynamics of the coupled system are described by a set of three nonlinear autonomous first-order ordinary differential equations given by

$$\begin{cases} \dot{x_1} = -bx_1 + y_1 z_1, \\ \dot{y_1} = -by_1 + (z_1 - a)x_1, \\ \dot{z_1} = 1 - x_1 y_1, \end{cases} \tag{2.1}$$

where a, b are real positive parameters. It is easy to know that model (2.1) has the following two fixed points:

$$E_1(\sqrt{\xi/2b}, \sqrt{2b/\xi}, \xi/2),\ E_2(-\sqrt{\xi/2b}, -\sqrt{2b/\xi}, \xi/2),$$

where

$$\xi = \sqrt{a + \sqrt{a^2 + 4b^2}}.$$

Since the eigenvalues of the linearized system at these points are

$$-2b, \pm\sqrt{(a\xi + 4b^2)/(b\xi)}i$$

local dynamics around two non-hyperbolic equilibria takes place on two-dimensional centre manifolds. In addition, the divergence of the system is $-2b$, which implies that the system is dissipative for $b > 0$, as the volume of the system reduces according to the Liouville formula. Moreover, system (2.1) is chaotic for classic parameter values $(a, b) = (5, 2)$ (see figure 1). The corresponding bifurcation diagrams for $a \in [0.1, 5.5]$ or $b \in [0.3, 2]$ is shown in figure 2.

Now, by introducing an additional state w_1 and coupling it to the second equation of the chaotic system (2.1), we obtain a new 4D system

$$\begin{cases} \dot{x_1} = -mx_1 + y_1 z_1, \\ \dot{y_1} = -my_1 + (z_1 - c)x_1 + w_1, \\ \dot{z_1} = 1 - x_1 y_1, \\ \dot{w_1} = -vx_1, \end{cases} \tag{2.2}$$

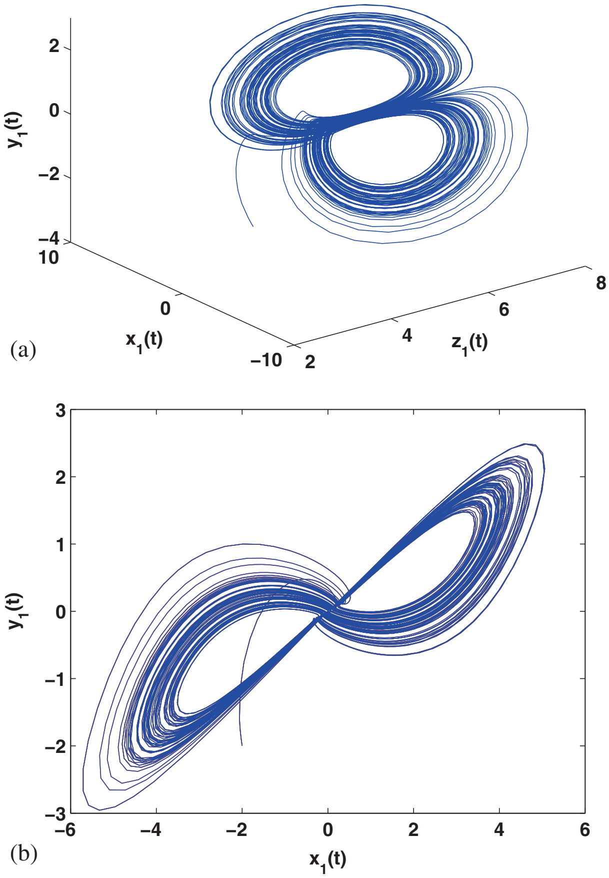

Figure 1. Coexistence of chaotic attractors with two non-hyperbolic equilibria of system (2.1) for the case $(a, b) = (5, 2)$ with initial conditions $(-2, -2, 3)$: (**a**) $x-y-z$ space; (**b**) $x-y$ plane.

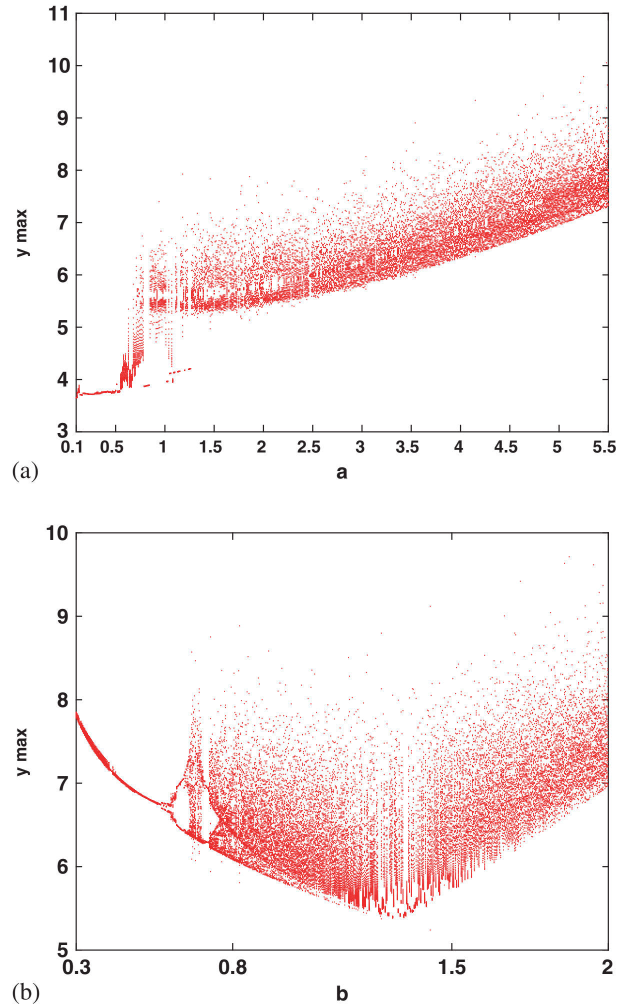

Figure 2. Bifurcation diagram of system (2.1): (**a**) versus parameter $a \in [0.1, 5.5]$ and (**b**) versus parameter $b \in [0.3, 2]$.

where m, c, v are constant parameters. It is easy to see that system (2.2) does not have any equilibrium. But when we set $c = 4$, $m = 1$, $v = 0.1$ and initial conditions are $(-2, -2, 3, 0)$, system (2.2) is hyperchaotic and its hidden attractor is shown. Then figure 3 shows the actually expected hyperchaotic behaviour as designed. In this case, system (2.2) has two positive Lyapunov exponents:

$$L_1 = 0.0999, \quad L_2 = 0.0241$$

and the other two are

$$L_3 = -0.0006, \quad L_2 = -2.1234.$$

Clearly, system (2.1) has a hidden hyperchaotic attractor coexisting without equilibrium. This implies that system (2.1) can generate hyperchaos without any Silnikov orbits.

As an important analysis technique, the Poincaré map can reflect bifurcation and folding properties of chaos. When $c = 4$, $m = 1$, $v = 0.1$, one may take $y = 0$ as the crossing plane. Figure 4 shows the Poincaré map in the $x-z$ 2D space and $x-w$ space. It is clear that some sheets are folded indicating that the system has extremely rich dynamics.

3. Adaptive reduced-order function projective synchronization between system (2.2) and Rikitake system

In the following, we shall design reduced-order projective synchronization scheme between hyperchaotic Rikitake system coexisting without equilibrium and the 3D chaotic Rikitake system coexisting with two non-hyperbolic equilibria. The projective subsystem is constructed by the first three equations of hyperchaotic Rikitake system. The response 3D chaotic Rikitake system is [40]

$$\begin{cases} \dot{x_2} = -bx_2 + y_2 z_2, \\ \dot{y_2} = -by_2 + (z_2 - a)x_2, \\ \dot{z_2} = 1 - x_2 y_2. \end{cases} \tag{3.1}$$

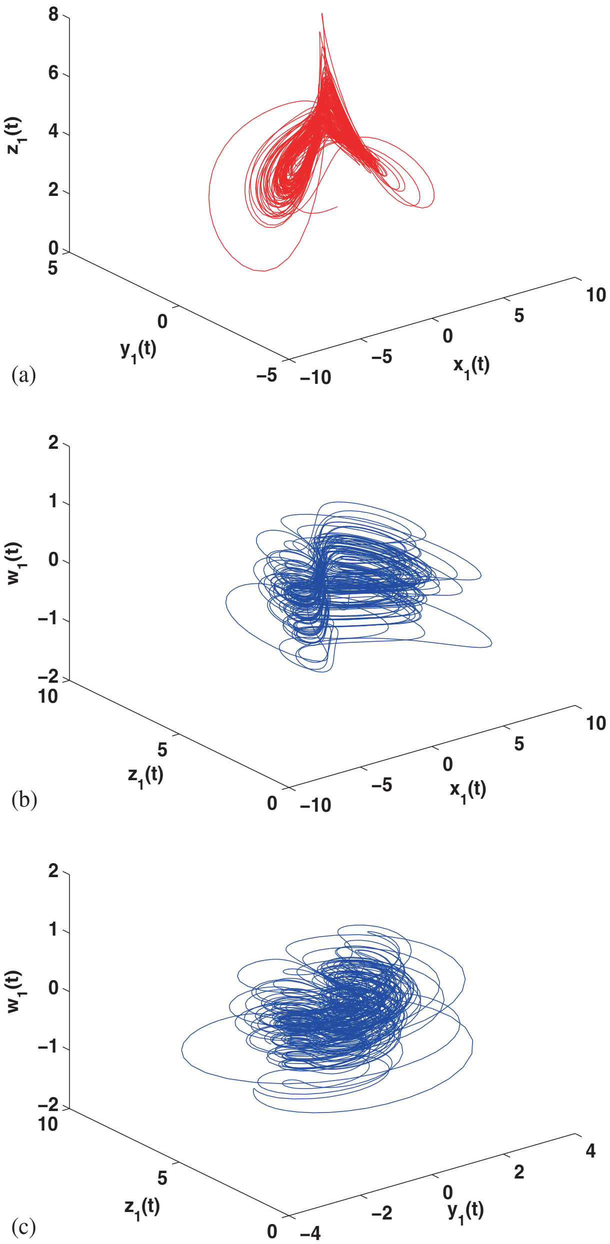

Figure 3. Hyperchaotic attractors of system (2.2) without equilibrium for $(m, c, v) = (4, 1, 0.1)$ with initial conditions $(-2, -2, 3, 0)$: (**a**) Hidden attractor projected in 3D space $x_1 - y_1 - z_1$; (**b**) hidden attractor projected in 3D space $x_1 - z_1 - w_1$ and (**c**) hidden attractor projected in 3D space $y_1 - z_1 - w_1$.

Note that the system is chaotic when $(a, b) = (5, 2)$. Now we give the response system

$$\begin{cases} \dot{x_2} = -bx_2 + y_2 z_2 + u_1, \\ \dot{y_2} = -by_2 + (z_2 - a)x_2 + u_2, \\ \dot{z_2} = 1 - x_2 y_2 + u_3, \end{cases} \tag{3.2}$$

where u_1, u_2 and u_3 are the controllers.

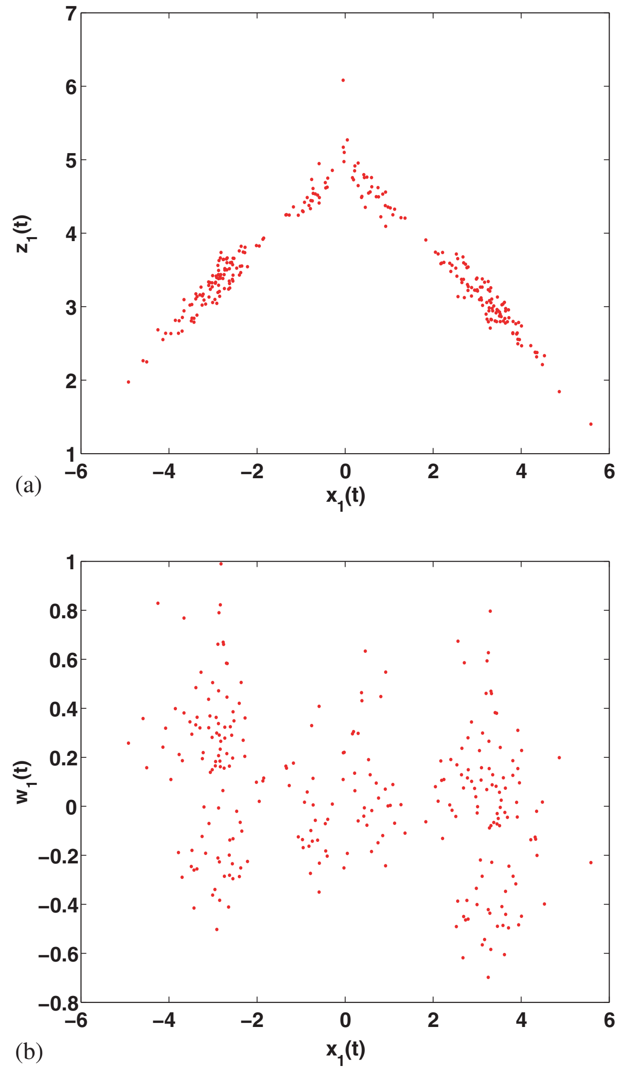

Figure 4. Poincaré map in the (**a**) $x-z$ 2D space and (**b**) $x-w$ 2D space with $(c, m, v) = (4, 1, 0.1)$.

DEFINITION 1

Systems (3.1) and (3.2) are said to achieve reduced-order projective synchronization if and only if the three following conditions satisfy:

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} |x_2(t) - nx_1(t)|,$$

$$\lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} |y_2(t) - ny_1(t)|,$$

$$\lim_{t\to\infty} e_3(t) = \lim_{t\to\infty} |z_2(t) - nz_1(t)|,$$

where n is the parameter and the error signals

$$e_1(t) = x_2(t) - nx_1(t), \quad e_2(t) = y_2(t) - ny_1(t),$$
$$e_3(t) = z_2(t) - nz_1(t).$$

By using system (3.2) and the error signals, the error dynamical system can be obtained as follows:

$$\begin{cases} \dot{e_1} = y_2 - ny_1z_1 - bx_2 + mnx_1 + u_1, \\ \dot{e_2} = x_2z_2 - nx_1z_1 - by_2 + mny_1 - ax_2 + cnx_1 \\ \qquad - nw_1 + u_2, \\ \dot{e_3} = -x_2y_2 + nx_1y_1 + u_3. \end{cases}$$

We define the controllers as follows:

$$u_1 = -y_2 + ny_1z_1 + bx_2 - mnx_1 - p_1e_1,$$

$$u_2 = -x_2z_2 + nx_1z_1 + by_2 - mny_1 + ax_2 - cnx_1 + nw_1 - p_2e_2,$$

$$u_3 = x_2y_2 - nx_1y_1 - p_3e_3.$$

Here p_1, p_2 and p_3 are negative constants representing the control gain. Let

$$\varepsilon_a = a_1(t) - a, \quad \varepsilon_b = b_1(t) - b, \quad \varepsilon_b = b_1(t) - a,$$

$$\varepsilon_m = m_1(t) - m.$$

Then the following estimates of parameters and update laws are set:

$$\dot{a_1} = -e_2x_2 - l\varepsilon_a, \quad \dot{b_1} = -e_2y_2 - e_1x_2 - l\varepsilon_b,$$

$$\dot{c_1} = ne_2x_1 - l\varepsilon_c, \quad \dot{m_1} = ne_1x_1 + ne_2y_2 - l\varepsilon_m,$$

where l is a positive parameter. Therefore, the reduced-order projective synchronization between the two systems (2.2) and (3.2) must satisfy

$$\lim_{t\to\infty} |\varepsilon_a| = \lim_{t\to\infty} |\varepsilon_b| = \lim_{t\to\infty} |\varepsilon_c| = \lim_{t\to\infty} |\varepsilon_m| = 0.$$

Hence, let us define the following Lyapunov function candidate:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \varepsilon_a^2 + \varepsilon_b^2 + \varepsilon_c^2 + \varepsilon_m^2),$$

and we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -p_1e_1^2 - p_2e_2^2 - p_3e_3^2 - l\varepsilon_a^2 - l\varepsilon_b^2 - l\varepsilon_c^2 - l\varepsilon_m^2.$$

This means that the above designed reduced-order projective synchronization scheme can be achieved. In order to demonstrate the validity of the above designed reduced-order projective synchronization scheme, the parameters of the system (2.2) are set to $c = 4$, $m = 1$, $v = 0.1$, and the parameter of the system (3.1) is set to $(a, b) = (5, 2)$. The initial conditions of the driving and the response systems are $(-2, -2, 3, 0)$ and $(-2, -2, 3)$. Choosing $l = p_i = 1$ $(i = 1, 2, 3)$ and $(-5, 10, 4, 2, 1, 4, -4)$ as the initial values of the parameters $e_i(i = 1, 2, 3)$ and figure 5a shows the trajectories of $e_i(i = 1, 2, 3)$, and as indicated, the error dynamical system tends to zero after control. Figure 5b shows that the estimates $a_1(t)$, $b_1(t)$, $c_1(t)$, $m_1(t)$ of the unknown parameters converge to $a = 5$, $b = 2$, $c = 4$ and $m = 1$ as $t \to \infty$.

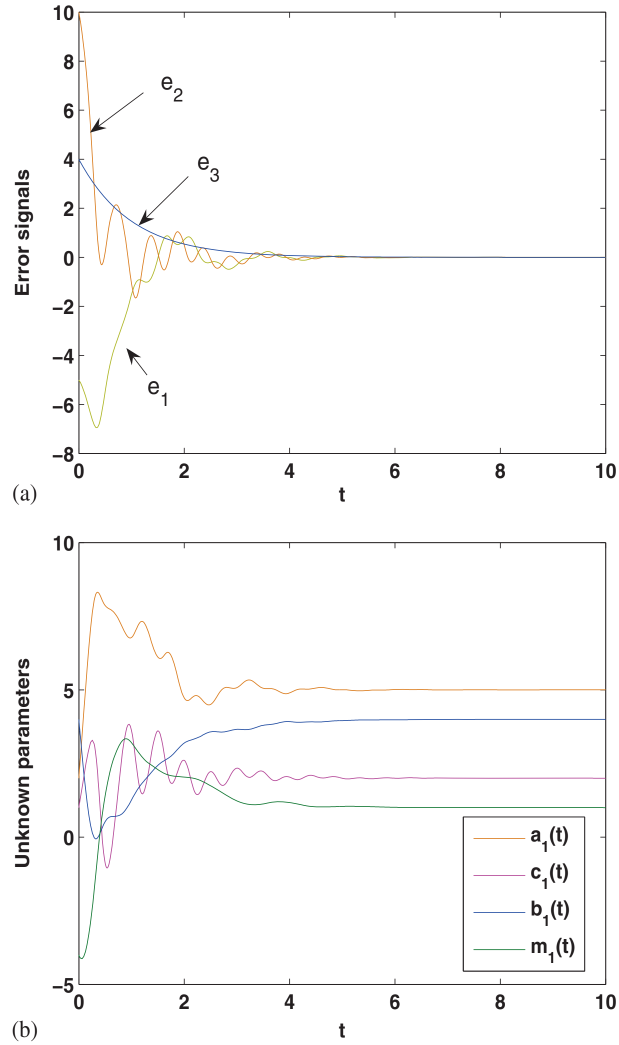

Figure 5. (**a**) The behaviour of the trajectories $e_i(i = 1, 2, 3)$ of the error system and (**b**) the estimates $a_1(t)$, $b_1(t)$, $c_1(t)$, $m_1(t)$ of the unknown parameters converge to 5, 2, 4, 1 as $t \to \infty$.

4. Conclusions and comments

In this paper, chaos synchronization between the hyperchaotic system without equilibrium and the chaotic system with only one stable equilibrium is shown by using the adaptive control technique. In addition, the generation of the new hyperchaotic system is confirmed through a novel electronic circuit design. A good qualitative agreement is illustrated between the simulation results and the real oscilloscope outputs. It is convenient to use the new system to purposefully generate hyperchaos in chaos applications. We believe that the unknown dynamical behaviours of the strange hyperchaotic attractors deserve further investigation and are very desirable for engineering applications such as secure communications in the near future.

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