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A review on the solution of Grad–Shafranov equation in the cylindrical coordinates based on the Chebyshev collocation technique

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Abstract. Equilibrium reconstruction consists of identifying, from experimental measurements, a distribution of the plasma current density that satisfies the pressure balance constraint. Numerous methods exist to solve the Grad–Shafranov equation, describing the equilibrium of plasma confined by an axisymmetric magnetic field. In this paper, we have proposed a new numerical solution to the Grad–Shafranov equation (an axisymmetric, magnetic field transformed in cylindrical coordinates solved with the Chebyshev collocation method) when the source term (current density function) on the right-hand side is linear. The Chebyshev collocation method is a method for computing highly accurate numerical solutions of differential equations. We describe a circular cross-section of the tokamak and present numerical result of magnetic surfaces on the IR-T1 tokamak and then compare the results with an analytical solution.

Keywords. Grad–Shafranov equation; Chebyshev collocation.

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1. Introduction

The need to operate tokamak plasmas in scenarios optimized for fusion by all available means calls for simultaneous real-time control of many parameters. This covers monitoring the shape of the plasma cross-section in view of its role in plasma stability and transport processes, novel divertor configurations to mitigate the erosion of plasma facing components, together with temperature, pressure and current profile tailoring, e.g. to maximize the bootstrap current or sustain internal transport barriers. A crucial component in this task is the magnetohydrodynamic (MHD) equilibrium that defines the geometry of the confining magnetic field and of the divertor. Knowledge of the immediate geometry may be directly used to feedback control the plasma shape and divertor parameters, and guide the choice of location of heating and current drive. In the design of tokamak, the system for controlling the plasma position and shape plays an important role, because it has direct implication in the power supply system. Any contact with material walls would instantaneously cool the plasma and must be avoided. In the axisymmetric toroidal vessel of a tokamak, hot plasma is confined primarily by magnetic Lorentz forces. Strong helical magnetic fields that trace out nested toroidal surfaces help to thermally insulate the plasma from the walls and support it against its own pressure gradient. The analysis of MHD equilibrium is essential to study macroscopic dynamics in magnetically confined plasma. Equilibria are the starting points for many transport and stability calculations. The extreme anisotropy and disparity in spatial characteristics associated with these calculations require accurate numerical methods and accurate equilibrium profiles [1–10]. Magnetostatic equilibria are described by the force balance equation, the magnetic divergence constraint, and the Ampere's law:

$$\mathrm{J} \times B = \nabla P, \tag{1}$$

$$\nabla \cdot B = 0, \tag{2}$$

$$\nabla \times B = \mu_0 J, \tag{3}$$

where μ_0 is the magnetic permeability, J is the current density, P is the plasma pressure, and B is the magnetic field. Equation (2) suggests that if the plasma geometry possesses a symmetry property (axial or helical), a stream function can be introduced to describe the magnetic field. In the case of an axisymmetric torus, this leads to the well-known Grad–Shafranov equation [11–20]:

$$\Delta^*\psi = -\mu_0 R^2 \frac{\mathrm{d}P}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi}, \tag{4}$$

where the Grad–Shafranov operator is defined as

$$\Delta^*\psi = \frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2}. \tag{5}$$

The flux function ψ is related to the physical poloidal flux, divided by a factor of 2π. The two functions $F^2(\psi)$ and $P(\psi)$ are either prescribed or determined from transport effects (not analysed here). Analytical solutions of Grad–Shafranov equation have been found for simple prescriptions of $F^2(\psi)$ and $P(\psi)$ [21–30]. These solutions have limited applicability to most of the experiments, and numerical solutions are often needed. Numerical techniques for solving the Grad–Shafranov equation are categorized according to the specification of the flux along the boundary. The Grad–Shafranov equation has been solved using finite difference [9], Green functions [10], linear finite elements [11] and Hermite cubic finite elements [12]. $F^2(\psi)$ and $P(\psi)$ are arbitrary functions of ψ which occur as source terms on the right-hand side of the Grad–Shafranov equation. When source terms are linear, eq. (4) becomes a linear differential equation. In this work, we have investigated a linear function and study a new numerical for computing highly accurate numerical solution of the Grad–Shafranov equation. Chebyshev collocation method for solving partial differential equations has become increasingly popular because of its ability to achieve high accuracy using relatively few grid points [31–41]. In this work, a method is proposed to compute highly accurate numerical solution of the Grad–Shafranov equation by using the Chebyshev collocation method. We show magnetic surfaces on the IR-T1 tokamak and then compare with an analytical solution. The paper is organized as follows: in §2, we propose Chebyshev collocation formulation of the Grad–Shafranov equation for a circular cross-section of the tokomak, in §3 and in §4, we shall show the result of the numerical solution and an analytical solution respectively and in §5, discussions and conclusion are given.

2. Chebyshev collocation formulation of the Grad–Shafranov equation

Grad–Shafranov equation can be written in axisymmetric coordinate system (R, Z), as

$$\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = \mu_0 R j_\Phi, \tag{6}$$

where j_Φ is the toroidal current density. The simplest solution of eq. (6) can be found by assuming

$$-\mu_0\frac{\mathrm{d}P}{\mathrm{d}\psi} = A_1, \quad -\frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi} = A_2, \tag{7}$$

where A_1 and A_2 are constants. This obviously reduces the set of possible current density profile shapes to

$$\mu_0 R j_\Phi = A_1 + A_2. \tag{8}$$

Multiplying the two sides of eq. (6) by R, it deduces to

$$R\frac{\partial^2\psi}{\partial R^2} - \frac{\partial\psi}{\partial R} + R\frac{\partial^2\psi}{\partial Z^2} = R(A_1 + A_2). \tag{9}$$

Chebyshev collocation method can numerically solve this kind of problem, if an appropriate transformation is applied. We consider a circular cross-section tokamak and use Chebyshev collocation method in this circular domain with respect to R and Z variables in the following domain:

$$\Omega = \{(R, Z) \in \mathbb{R}^2 | (R - R_0)^2 + Z^2 \le a^2\} \tag{10}$$

or

$$\Omega = \{(R, Z) \in \mathbb{R}^2 | -a \le Z \le a,\ R_0 - \sqrt{a^2 - Z^2} \le R \le R_0 + \sqrt{a^2 - Z^2}\}, \tag{11}$$

where R_0 and a are respectively the major radius and minor radius of the torus. It is recalled that in one-dimensional Chebyshev collocation method, we use a computational domain given by [–1, 1]. A bijective mapping is defined for Ω in (R, Z) to a square domain $\tilde{\Omega} = [-1, 1] \times [-1, 1]$ in the (ξ, η) plane. This yields $R = R(\xi, \eta)$ and $Z = Z(\xi, \eta)$. The collection points in (R, Z) are mapped to the Chebyshev points in (ξ, η). To solve eq. (9), in the first step we transform the physical domain [–a, a] to [–1, 1] by

$$Z_i = a\eta_i, \quad i = 1, 2, ..., N_Z, \tag{12}$$

where η_i is given as follows [13]:

$$\eta_i = -\cos\left(\frac{(i-1)\pi}{N_Z - 1}\right), \quad i = 1, 2, ..., N_Z, \tag{13}$$

and N_Z is the number of the collocation points in [–a, a]. In the second step, we transform the physical domain $[R_j^+(Z_i), R_j^-(Z_i)]$ to [–1, 1] by

$$R_j = \frac{R_j^+(Z_i) - R_j^-(Z_i)}{2}\xi_j + \frac{R_j^+(Z_i) + R_j^-(Z_i)}{2}, \tag{14}$$

where

$$R_j^+(Z_i) = R_0 + \sqrt{a^2 - Z_i^2},$$

$$R_j^-(Z_i) = R_0 - \sqrt{a^2 - Z_i^2},\quad i = 1, 2, ..., N_Z,\quad j = 1, 2, ..., N_R, \tag{15}$$

and ξ_j is given by

$$\xi_j = -\cos\left(\frac{(j-1)\pi}{N_R - 1}\right),\quad i = 1, 2, ..., N_R, \tag{16}$$

and N_R is the number of collocation points in $[R_j^-(Z_i),\ R_j^+(Z_i)]$.

Using the bivariate Lagrange interpolation polynomials

$$\psi(\xi, \eta) \cong \sum_{i=1}^{N_R} \sum_{j=1}^{N_Z} L_i(\xi) L_j(\eta) \psi(\xi_i, \eta_j). \tag{17}$$

The functions $L_i(\xi)$ and $L_j(\eta)$ are defined as

$$L_i(\xi) = \prod_{\substack{l=1\\ i\neq l}}^{N_R} \frac{\xi - \xi_l}{\xi_i - \xi_l},\quad i = 1, 2, ..., N_R, \tag{18}$$

and

$$L_i(\eta) = \prod_{\substack{l=1\\ i\neq l}}^{N_Z} \frac{\eta - \eta_l}{\eta_i - \eta_l},\quad i = 1, 2, ..., N_Z, \tag{19}$$

with

$$L_j(\xi_i) = \delta_{ij},\quad i, j = 1, 2, ..., N_R,$$
$$L_j(\eta_i) = \delta_{ij},\quad i, j = 1, 2, ..., N_Z, \tag{20}$$

and δ_{ij} is called the Kronecker δ, defined as

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{21}$$

In the next step, one needs the partial derivatives of ξ and η with respect to R and Z which can be obtained as follows [14]:

$$\frac{\partial \xi}{\partial R} = \frac{1}{J}\frac{\partial Z}{\partial \eta},\quad \frac{\partial \xi}{\partial Z} = -\frac{1}{J}\frac{\partial R}{\partial \eta},$$
$$\frac{\partial \eta}{\partial Z} = -\frac{1}{J}\frac{\partial R}{\partial \xi},\quad \frac{\partial \eta}{\partial R} = \frac{1}{J}\frac{\partial Z}{\partial \xi}, \tag{22}$$

where J is the determinant of the Jocobian matrix of (R, Z) with respect to (ξ, η) defined by

$$J = \begin{vmatrix} \dfrac{\partial R}{\partial \xi} & \dfrac{\partial R}{\partial \eta} \\ \dfrac{\partial Z}{\partial \xi} & \dfrac{\partial Z}{\partial \eta} \end{vmatrix}. \tag{23}$$

The Jacobian of transformation is obtained directly from the equations

$$Z = a\eta \tag{24}$$

and

$$R = \sqrt{a^2 - Z^2}\xi + R_0. \tag{25}$$

Using composite differentiation (chain rule), and considering relation (22) and (23), eq. (9) deduces to

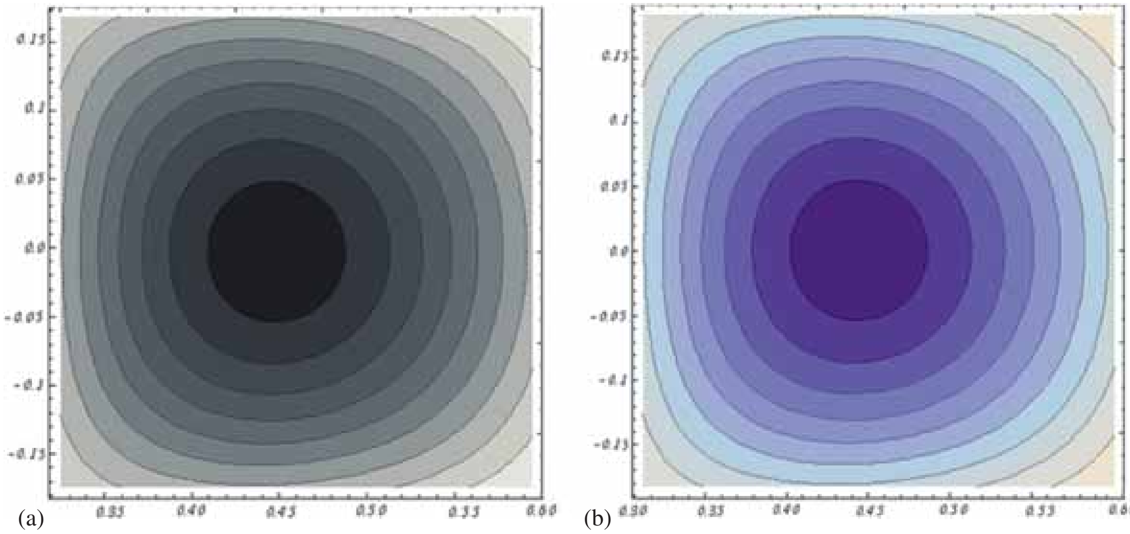
$$B(\xi, \eta)\frac{\partial^2 \psi(\xi, \eta)}{\partial \xi^2} + C(\xi, \eta)\frac{\partial^2 \psi(\xi, \eta)}{\partial \eta^2} + D(\xi, \eta)\frac{\partial \psi(\xi, \eta)}{\partial \xi} + E(\xi, \eta)\frac{\partial^2 \psi(\xi, \eta)}{\partial \xi \partial \eta} = F(\xi, \eta)(A_1 + A_2) \tag{26}$$

where

$$B(\xi, \eta) = \left(a\xi\sqrt{1-\eta^2} + R_0\right)(1 - \eta^2 + \eta^2\xi^2),$$
$$C(\xi, \eta) = \left(a\xi\sqrt{1-\eta^2} + R_0\right)(1 - \eta^2),$$
$$D(\xi, \eta) = -a(1-\eta^2)\sqrt{1-\eta^2} + \xi\left(a\xi\sqrt{1-\eta^2} + R_0\right)(1 + 2\eta^2),$$
$$E(\xi, \eta) = 2\eta\xi(1-\eta^2)\left(a\xi\sqrt{1-\eta^2} + R_0\right),$$
$$F(\xi, \eta) = a^2(1-\eta^2)^2\left(a\xi\sqrt{1-\eta^2} + R_0\right). \tag{27}$$

In the next step, by replacing (ξ, η) by (ξ_j, η_i), and simplifying, eq. (26) can be

$$B(\xi_j, \eta_i)\sum_{k=1}^{N_R}\sum_{l=1}^{N_Z} L_k''(\xi_j)\, L_l(\eta_i)\psi_{ij} + C(\xi_j, \eta_i)\sum_{k=1}^{N_R}\sum_{l=1}^{N_Z} L_k(\xi_j) L_l''(\eta_i)\,\psi_{ij} + D(\xi_j, \eta_i)\sum_{k=1}^{N_R}\sum_{l=1}^{N_Z} L_k'(\xi_j) L_l(\eta_i)\,\psi_{ij} + E(\xi_j, \eta_i)\sum_{k=1}^{N_R}\sum_{l=1}^{N_Z} L_k'(\xi_j) L_l'(\eta_i)\,\psi_{ij} = F(\xi_j, \eta_i)(A_1 + A_2),\quad i = 1, 2, ..., N_Z,\quad j = 1, 2, ..., N_R, \tag{28}$$



(a) (b)

Figure 1. Magnetic contour approximated by (**a**) analytical solution method on IR-T1 tokamak with $j_\Phi = 0.62\ \text{A/m}^2$ and by (**b**) Chebyshev collocation method on tokamak IR-T1, with $j_\Phi = 0.62\ \text{A/m}^2$.

with

$$\psi_{ij} = \psi(\xi_j, \eta_i), \;\; i = 1, 2, ..., N_Z, \;\; j = 1, 2, ..., N_R. \tag{29}$$

By simplifying the above equations one obtains

$$\begin{aligned} &B(\xi_j, \eta_i) \sum_{k=1}^{N_R} \hat{D}^{(2)}_{jk} \psi_{lj} + C(\xi_j, \eta_i) \sum_{l=1}^{N_Z} D^{(2)}_{il} \psi_{ik} \\ &\quad + D(\xi_j, \eta_i) \sum_{k=1}^{N_R} \hat{D}_{jk} \psi_{lj} \\ &\quad + E(\xi_j, \eta_i) \sum_{k=1}^{N_R} \sum_{l=1}^{N_Z} \hat{D}_{jk} D_{il} \psi_{lj} \\ &\quad = F(\xi_j, \eta_i)(A_1 + A_2), \\ &i = 1, 2, \ldots, N_Z, \quad j = 1, 2, \ldots, N_R. \end{aligned} \tag{30}$$

where $\tilde{D} = (\tilde{D}_{ij})_{N\times N}$ is the first-order Chebyshev differentiation matrix, $\tilde{D}^2 = (D^2_{ij})_{N_R\times N_R} = \tilde{D}^2$ is the second-order differentiation matrix with respect to ξ, and $D^2 = (D^2_{ij})_{N_Z\times N_Z}$ is the second-order differentiation matrix with respect to η (see [13]).

3. Numerical solution of the Grad–Shafranov equation

The IR-T1 tokamak is a small air-core transformer tokamak with circular cross-section and without the conducting shell and the divertor where $R_0 = 45$ cm, $a = 12.5$ cm. For simplicity, in eq. (30), the constant A_2 is set to zero. The plasma current can be clearly measured by Rogowski coil [16]. So the plasma current can be expressed by

$$\mu_0 I_{\text{p}} = \iint R A_1 \text{d}R\, \text{d}Z. \tag{31}$$

After integration for IR-T1 tokamak, we have

$$A_1 = 27.631 \mu_0 I_{\text{p}}. \tag{32}$$

We assumed that the plasma pressure vanishes at the boundary, i.e. $\psi = 0$ in boundary points [17]. The magnetic surfaces on the circular cross-section of tokamak IR-T1 is presented in figure 1b, by considering $N_R = N_Z = 70$.

4. Analytical solution of the Grad–Shafranov equation

There are different solutions depending on source functions selected for plasma pressure and poloidal current. The simplest solution to the inhomogeneous Grad–Shafranov equation is the well-known Solovev equilibrium [15]. This equilibrium corresponds to linear source functions. Analytical solutions of the Grad–Shafranov equation [18,19] are very useful for theoretical studies of plasma equilibrium, transport, and MHD stability. The shape of the current profile is essentially flat. The poloidal magnetic flux ψ is obtained as given in [15]:

$$\begin{aligned} \psi &= c_1 + c_2 R^2 + c_3(R^4 - 4R^2 Z^2) \\ &\quad + c_4(R^2 \ln R - Z^2) + \frac{A_1}{8} R^4 - \frac{A_2}{2} Z^2. \end{aligned} \tag{33}$$

The equation can be used for tokamaks that have up-down symmetric or D-shape plasma cross-section. The resulting magnetic surfaces on tokamak IR-T1 is shown in figure 1a.

5. Discussions and conclusions

In this research, the Chebyshev collocation method (a method for computing highly accurate numerical solutions of differential equations) has been applied to solve Grad–Shafranov equation in cylindrical coordinates. We have used Solovev's assumption. In this case, the right-hand side of the Grad–Shafranov equation becomes independent of ψ by choosing linear profiles $P(\psi)$ and $F^2(\psi)$ and then comparing with an analytical solution. The comparison of two figures shows that the values of magnetic surfaces and the displacement of Shafranov shift in figure 1b are the same as that in figure 1a.

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