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Multiswitching combination–combination synchronization of chaotic systems

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Abstract. In this paper, a novel synchronization scheme is investigated for a class of chaotic systems. The multiswitching synchronization scheme is extended to the combination–combination synchronization scheme such that the combination of state variables of two drive systems synchronize with different combination of state variables of two response systems, simultaneously. The new scheme, multiswitching combination–combination synchronization (MSCCS), is a notable extension of the earlier multiswitching schemes concerning only the single drive–response system model. Various multiswitching modified projective synchronization schemes are obtained as special cases of MSCCS, for a suitable choice of scaling factors. Suitable controllers have been designed and using Lyapunov stability theory sufficient condition is obtained to achieve MSCCS between four hyperchaotic systems and the corresponding theoretical proof is given. Numerical simulations are performed to validate the theoretical results.

Keywords. Chaos synchronization; multiswitching synchronization; nonlinear control; Lyapunov stability theory.

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1. Introduction

Chaos synchronization has developed into a very important area of research since the concept was first proposed by Pecora and Carroll in 1990 [1]. The topic has been widely studied due to its potential interdisciplinary applications across areas such as physics, biological systems, electrical engineering, information processing, communication theory etc. [2–5]. Over the years many types of synchronization methods such as complete synchronization [6], antisynchronization [7], projective synchronization [8], lag synchronization [9], phase synchronization [10], reduced order synchronization [11], increased order synchronization [12], etc. have been investigated. Various methods such as active control method [13], function cascade [14], predictive control method [15], adaptive control method [16], sliding mode control [17], active backstepping method [18] etc. have been developed and studied to achieve synchronization.

Majority of the synchronization studies have been restricted to single drive–response system model. It is therefore interesting to ask whether these studies can be extended to multiple drive–response system models involving three or more chaotic systems. In recent years, new synchronization schemes wherein three or more chaotic systems are involved, such as combination synchronization [19–21], combination–combination synchronization [22], compound synchronization [23,24], and double compound synchronization [25], have been presented. In addition to their own intrinsic interest, these schemes are significant in enhancing the security of information transmitted via chaotic signals because of the complexity which they bring in transmitted signal.

Combination–combination synchronization scheme involving two drive and two response systems was recently investigated by Sun *et al* [22] to overcome the disadvantage of combination synchronization which involves one response system. This scheme of combination–combination synchronization is an extension of the combination synchronization scheme. In this scheme, a universal controller is designed to realize synchronization between a combination of two drive systems with combination of two response systems. Further studies in this direction have been reported in [26–28].

Multiswitching synchronization scheme proposed by Ucar *et al* [29] is an important and interesting extension of the existing synchronization schemes. In the multiswitching synchronization scheme, a wide range of possible synchronization directions exist as the different states of the drive system are synchronized with different states of the response system. The potential application of such schemes to chaos communication makes them an interesting area of research [30–32]. Multiswitching synchronization involving multiple drive and response systems is an untouched area of research. Only a handful of results exist for such studies in the chaos synchronization literature [33].

In view of the above discussions, in this paper, we present a new multiswitching combination–combination synchronization (MSCCS) scheme, wherein two drive chaotic systems are combined in different ways to form a resultant signal which is then synchronized with some combination of two response chaotic systems. To the best of our knowledge, study on multiswitching synchronization involving two response systems has not been reported in any earlier work. Using Chen hyperchaotic system [34] and Liu hyperchaotic system [35] as drive systems and Lorenz hyperchaotic system [36] and Qi hyperchaotic system [37] as response systems, we illustrate the scheme of MSCCS. Suitable nonlinear controllers are designed and, with the help of Lyapunov stability theory, we realize the desired synchronization.

The paper is organized as follows. In §2, the scheme of MSCCS is developed. In §3, synchronization is achieved among four non-identical hyperchaotic systems and theoretical results are validated by performing necessary numerical simulations. Finally, conclusions are drawn in §4.

2. The scheme of multiswitching combination–combination synchronization

In this section, we construct the scheme of MSCCS of chaotic systems. We need two drive systems and two response systems. Let the drive systems be described by

$$\dot{x} = f_1(x), \tag{1}$$

$$\dot{y} = f_2(y), \tag{2}$$

and the response systems be given by

$$\dot{z} = g_1(z) + u, \tag{3}$$

$$\dot{w} = g_2(w) + u^*, \tag{4}$$

where $x = (x_1, x_2, x_3, ..., x_n)^T$, $y = (y_1, y_2, y_3, ..., y_n)^T$, $z = (z_1, z_2, z_3, ..., z_n)^T$, and $w = (w_1, w_2, w_3, ..., w_n)^T$ are state vectors of systems (1)–(4) respectively; f_1, f_2, g_1, g_2: $R^n \to R^n$ are four continuous vector functions and u, u^*: $R^n \times R^n \times R^n \times R^n \to R^n$ are two controllers to be designed for the response systems (3) and (4), respectively. To solve the combination–combination synchronization problem the error is defined as $e = Ax + By - Cz - Dw$.

DEFINITION 1 (See [22])

If there exist four constant matrices $A, B, C, D \in R^n$ and $C \neq 0$ or $D \neq 0$ such that

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|Ax + By - Cz - Dw\| = 0, \tag{5}$$

then the drive systems (1) and (2) realize combination–combination synchronization with the response systems (3) and (4), where $\|\cdot\|$ represents the vector norm.

Remark 1. The constant matrices A, B, C, D are called the scaling matrices. These matrices can be extended to functional matrices of state variables x, y, z and w.

Remark 2. For the convenience of our discussion, if we assume $A = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$, $B = \mathrm{diag}(\beta_1, \beta_2, \beta_3, ..., \beta_n)$, $C = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n)$, and $D = \mathrm{diag}(\delta_1, \delta_2, \delta_3, ..., \delta_n)$, then the components of the error vector e are obtained as

$$e_{ijkl} = \alpha_i x_i + \beta_j y_j - \gamma_k z_k - \delta_l w_l. \tag{6}$$

In relation to Definition 1, the indices of the error states are strictly chosen to satisfy $i = j = k = l$ $(i, j, k, l = 1, 2, ..., n)$.

DEFINITION 2

If the error states in relation to Definition 1 are redefined such that $i = j = k \neq l$ or $i = j = l \neq k$ or $i = k = l \neq j$ or $j = k = l \neq i$; or $i = j \neq k = l$ or $i = k \neq j = l$ or $i = l \neq j = k$; or $i = j \neq k \neq l$ or

$i=k\neq j\neq l$ or $i=l\neq k\neq j$ or $i\neq j=k\neq l$ or $i\neq j\neq k=l$ or $i\neq k\neq j=l$; or $i\neq j\neq k\neq l$ and

$$\lim_{t\to\infty}\|e\|=\lim_{t\to\infty}\|Ax+By-Cz-Dw\|=0, \tag{7}$$

then the drive systems (1) and (2) are said to be in multiswitching combination–combination synchronization with systems (3) and (4), where $\|\cdot\|$ represents the vector norm.

Using eqs (1)–(4) and eq. (7) the error dynamics is obtained as

$$\begin{aligned}\dot{e} = {}& Af_1(x)+Bf_2(y)-Cg_1(z)\\ &-Cu-Dg_2(w)-Du^*.\end{aligned} \tag{8}$$

Our goal is to determine the suitable control input $U=Cu+Du^*$, of the general form

$$U=Af_1(x)+Bf_2(y)-Cg_1(z)-Dg_2(w)+pe, \tag{9}$$

such that systems (1)–(4) achieve MSCCS in accordance with Definition 2. Here p is a constant that influences the rate of convergence.

Remark 3. If $C=0$ or $D=0$, then the MSCCS problem reduces to multiswitching combination synchronization (MSCS) problem.

Remark 4. If $A=D=0$, $C=I$ or $A=C=0$, $D=I$ or $B=D=0$, $C=I$ or $B=C=0$, $D=I$, then the MSCCS problem reduces to multiswitching modified projective synchronization, where I is an $n\times n$ identity matrix.

Remark 5. If $A=B=C=0$ or $A=B=D=0$ then the synchronization problem turns into a chaos control problem.

3. Multiswitching combination–combination synchronization among four non-identical hyperchaotic systems

In this section, we realize the MSCCS among four non-identical hyperchaotic systems. The Chen hyperchaotic system and Liu hyperchaotic system are taken as drive systems and are, respectively, described as follows:

$$\begin{cases}\dot{x}_1 = a_1(x_2-x_1)+x_4,\\ \dot{x}_2 = d_1x_1-x_1x_3+c_1x_2,\\ \dot{x}_3 = x_1x_2-b_1x_3,\\ \dot{x}_4 = x_2x_3+rx_4,\end{cases} \tag{10}$$

$$\begin{cases}\dot{y}_1 = a_2(y_2-y_1),\\ \dot{y}_2 = b_2y_1-ky_1y_3+y_4,\\ \dot{y}_3 = -c_2y_3+hy_1^2,\\ \dot{y}_4 = -d_2y_1.\end{cases} \tag{11}$$

The Lorenz hyperchaotic system and Qi hyperchaotic system are taken as response systems and are, respectively, given by

$$\begin{cases}\dot{z}_1 = a_3(z_2-z_1)+u_1,\\ \dot{z}_2 = c_3z_1-z_1z_3+z_2-z_4+u_2,\\ \dot{z}_3 = z_1z_2-b_3z_3+u_3,\\ \dot{z}_4 = d_3z_2z_3+u_4,\end{cases} \tag{12}$$

$$\begin{cases}\dot{w}_1 = a_4(w_2-w_1)+w_2w_3+u_1^*,\\ \dot{w}_2 = b_4(w_1+w_2)-w_1w_3+u_2^*,\\ \dot{w}_3 = -c_4w_3-fw_4+w_1w_2+u_3^*,\\ \dot{w}_4 = -d_4w_4+gw_3+w_1w_2+u_4^*,\end{cases} \tag{13}$$

where $u=(u_1,u_2,u_3,u_4)$ and $u^*=(u_1^*,u_2^*,u_3^*,u_4^*)$ are the controllers to be designed.

By the conditions on indices $i,j,k,l=1,2,3,4$ stated in Definition 2, several multiswitching combinations exist for defining the error states for the drive–response systems (10)–(13), some of which are as follows:

For $i=j=k\neq l$, we have:

$e_{1112}, e_{1113}, e_{2221}, e_{2224}, e_{3332}, e_{3334}, e_{4441}, e_{4442}$ etc.

For $i=j\neq k\neq l$, we have:

$e_{1123}, e_{1143}, e_{2214}, e_{2241}, e_{3314}, e_{3342}, e_{4421}, e_{4432}$ etc.

For $i\neq j\neq k\neq l$, we have:

$e_{1234}, e_{1432}, e_{2314}, e_{2431}, e_{3124}, e_{3421}, e_{4132}, e_{4321}$ etc.

Here we discuss results for one randomly selected error space vector combination formed out of 63^4 multiswitching possibilities. The results for the rest of the possible combinations can be obtained in a similar manner. If we assume $A=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $B=\mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4)$, $C=\mathrm{diag}(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$, and $D=\mathrm{diag}(\delta_1,\delta_2,\delta_3,\delta_4)$, then the switched error states are given by

$$\begin{cases}e_{4321} = \alpha_4x_4+\beta_3y_3-\gamma_2z_2-\delta_1w_1,\\ e_{3212} = \alpha_3x_3+\beta_2y_2-\gamma_1z_1-\delta_2w_2, \quad \text{Switch 1}\\ e_{2133} = \alpha_2x_2+\beta_1y_1-\gamma_3z_3-\delta_3w_3,\\ e_{1444} = \alpha_1x_1+\beta_4y_4-\gamma_4z_4-\delta_4w_4,\end{cases} \tag{14}$$

where we refer eq. (14) as Switch 1 for simplicity. The notations $\alpha_i,\beta_j,\gamma_k,\delta_l$ $(i,j,k,l=1,2,3,4)$ representing the scaling factors are set for convenience and may assume different or same values in applications.

For Switch 1, the error dynamical system is obtained as follows:

$$\begin{cases} \dot{e}_{4321} = \alpha_4\dot{x}_4 + \beta_3\dot{y}_3 - \gamma_2\dot{z}_2 - \delta_1\dot{w}_1, \\ \dot{e}_{3212} = \alpha_3\dot{x}_3 + \beta_2\dot{y}_2 - \gamma_1\dot{z}_1 - \delta_2\dot{w}_2, \\ \dot{e}_{2133} = \alpha_2\dot{x}_2 + \beta_1\dot{y}_1 - \gamma_3\dot{z}_3 - \delta_3\dot{w}_3, \\ \dot{e}_{1444} = \alpha_1\dot{x}_1 + \beta_4\dot{y}_4 - \gamma_4\dot{z}_4 - \delta_4\dot{w}_4. \end{cases} \tag{15}$$

Using eqs (10)–(13) the error system (15) is transformed into the following form:

$$\begin{cases} \dot{e}_{4321} = \alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ \qquad - \gamma_2(c_3z_1 - z_1z_3 + z_2 - z_4 + u_2) \\ \qquad - \delta_1(a_4(w_2 - w_1) + w_2w_3 + u_1^*), \\ \dot{e}_{3212} = \alpha_3(x_1x_2 - b_1x_3) + \beta_2(b_2y_1 - ky_1y_3 + y_4) \\ \qquad - \gamma_1(a_3(z_2 - z_1) + u_1) - \delta_2(b_4(w_1 + w_2) \\ \qquad - w_1w_3 + u_2^*), \\ \dot{e}_{2133} = \alpha_2(d_1x_1 - x_1x_3 + c_1x_2) + \beta_1a_2(y_2 - y_1) \\ \qquad - \gamma_3(z_1z_2 - b_2z_3 + u_3) \\ \qquad - \delta_3(-c_4w_3 - fw_4 + w_1w_2 + u_3^*), \\ \dot{e}_{1444} = \alpha_1(a_1(x_2 - x_1) + x_4) + \beta_4(-d_2y_1) \\ \qquad - \gamma_4(d_3z_2z_3 + u_4) \\ \qquad - \delta_4(-d_4w_4 + gw_3 + w_1w_2 + u_4^*). \end{cases} \tag{16}$$

Denote

$$\begin{cases} U_1 = \gamma_2u_2 + \delta_1u_1^*, \\ U_2 = \gamma_1u_1 + \delta_2u_2^*, \\ U_3 = \gamma_3u_3 + \delta_3u_3^*, \\ U_4 = \gamma_4u_4 + \delta_4u_4^*. \end{cases} \tag{17}$$

Theorem 1. *The drive systems* (10) *and* (11) *will achieve multiswitching combination–combination synchronization with response systems* (12) *and* (13) *if the control functions* U_1, U_2, U_3, *and* U_4 *are chosen such that*

$$\begin{cases} U_1 = \alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ \qquad - \gamma_2(c_3z_1 - z_1z_3 + z_2 - z_4) \\ \qquad - \delta_1(a_4(w_2 - w_1) + w_2w_3) \\ \qquad + (\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1) \\ \qquad + a_1(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2) \\ \qquad - a_2(\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4 - \delta_4w_4), \\ U_2 = \alpha_3(x_1x_2 - b_1x_3) + \beta_2(b_2y_1 - ky_1y_3 + y_4) \\ \qquad - \gamma_1(a_3(z_2 - z_1)) - \delta_2(b_4(w_1 + w_2) - w_1w_3) \\ \qquad + (\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2) \\ \qquad - a_1(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1) \\ \qquad + a_3(\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3 - \delta_3w_3), \\ U_3 = \alpha_2(d_1x_1 - x_1x_3 + c_1x_2) + \beta_1a_2(y_2 - y_1) \\ \qquad - \gamma_3(z_1z_2 - b_3z_3) - \delta_3(-c_4w_3 - fw_4 + w_1w_2) \\ \qquad + (\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3 - \delta_3w_3) \\ \qquad - a_3(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2), \\ U_4 = \alpha_1(a_1(x_2 - x_1) + x_4) + \beta_4(-d_2y_1) \\ \qquad - \gamma_4(d_3z_2z_3) - \delta_4(-d_4w_4 + gw_3 + w_1w_2) \\ \qquad + (\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4 - \delta_4w_4) \\ \qquad + a_2(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1). \end{cases} \tag{18}$$

Proof. Let $E_1 = e_{4321}$, $E_2 = e_{3212}$, $E_3 = e_{2133}$, and $E_4 = e_{1444}$. From (16) and (17) we get

$$\begin{cases} \dot{E}_1 = \alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ \qquad - \gamma_2(c_3z_1 - z_1z_3 + z_2 - z_4) \\ \qquad - \delta_1(a_4(w_2 - w_1) + w_2w_3) - U_1, \\ \dot{E}_2 = \alpha_3(x_1x_2 - b_1x_3) + \beta_2(b_2y_1 - ky_1y_3 + y_4) \\ \qquad - \gamma_1a_3(z_2 - z_1) \\ \qquad - \delta_2(b_4(w_1 + w_2) - w_1w_3) - U_2, \\ \dot{E}_3 = \alpha_2(d_1x_1 - x_1x_3 + c_1x_2) + \beta_1a_2(y_2 - y_1) \\ \qquad - \gamma_3(z_1z_2 - b_2z_3) \\ \qquad - \delta_3(-c_4w_3 - fw_4 + w_1w_2) - U_3, \\ \dot{E}_4 = \alpha_1(a_1(x_2 - x_1) + x_4) + \beta_4(-d_2y_1) - \gamma_4(d_3z_2z_3) \\ \qquad - \delta_4(-d_4w_4 + gw_3 + w_1w_2) - U_4. \end{cases} \tag{19}$$

Consider the Lyapunov function in the form of

$$V(E(t)) = \frac{1}{2}(E_1^2 + E_2^2 + E_3^2 + E_4^2).$$

The derivative of V along the trajectories of (19) is obtained as

$$\begin{aligned} \dot{V}(E(t)) = &\ E_1\dot{E}_1 + E_2\dot{E}_2 + E_3\dot{E}_3 + E_4\dot{E}_4 \\ = &\ E_1(\alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ & - \gamma_2(c_3z_1 - z_1z_3 + z_2 - z_4) \\ & - \delta_1(a_4(w_2 - w_1) + w_2w_3) - U_1) \\ & + E_2(\alpha_3(x_1x_2 - b_1x_3) \\ & + \beta_2(b_2y_1 - ky_1y_3 + y_4) - \gamma_1a_3(z_2 - z_1) \\ & - \delta_2(b_4(w_1 + w_2) - w_1w_3) - U_2) \\ & + E_3(\alpha_2(d_1x_1 - x_1x_3 + c_1x_2) \\ & + \beta_1a_2(y_2 - y_1) - \gamma_3(z_1z_2 - b_2z_3) \\ & - \delta_3(-c_4w_3 - fw_4 + w_1w_2) - U_3) \\ & + E_4(\alpha_1(a_1(x_2 - x_1) + x_4) \\ & + \beta_4(-d_2y_1) - \gamma_4(d_3z_2z_3) \\ & - \delta_4(-d_4w_4 + gw_3 + w_1w_2) - U_4). \end{aligned} \tag{20}$$

Substituting the values of U_1, U_2, U_3, and U_4 in (20) we get

$$\begin{aligned} \dot{V}(E(t)) = &\ E_1(-(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1) \\ & - a_1(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2) \\ & + a_2(\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4 - \delta_4w_4)) \\ & + E_2(-(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2) \\ & + a_1(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1) \\ & - a_3(\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3 - \delta_3w_3)) \\ & + E_3(-(\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3 - \delta_3w_3) \\ & + a_3(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1 - \delta_2w_2)) \\ & + E_4(-(\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4 - \delta_4w_4) \\ & - a_2(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2 - \delta_1w_1)). \end{aligned}$$

This can be further written as

$$\begin{aligned}\dot{V}(E(t)) &= E_1(-E_1 - a_1E_2 + a_2E_4) \\ &\quad + E_2(-E_2 + a_1E_1 - a_3E_3) \\ &\quad + E_3(-E_3 + a_3E_2) + E_4(-E_4 - a_2E_1) \\ &= -E_1^2 - E_2^2 - E_3^2 - E_4^2 \\ &= -E^TE,\end{aligned}$$

where $E^T = (E_1, E_2, E_3, E_4)^T$. Thus, we see that $\dot{V}(E(t))$ is negative definite. According to Lyapunov stability theory, we know $E_i \to 0\,(i = 1, 2, 3, 4)$, that is, $\lim_{t\to\infty}\|E\| = 0$, which means that the drive systems (10)–(12) will achieve MSCCS with the response system (13). □

We perform numerical simulations to illustrate the results. In simulation process we assume $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, and $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$. The system parameters are taken as $a_1 = 35, b_1 = 3, c_1 = 12, d_1 = 7, r = 0.5, a_2 = 10, b_2 = 40, c_2 = 2.5, d_2 = 10.6, h = 4, k = 1, a_3 = 10, b_3 = 8/3, c_3 = 28, d_3 = 0.1, a_4 = 50, b_4 = 24, c_4 = 13, d_4 = 8, f = 33$, and $g = 30$. The initial states of the drive and the response systems are arbitrarily chosen as $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-5, 1, -4, 6)$, $(y_1(0), y_2(0), y_3(0), y_4(0)) = (7, -2, 0, 0)$, $(z_1(0), z_2(0), z_3(0), z_4(0)) = (5, 4, -2, 3)$, and $(w_1(0), w_2(0), w_3(0), w_4(0)) = (-14, 3, -1.5, 4.5)$. Figure 1 displays the time response of synchronization errors e_{4321}, e_{3212}, e_{2133}, and e_{1444}. Figures 2–5 illustrate the time response of the synchronized states $x_1 + y_3$ and $z_2 + w_1$, $x_3 + y_2$ and $z_1 + w_2$, $x_2 + y_1$ and $z_3 + w_3$, $x_1 + y_4$ and $z_4 + w_4$ of the drive systems (10), (11) and response systems (12), (13) respectively.

The following corollaries are easily obtained from Theorem 1 and their proofs are omitted here.

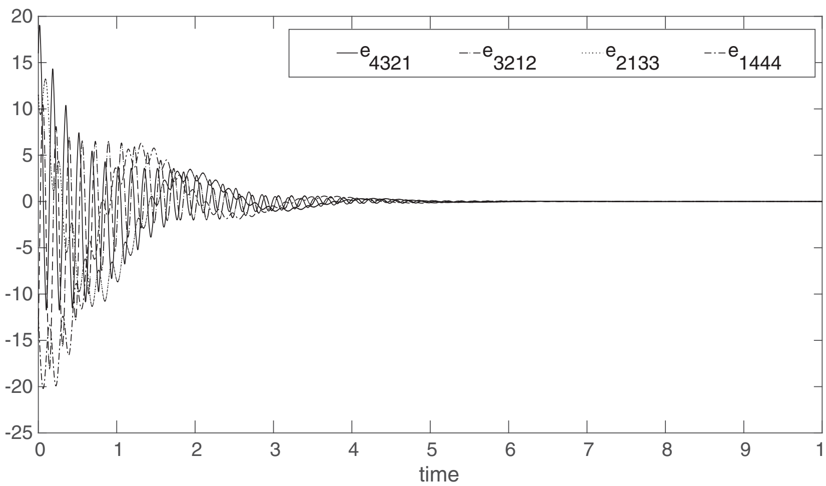

Figure 1. Synchronization errors e_{4321}, e_{3212}, e_{2133}, and e_{1444} between drive systems (10), (11) and response systems (12), (13).

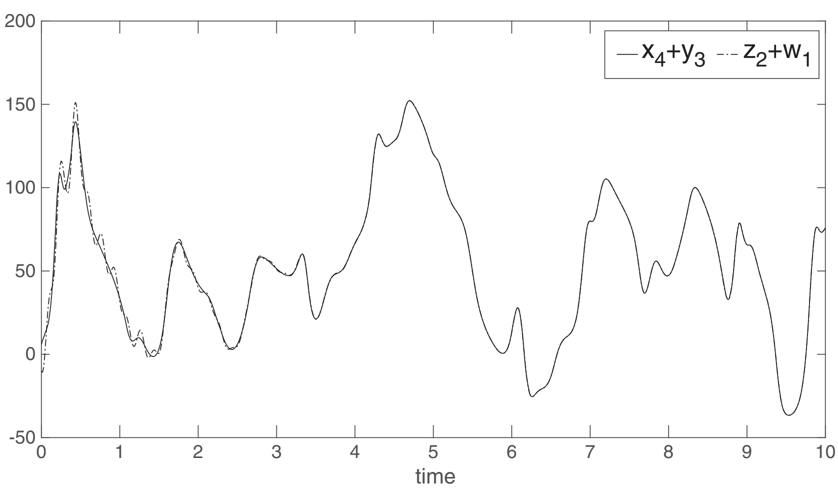

Figure 2. Response for states $x_4 + y_3$ and $z_2 + w_1$ between drive systems (10), (11) and response systems (12), (13).

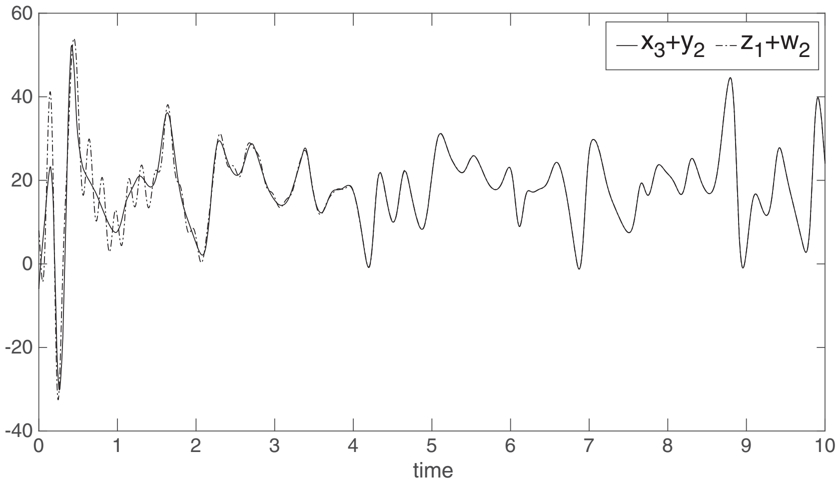

Figure 3. Response for states $x_3 + y_2$ and $z_1 + w_2$ between drive systems (10), (11) and response systems (12), (13).

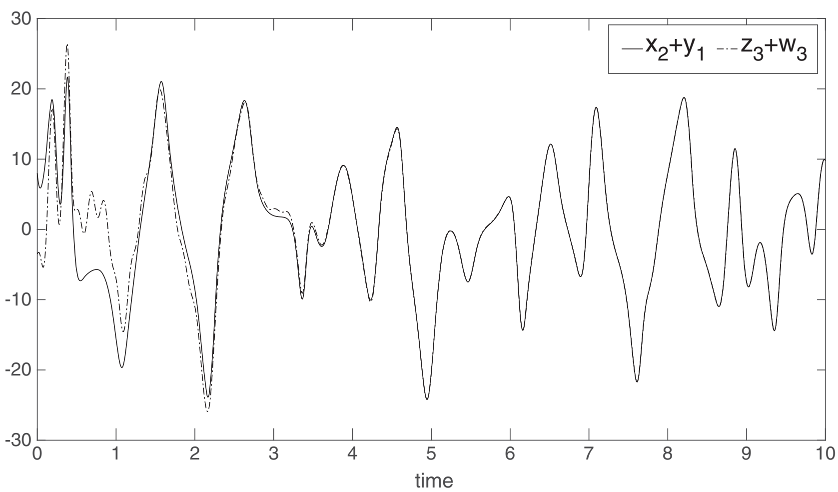

Figure 4. Response for states $x_2 + y_1$ and $z_3 + w_3$ between drive systems (10), (11) and response systems (12), (13).

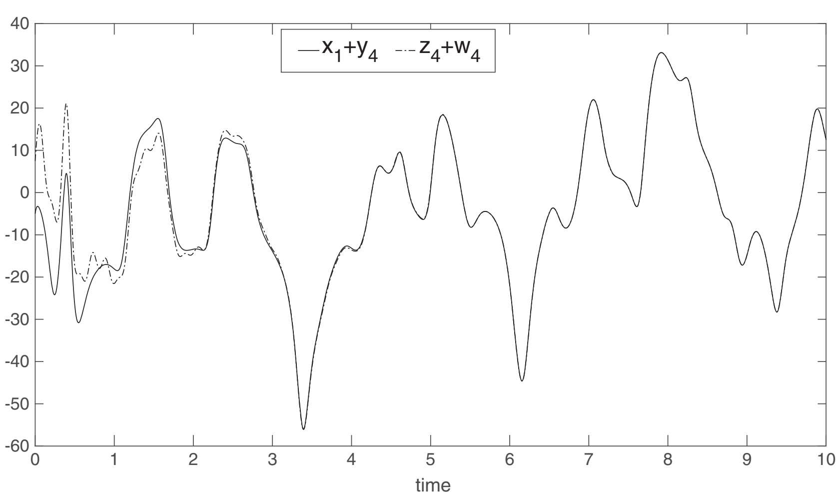

Figure 5. Response for states $x_1 + y_4$ and $z_4 + w_4$ between drive systems (10), (11) and response systems (12), (13).

COROLLARY 1

(i) *If* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, *then the drive systems* (10) *and* (11) *will achieve multiswitching combination synchronization with response system* (12), *provided the control functions are chosen as*

$$\begin{cases} u_1 = \dfrac{1}{\gamma_1}(\alpha_3(x_1x_2 - b_1x_3) + \beta_2(b_2y_1 - ky_1y_3 + y_4) \\ \quad + (\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1) \\ \quad - a_1(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2) \\ \quad + a_3(\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3)) - a_3(z_2 - z_1), \\ u_2 = \dfrac{1}{\gamma_2}(\alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ \quad + (\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2) \\ \quad + a_1(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1) \\ \quad - a_2(\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4)) \\ \quad - (c_3z_1 - z_1z_3 + z_2 - z_4), \\ u_3 = \dfrac{1}{\gamma_3}(\alpha_2(d_1x_1 - x_1x_3 + c_1x_2) + \beta_1a_2(y_2 - y_1) \\ \quad + (\alpha_2x_2 + \beta_1y_1 - \gamma_3z_3) \\ \quad - a_3(\alpha_3x_3 + \beta_2y_2 - \gamma_1z_1)) - (z_1z_2 - b_3z_3), \\ u_4 = \dfrac{1}{\gamma_4}(\alpha_1(a_1(x_2 - x_1) + x_4) + \beta_4(-d_2y_1) \\ \quad + (\alpha_1x_1 + \beta_4y_4 - \gamma_4z_4) \\ \quad + a_2(\alpha_4x_4 + \beta_3y_3 - \gamma_2z_2)) - d_3z_2z_3. \end{cases}$$

(ii) *If* $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, *then the drive systems* (10) *and* (11) *will achieve multiswitching combination synchronization with response system* (13), *provided the control functions are chosen as*

$$\begin{cases} u_1^* = \dfrac{1}{\delta_1}(\alpha_4(x_2x_3 + rx_4) + \beta_3(-c_2y_3 + hy_1^2) \\ \quad + (\alpha_4x_4 + \beta_3y_3 - \delta_1w_1) \\ \quad + a_1(\alpha_3x_3 + \beta_2y_2 - \delta_2w_2) \\ \quad - a_2(\alpha_1x_1 + \beta_4y_4 - \delta_4w_4)) \\ \quad - (a_4(w_2 - w_1) + w_2w_3), \\ u_2^* = \dfrac{1}{\delta_2}(\alpha_3(x_1x_2 - b_1x_3) + \beta_2(b_2y_1 - ky_1y_3 + y_4) \\ \quad + (\alpha_3x_3 + \beta_2y_2 - \delta_2w_2) \\ \quad - a_1(\alpha_4x_4 + \beta_3y_3 - \delta_1w_1) \\ \quad + a_3(\alpha_2x_2 + \beta_1y_1 - \delta_3w_3)) \\ \quad - (b_4(w_1 + w_2) - w_1w_3), \\ u_3^* = \dfrac{1}{\delta_3}(\alpha_2(d_1x_1 - x_1x_3 + c_1x_2) + \beta_1a_2(y_2 - y_1) \\ \quad + (\alpha_2x_2 + \beta_1y_1 - \delta_3w_3) \\ \quad - a_3(\alpha_3x_3 + \beta_2y_2 - \delta_2w_2)) \\ \quad - (-c_4w_3 - fw_4 + w_1w_2), \\ u_4^* = \dfrac{1}{\delta_4}(\alpha_1(a_1(x_2 - x_1) + x_4) + \beta_4(-d_2y_1) \\ \quad + (\alpha_1x_1 + \beta_4y_4 - \delta_4w_4) \\ \quad + a_2(\alpha_4x_4 + \beta_3y_3 - \delta_1w_1)) \\ \quad - (-d_4w_4 + gw_3 + w_1w_2). \end{cases}$$

COROLLARY 2

(i) *If* $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, *and* $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, *then the drive system* (10) *will achieve multiswitching modified projective synchronization with response system* (12), *provided the control functions are chosen as*

$$\begin{cases} u_1 = \alpha_3(x_1x_2 - b_1x_3) - a_3(z_2 - z_1) \\ \quad + (\alpha_3x_3 - z_1) - a_1(\alpha_4x_4 - z_2) \\ \quad + a_3(\alpha_2x_2 - z_3), \\ u_2 = \alpha_4(x_2x_3 + rx_4) - (c_3z_1 - z_1z_3 + z_2 - z_4), \\ \quad + (\alpha_4x_4 - z_2) + a_1(\alpha_3x_3 - z_1) \\ \quad - a_2(\alpha_1x_1 - z_4), \\ u_3 = \alpha_2(d_1x_1 - x_1x_3 + c_1x_2) - (z_1z_2 - b_3z_3) \\ \quad + (\alpha_2x_2 - z_3) - a_3(\alpha_3x_3 - z_1), \\ u_4 = \alpha_1(a_1(x_2 - x_1) + x_4) - d_3z_2z_3 \\ \quad + (\alpha_1x_1 - z_4) + a_2(\alpha_4x_4 - z_2). \end{cases}$$

(ii) *If* $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, *and* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$, *then the drive system* (10) *will achieve multiswitching modified projective synchronization with response system* (13), *provided the control functions are chosen as*

$$\begin{cases} u_1^* = \alpha_4(x_2x_3 + rx_4) - (a_4(w_2 - w_1) + w_2w_3) \\ \quad + (\alpha_4x_4 - w_1) + a_1(\alpha_3x_3 - w_2) \\ \quad - a_2(\alpha_1x_1 - w_4), \\ u_2^* = \alpha_3(x_1x_2 - b_1x_3) - (b_4(w_1 + w_2) - w_1w_3) \\ \quad + (\alpha_3x_3 - w_2) - a_1(\alpha_4x_4 - w_1) \\ \quad + a_3(\alpha_2x_2 - w_3), \\ u_3^* = \alpha_2(d_1x_1 - x_1x_3 + c_1x_2) \\ \quad - (-c_4w_3 - fw_4 + w_1w_2) \\ \quad + (\alpha_2x_2 - w_3) - a_3(\alpha_3x_3 - w_2), \\ u_4^* = \alpha_1(a_1(x_2 - x_1) + x_4) - (-d_4w_4 + gw_3 + w_1w_2) \\ \quad + (\alpha_1x_1 - w_4) + a_2(\alpha_4x_4 - w_1). \end{cases}$$

(iii) *If* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, *and* $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, *then the drive system* (11) *will achieve multiswitching modified projective synchronization with response system* (12), *provided the control functions are chosen as*

$$\begin{cases} u_1 = \beta_2(b_2y_1 - ky_1y_3 + y_4) - a_3(z_2 - z_1) \\ \quad + (\beta_2y_2 - z_1) - a_1(\beta_3y_3 - z_2) \\ \quad + a_3(\beta_1y_1 - z_3), \\ u_2 = \beta_3(-c_2y_3 + hy_1^2) - (c_3z_1 - z_1z_3 + z_2 - z_4), \\ \quad + (\beta_3y_3 - z_2) + a_1(\beta_2y_2 - z_1) \\ \quad - a_2(\beta_4y_4 - z_4), \\ u_3 = \beta_1a_2(y_2 - y_1) - (z_1z_2 - b_3z_3) \\ \quad + (\beta_1y_1 - z_3) - a_3(\beta_2y_2 - z_1), \\ u_4 = \beta_4(-d_2y_1) - d_3z_2z_3 \\ \quad + (\beta_4y_4 - z_4) + a_2(\beta_3y_3 - z_2). \end{cases}$$

(iv) *If* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, *and* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$, *then the drive system* (11) *will achieve multiswitching modified projective synchronization with response system* (13), *provided the control functions are chosen as*

$$\begin{cases} u_1^* = \beta_3(-c_2y_3 + hy_1^2) - (a_4(w_2 - w_1) + w_2w_3) \\ \quad + (\beta_3y_3 - w_1) + a_1(\beta_2y_2 - w_2) \\ \quad - a_2(\beta_4y_4 - w_4), \\ u_2^* = \beta_2(b_2y_1 - ky_1y_3 + y_4) - (b_4(w_1 + w_2) - w_1w_3) \\ \quad + (\beta_2y_2 - w_2) - a_1(\beta_3y_3 - w_1) \\ \quad + a_3(\beta_1y_1 - w_3), \\ u_3^* = \beta_1a_2(y_2 - y_1) - (-c_4w_3 - fw_4 + w_1w_2) \\ \quad + (\beta_1y_1 - w_3) - a_3(\beta_2y_2 - w_2), \\ u_4^* = \beta_4(-d_2y_1) - (-d_4w_4 + gw_3 + w_1w_2) \\ \quad + (\beta_4y_4 - w_4) + a_2(\beta_3y_3 - w_1). \end{cases}$$

COROLLARY 3

(i) *If* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, *and* $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, *then the equilibrium point* $(0, 0, 0, 0)$ *of the response system* (12) *is asymptotically stable*, *provided the control functions are chosen as*

$$\begin{cases} u_1 = -a_3(z_2 - z_1) - z_1 + a_1z_2 - a_3z_3, \\ u_2 = -(c_3z_1 - z_1z_3 + z_2 - z_4) - z_2 - a_1z_1 + a_2z_4, \\ u_3 = -(z_1z_2 - b_3z_3) - z_3 + a_3z_1, \\ u_4 = -d_3z_2z_3 - z_4 - a_2z_2. \end{cases}$$

(ii) *If* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, *and* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$, *then the equilibrium point* $(0, 0, 0, 0)$ *of the response system* (13) *is asymptotically stable*, *provided the control functions are chosen as*

$$\begin{cases} u_1^* = -(a_4(w_2 - w_1) + w_2w_3) - w_1 - a_1w_2 + a_2w_4, \\ u_2^* = -(b_4(w_1 + w_2) - w_1w_3) - w_2 + a_1w_1 - a_3w_3, \\ u_3^* = -(-c_4w_3 - fw_4 + w_1w_2) - w_3 + a_3w_2, \\ u_4^* = -(-d_4w_4 + gw_3 + w_1w_2) - w_4 - a_2w_1. \end{cases}$$

4. Conclusions

In this paper, we propose MSCCS among four nonidentical hyperchaotic systems. The presented scheme extends the concept of multiswitching combination synchronization to the combination of two drive and two response systems. Suitable controllers are designed to realize combination–combination synchronization in multiswitching manner. The simulation results verify that the proposed controllers work effectively for synchronizing the combination of drive systems and combination of two response systems. Apart from strengthening the security of transmitted signal in chaos communication, the proposed scheme may have potential advantage in completing intelligent synchronization. Further work may be carried out to find more applications for the presented scheme.

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