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Isgur–Wise function in a QCD-inspired potential model with WKB approximation

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Abstract. We use Wentzel–Kramers–Brillouin (WKB) approximation for calculating the slope and curvature of Isgur–Wise function in a QCD-inspired potential model. This work is an extension of the approximation methods to the QCD-inspired potential model. The approach hints at an effective range of distance for calculating the slope and curvature of Isgur–Wise function. Comparison is also made with those of Dalgarno method and variationally improved perturbation theory (VIPT) as well as other models to show the advantages of using WKB approximation.

Keywords. Wentzel–Kramers–Brillouin approximation; Isgur–Wise function; slope and curvature.

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1. Introduction

The wavefunction is a very important entity in the physics. Although its origin is in quantum mechanics, it is useful in every area of physics from condensed matter to high-energy physics. This means that the appropriate solution of Schrödinger equation is very important. Whenever the exact solution is not possible, we use different approximation methods like the usual perturbation theory, variational method, Wentzel–Kramers–Brillouin approximation (WKBA), a comparatively a newer one – the variationally improved perturbation theory (VIPT) etc. All these methods have their own merits and demerits and depending on their effectiveness we use them in different situations.

Use of WKBA in the calculation of slope and curvature of Isgur–Wise (IW) function can be considered as an extension of the applicability of quantum mechanical approximation methods in a QCD-inspired model pursued by us in recent years [1–9]. We have treated this approach as a first step within this potential model of QCD [1]. In this model, we use the Cornell potential, i.e. the linear plus Coulomb potential [2], in calculating various static and dynamic properties of heavy flavoured mesons like elastic form factor, charge radii, Isgur–Wise function etc. [1,3–9] where one needs an accurate wavefunction. Different approximation methods like the Dalgarno method of perturbation theory [10], VIPT [11–13] etc. are used by us in finding the wavefunction accurately. In this process [3–9,13], while using the Dalgarno method or VIPT in the Schrödinger equation with the Cornell potential, we are left with two options of choosing the parent or the perturbation. Further, these two methods, i.e., Dalgarno method (DM) and VIPT, are constrained by the essence of either large strong coupling constant α_s or small confinement parameter b for heavier mesons [3–7] and cut-off in the power of interquark distance r or numerical calculation [8,9] in the process.

The reason for choosing this method in the calculation of IW function is straightforward as is clear from above. The WKBA can give direct solution of Schrödinger equation for the complete potential instead of choosing one part (say Coulombic term) as the parent and the other (linear term) as the perturbation and at the same time can be put in a one-dimensional form thereby making the solution simpler. Also we would hope to get rid of constraints as has happened for DM and VIPT.

Under such circumstances, using WKBA definitely makes sense which will enhance the applicability of not only our QCD potential model but also the method itself. There are a few works [14–16], where linear cum

Coulomb potential is used under WKBA to calculate the energy spectrum, decay width etc. With the success of those models, here we try to extend WKBA for calculating the slope and the curvature of IW function to sort out the advantages as well as the limitations of the method.

The rest of the paper is organized as follows: Section 2 contains the formalism, §3 the result and calculation; §4 includes the discussion and conclusion.

2. Formalism

We first start with the one-dimensional (say x) Schrödinger equation [10]:

$$\frac{d^2\Psi}{dx^2} + 2\mu[E - V(x)]\Psi = 0. \quad (1)$$

With

$$k^2(x) = 2\mu[E - V(x)] = -k'^2(x) \quad (2)$$

the above equation becomes

$$\frac{d^2\Psi}{dx^2} + k^2(x)\Psi = 0. \quad (3)$$

We note that the energy E may be $E > V(x)$ or $E < V(x)$ and depending on this the solutions are either oscillatory or exponential [10].

The WKB quantization condition for the allowed bound state energy is [10]

$$\left(n_r + \frac{1}{2}\right) = \int_A^B k(x)dx, \quad (4)$$

where A and B represent the turning points at which $E = V(x)$ and $n_r \geq 0$.

Now let us consider the three-dimensional radial Schrödinger equation containing an additional centrifugal potential $l(l+1)/2\mu r^2$ to the original potential $V(r)$, the independent variable r is converted to x to give the three-dimensional equation a one-dimensional form and in the process, instead of the integral $\int_A^B k(x)dx$, we are left with the integral given as [10]

$$\int_A^B k(r)dr = \int_A^B \left(E - V(r) - \frac{(l+\frac{1}{2})^2}{2\mu r^2}\right)^{1/2} dr. \quad (5)$$

We note that the term $l(l+1)/2\mu r^2$ is changed to $(l+\frac{1}{2})^2/2\mu r^2$ and the WKB solution is applicable to radial Schrödinger equation like the one-dimensional Schrödinger equation in the variable x [10].

With the original linear cum Coulombic potential $V(r) = (-4\alpha_s/3r) + br$, the WKB quantization condition in this case (eq. (5)) for the ground state ($n_r = 1, l = 0$) is

$$\left(n + \frac{1}{2}\right) = \int_A^B \left(E - br + \frac{\alpha}{r} - \frac{1}{8\mu r^2}\right)^{1/2} dr. \quad (6)$$

As stated above, the limits A and B are the turning points (figure 1) given by the positive roots of the cubic equation:

$$br^3 - Er^2 - \alpha r + \frac{1}{8\mu} = 0. \quad (7)$$

We also note that

$$\alpha = \frac{4\alpha_s}{3}. \quad (8)$$

At the turning points, the WKB solutions become invalid and we find wave functions only in the regions $r < A$ $(E < V)$, $A < r < B$ $(E > V)$ and $r > B$ $(E < V)$. We use the leading-order expressions for energy given by eq. (11) of ref. [15] with W, μ replaced by E and b respectively:

$$E = \left(\frac{3\pi b}{4}\right)^{2/3} \left(n + \frac{3}{2}\right)^{2/3}. \quad (9)$$

The solutions for the respective regions are [10]

$$\Psi_1(r) = \frac{N_1}{2\sqrt{\pi k' r}} e^{\int k' dr}, \quad r < A, \quad (10)$$

$$\Psi_2(r) = \frac{N_2}{2\sqrt{\pi k r}} \cos\left(\int k dr - \frac{\pi}{4}\right), \quad A < r < A, \quad (11)$$

$$\Psi_3(r) = \frac{N_3}{2\sqrt{\pi k' r}} e^{-\int k' dr}, \quad r > B, \quad (12)$$

where N_1, N_2 and N_3 are the normalization constants in the respective regions while k' and k are given respectively by

$$k' = \sqrt{2\mu(V - E)} = \sqrt{2\mu\left(br^2 - Er - \frac{\alpha}{r} + \frac{1}{4r^2}\right)} \quad (13)$$

and

$$k = \sqrt{2\mu(E - V)} = \sqrt{2\mu\left(Er - br^2 + \frac{\alpha}{r} - \frac{1}{4r^2}\right)}. \quad (14)$$

The relativistic modification to these wavefunctions are done through standard Dirac modification [17,18] by multiplying them with $(r/a_0)^{-\epsilon}$, where

$$a_0 = \frac{3}{4\mu\alpha_s} \tag{15}$$

and

$$\epsilon = 1 - \sqrt{1 - \left(\frac{4\alpha_s}{3}\right)^2}. \tag{16}$$

The IW function and its slope and curvature can be calculated as [3,19,20]

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 \, |\psi(r)|^2 \cos pr\mathrm{d}r \tag{17}$$

$$= 1 - \rho^2(y-1) + C(y-1)^2 + \cdots, \tag{18}$$

where

$$p^2 = 2\mu^2(y-1). \tag{19}$$

The limits 0 and ∞ come from the fact that our system (meson) is analogous to that of H-atom where (radial) interparticle distance r ranges from 0 to ∞ in the spherically symmetric coordinate system. Consideration of ∞ as the upper limit is not mandatory as one can consider any range of r as the region of interest in QCD. Further, taking ∞ as the upper limit has certain disadvantages like the divergence of IW function whenever the integrand in eq. (17) is proportional to r or its higher order.

So, it is not a bad idea to consider the limit between two finite values of r, say r_1 and r_2, with r_1 (r_2) representing the lower (upper) limit of the effective range of distance. There are examples in which people have restricted upto a distance of 1 fermi $\sim$ 5.076 GeV^{-1} (i.e. 0.1 fermi $< r <$ 1 fermi) for the Cornell potential [21]. This would facilitate us to not only avoid diverging results but also to observe a distance-dependent IW function as we have encountered three regions of interest in WKB approximation.

Thus, we express the IW function as

$$\xi\,(y) = \int_{r_1}^{r_2} 4\pi r^2 \, |\psi(r)|^2 \cos pr\mathrm{d}r. \tag{20}$$

As we have three regions in this method, namely $r < A$ $(E < V)$, $A < r < B$ $(E > V)$ and $r > B$ $(E < V)$, we can think of the total IW function as the contributions from all these regions with appropriate choice of r_1 and r_2.

$$\xi(y) = \int_0^{A} 4\pi r^2 \, |\psi(r)|^2 \cos pr\mathrm{d}r + \int_A^{B} 4\pi r^2 \, |\psi(r)|^2 \cos pr\mathrm{d}r + \int_B^{r_2} 4\pi r^2 \, |\psi(r)|^2 \cos pr\mathrm{d}r. \tag{21}$$

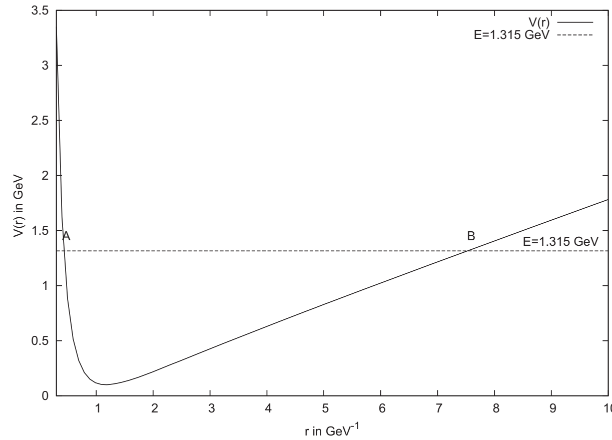

Figure 1. Variation of $V(r)$ vs. r for the energy E given by eq. (9). The points of intersection between $V(r)$ (solid curve) and E (dashed curve) are the turning points A and B – the positive roots of eq. (7).

For the region $r > B$, we choose a particular value of r_2 which is slightly above the value of B for each meson which thus represents the upper limit of range for that meson within the model. The regions can be located from figure 1 and they are listed in table 1. From this table, it is obvious that if we restrict ourselves up to a distance of 1 fermi $\sim$ 5.076 GeV^{-1} as adopted in ref. [21], we can no longer think of the region $r > B$ as $B >$ 5.076 GeV^{-1}. In that case our IW function is calculated within a distance $0 < r <$ 5.076 GeV^{-1}, i.e. completely ignoring the contribution from the region $r > B$ (table 4).

The integration involved in eqs (10)–(12) lead to the well-known elliptic integrals [15]. Instead of using the elliptic functions we have integrated them numerically using *Mathematica* software.

We use both the $\overline{\mathrm{MS}}$ [4,7] and V-schemes [22–24] to see the role of α_s in WKBA in calculating the IW function.

Table 1. List of the turning points A, B in GeV^{-1}. The value $b = 0.183$ GeV^2 is taken from ref. [25].

	$\overline{\mathrm{MS}}$ scheme		V scheme	
Mesons	A	B	A	B
D	0.431	7.52	0.337	7.79
D_s	0.353	7.53	0.269	7.80
B^0/B^+	0.461	7.36	0.444	7.40
B_s	0.374	7.38	0.357	7.41
B_c	0.195	7.39	0.182	7.43

Table 2. Values of ρ^2 and C under the $\overline{\text{MS}}$-scheme. Subscript 'rel' refers to the inclusion of relativistic effect. The maximum finite distance in the region $r > B$, taken for each meson is slightly above B for that meson.

	Region $r < A$				Region $A < r < B$				Region $r > B$			
Meson	ρ^2	C	ρ^2_{rel}	C_{rel}	ρ^2	C	ρ^2_{rel}	C_{rel}	ρ^2	C	ρ^2_{rel}	C_{rel}
D	3.20×10^{-3}	4.92×10^{-6}	1.20×10^{-3}	1.61×10^{-6}	0.357	4.34×10^{-3}	0.142	1.80×10^{-3}	5.09	0.065	5.06	0.064
D_s	3.78×10^{-3}	6.66×10^{-6}	1.51×10^{-3}	2.33×10^{-6}	0.533	0.012	0.190	4.30×10^{-3}	9.07	0.205	9.02	0.204
B^0/B^+	4.56×10^{-3}	1.03×10^{-5}	2.80×10^{-3}	5.96×10^{-6}	0.416	6.87×10^{-3}	0.258	4.26×10^{-3}	6.41	0.106	6.38	0.105
B_s	5.80×10^{-3}	1.62×10^{-5}	3.72×10^{-3}	9.74×10^{-3}	0.733	0.024	0.436	0.014	12.6	0.405	12.5	0.404
B_c	1.24×10^{-2}	6.46×10^{-4}	9.10×10^{-3}	4.41×10^{-5}	3.07	0.712	1.52	0.353	90.8	21.1	90.5	21.0

3. Calculation and result

We have calculated different parameters and listed them in different tables. For example, we have listed the turning points A and B (eq. (7)) in table 1. Table 2 comprises the slope ρ^2 and curvature C for all the three regions (eq. (21)) under $\overline{\text{MS}}$ with and without relativistic effect while table 3 shows the same for V-scheme. In table 4, we have recorded only the contribution (contribution from the region $r < A$ will remain the same as above) in the region $A < r < 5.076\ \text{GeV}^{-1}$ by adopting our effective region as 0.1 fermi $< r <$ 1 fermi (with 1 fermi $\sim 5.076\ \text{GeV}^{-1}$) from the work of Bali [21] for both the $\overline{\text{MS}}$ and V-schemes. In table 5, we have recorded the predictions of the other models to make a comparison with our work.

4. Discussion and conclusion

In this analysis of Isgur–Wise function we have noticed that the contributions for slope and curvature of IW function for the regions $r < A$ and $A < r < B$ together lead to comparable and satisfactory results with other models. The region $r > B$ (by setting the maximum distance slightly above B) results in very large values of slope and curvature which we have to discard. In other sense, sticking ourselves upto the regions $r < A$ and $A < r < B$ leaving aside the region $r > B$ is a clear indication that IW function solely depends on distance r and we have to allow the Cornell potential valid upto a particular (finite) distance. This concept surely helps us to restrict our potential upto a suitable distance instead of infinity and to observe the dependence of IW

Table 3. Values of ρ^2 and C under the V-scheme. Subscript 'rel' refers to the inclusion of relativistic effect. The maximum finite distance in the region $r > B$, taken for each meson is slightly above B for that meson.

	Region $r < A$				Region $A < r < B$				Region $r > B$			
Meson	ρ^2	C	ρ^2_{rel}	C_{rel}	ρ^2	C	ρ^2_{rel}	C_{rel}	ρ^2	C	ρ^2_{rel}	C_{rel}
D	1.87×10^{-3}	1.62×10^{-6}	1.66×10^{-4}	8.54×10^{-8}	0.296	3.76×10^{-3}	0.032	4.07×10^{-4}	5.33	0.070	5.27	0.069
D_s	2.10×10^{-3}	1.99×10^{-6}	2.48×10^{-3}	1.40×10^{-7}	0.432	9.75×10^{-3}	0.036	8.15×10^{-4}	9.50	0.215	9.41	0.213
B^0/B^+	4.08×10^{-3}	8.28×10^{-6}	2.26×10^{-3}	4.25×10^{-6}	0.431	7.11×10^{-3}	0.244	4.01×10^{-3}	6.46	0.107	6.43	0.106
B_s	5.10×10^{-3}	1.25×10^{-5}	2.98×10^{-3}	6.73×10^{-6}	0.677	0.022	0.354	0.011	12.7	0.410	12.6	0.407
B_c	1.20×10^{-2}	5.71×10^{-5}	8.20×10^{-3}	3.61×10^{-5}	3.18	0.741	1.32	0.310	91.5	21.2	91.1	21.2

Table 4. Contributions of ρ^2 and C for the region $A < r < 5.076$ GeV^{-1} [21] under both $\overline{\text{MS}}$ and V-schemes with and without relativistic effect.

	$\overline{\text{MS}}$-scheme				V-scheme			
Meson	ρ^2	C	ρ^2_{rel}	C_{rel}	ρ^2	C	ρ^2_{rel}	C_{rel}
D	0.160	2.04×10^{-3}	0.086	1.09×10^{-3}	0.127	1.62×10^{-3}	0.027	0.346×10^{-3}
D_s	0.237	0.005	0.115	2.61×10^{-3}	0.183	0.004	0.031	0.704×10^{-3}
B^0/B^+	0.216	3.56×10^{-3}	0.142	2.34×10^{-3}	0.206	3.39×10^{-3}	0.138	2.28×10^{-3}
B_s	0.344	0.011	0.238	0.008	0.319	0.010	0.201	0.006
B_c	1.42	0.329	0.830	0.193	1.42	0.329	0.737	0.171

function with distance. This restriction is in agreement with a few works like ref. [21] and considering the effective range cited there we have observed a good agreement between our result (table 4) and other models (table 5). However, this observation has made the region $r > B$ quite insignificant as it leads to large values of slope and curvature.

We have also noticed that ρ^2 and C decrease with the increase in α_s as our results are larger with $\overline{\text{MS}}$-scheme which has smaller α_s than V-scheme. Further, the relativistic effect has reduced the values of ρ^2 and C as observed earlier [3–7]. However, it is interesting to observe that for the region $r > B$, there is a very little effect of α_s as well as relativistic consideration introduced through ϵ hinting at the necessity of an effective range here too for the calculation of IW function.

However, it is important to note that this method of calculating IW function is handicapped at the turning points where WKBA becomes invalid. Further, consideration of only the leading term of eq. (11) of ref. [15] may also be seen as another concern and hence exploration of approximate expression for energy (eq. (12) of ref. [15]) requires attention in this model.

To conclude, this work based on WKBA opens the idea of an effective range of Cornell potential while using in the calculation of IW function showing its distance dependence and usefulness of only the first two

Table 5. Predictions of the slope and curvature of the IW function in various models.

Model	Value of ρ^2	Value of curvature C
VIPT [6]	0.43	0.516
VIPT [9]	0.604	0.0033
Dalgarno Method [8]	0.896	0.0031
Dalgarno Method [5]	1.136	5.377
Le Yauanc *et al* [26]	≥ 0.75	–
Le Yauanc *et al* [27]	≥ 0.75	≥ 0.47
Rosner [28]	1.66	2.76
Mannel *et al* [29,30]	0.98	0.98
Pole ansatz [31]	1.42	2.71
Simple quark model [32]	1	1.11
Skryme model [33]	1.3	0.85
QCD sum rule [34]	0.65	0.47
Relativistic three-quark model [35]	1.35	1.75
UKQCD [36]	0.83^{+15+24}_{-11-22}	–
CLEO [37]	$0.76 \pm 0.16 \pm 0.08$	–
BELLE [38]	0.69 ± 0.14	–

regions (i.e. $r < A$ and $A < r < B$) however ignoring the third region (i.e. $r > B$). This can be an alternative consideration that one may have to take to improve our earlier work [3–9].

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