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Hulthén potential models for $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ elastic scattering

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Abstract. Simple Hulthén-type potential models are proposed to treat the $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ elastic scattering. The merit of our approach is examined by computing elastic scattering phases through the judicious use of the phase function method. Reasonable agreements in scattering phase shifts are obtained with the standard data.

Keywords. Nuclear Hulthén potential; phase function method; scattering phase shifts; $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ systems.

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1. Introduction

The $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ elastic scattering have been studied quite extensively by a number of groups [1–12]. These studies provide lots of reliable data. In most of these models, the effective interaction, considered between each pair of particles, is a finite-depth central potential. As the α-particles are tightly bound, the low-lying states of such systems can be determined fairly well through the relative motion of α-particles. The general procedure to understand such a system is to make use of the partial wave analysis of the phase shifts δ_ℓ for a given partial wave with angular momentum ℓ. As the spin of the α particle is zero and it possesses high internal binding energy, the phase shifts can easily be reduced to a minimum and one can analyse the scattering in terms of real phase shifts up to a laboratory bombarding energy of 25 MeV. The $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ interactions are combinations of the Coulomb potential V_C and some short-range interaction V_S. The short-range interaction is of nuclear origin while the Coulomb potential takes care of the charges. Relatively recently, we [13,14] have studied the nucleon–nucleon and alpha–nucleon systems through different two-particle interactions and obtained good results in scattering phase shifts. Inspired by this, we try to extend our study for $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ systems within the framework of nuclear Hulthén-type interaction model. The phase function method (PFM) [15] is adapted to examine the merit of our proposed interactions in computing the scattering phase shifts for the systems under consideration. In §2 we propose our interaction and briefly outline the phase function method for dealing with local potentials. Section 3 is devoted for results and discussions and finally in §4 we give some concluding remarks.

2. Proposed interactions and the phase function method

Darriulat *et al* [16] attempted to fit the real part of the $\alpha-\alpha$ scattering phase shifts by introducing a complex Woods–Saxon potential of the form

$$V_{\alpha\alpha}(r) = \frac{u_1}{1+\mathrm{e}^{(r-r_1)/a_1}} - \frac{u_2}{1+\mathrm{e}^{(r-r_2)/a_2}} - \frac{i\,W}{1+\mathrm{e}^{(r-r_3)/a_3}} + V_\mathrm{C}(r), \quad (1)$$

where the first term describes the repulsive core, the second one is the long-ranged attractive part, the third term accounts for the inelastic processes and the last term represents the electromagnetic interaction. They have used different sets of parameters for each partial wave.

Arnold and MacKellar [17] parametrized the nuclear Hulthén potential to fit the scattering length and the deuteron binding energy. Thus, Hulthén potential describes the interaction between two nucleons in

deuteron quite effectively. Recently, we have applied the nuclear Hulthén potential to describe the alpha–nucleon systems [13] within the framework of supersymmetric quantum mechanics [18–20] and the PFM [15]. The present text addresses itself to the study of $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ elastic scattering by considering the two-parameter simple potential model of Hulthén type. Rahaman *et al* [21] have used nonlocal potentials to fit the $\alpha-\alpha$ scattering phases for the partial waves $\ell = 0, 2, 4$. This nonlocal potential is of rank two (with four parameters) for S-wave with an attractive and a repulsive component whereas for D- and G-waves, it is of rank one (two parameters) and attractive. In analogy with the nonlocal potential model [21] we also argue for our local potential model as follows.

As the S-wave $\alpha-\alpha$ scattering phase shift changes its sign at about 20 MeV, a two-term potential is indispensible to account for the change in sign in the scattering phases. However, the D- and G-wave interactions may be represented by one-term potential because they produce positive phase shifts only. In such cases, the repulsive cores remain absent in the related potentials. Similarly, for the $\alpha-\mathrm{He}^3$ system, the entire S-wave phase shifts are negative and those for the P-wave states are positive. Therefore, a one-term nuclear potential may be of considerable interest for the $\alpha-\mathrm{He}^3$ system. In this context, we propose the following two-parameter interaction model for the systems under consideration:

$$V_1(r) = V_A(r) - (\beta^2 - \rho^2)\frac{\mathrm{e}^{-\beta r}}{(\mathrm{e}^{-\rho r} - \mathrm{e}^{-\beta r})} + C(\beta - \rho)^2 \frac{\mathrm{e}^{-(\beta+\rho)r}}{(\mathrm{e}^{-\rho r} - \mathrm{e}^{-\beta r})^2}. \quad (2)$$

For both the $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ systems we consider two sets of interactions, first one with $C = 1$, i.e. the potentials are of two-term and the second one, with $C = 0$ for $\ell = 2, 4$ (the D- and G-wave interactions are of one-term only) for $\alpha-\alpha$ system and $C = 0$ for $\ell = 0, 1$ for the $\alpha-\mathrm{He}^3$ system.

In reality, the Coulomb potential does not exist in nature and becomes somewhat screened at a certain distance. In contrast to the behaviour of the wave function for a well-behaved potential, that in the presence of the slowly decreasing Coulomb field contains logarithmically oscillating phase contributions. Thus, the traditional approach to the PFM for the local potential does not hold good for the pure Coulomb interaction and it needs separate treatment. The Hulthén potential at small values of r behaves like a Coulomb potential, whereas for large values of r it decreases exponentially so that its capacity for bound states is smaller than that of the Coulomb potential. Thus, to compute scattering phase shifts using PFM, the Coulomb potential is replaced by a screened Coulomb one $V_A(r)$, the atomic Hulthén potential, which is read as

$$V_A(r) = V_0 \frac{\mathrm{e}^{-r/a}}{1 - \mathrm{e}^{-r/a}}, \quad (3)$$

where V_0 is the strength and a is the screening radius. The merits of our proposed interactions will be examined by computing scattering phase shifts through the judicious use of the PFM [15].

Phase function method is an efficient approach for computing the scattering phase shifts for quantum mechanical problems involving local [15] and nonlocal interactions [22,23]. For a local potential, the phase function $\delta_\ell(k, r)$ satisfies the first-order non-linear differential equation which is expressed as

$$\delta'_\ell(k, r) = -k^{-1} V(r) [\hat{j}_\ell(kr) \cos \delta_\ell(k, r) - \hat{\eta}_\ell(kr) \sin \delta_\ell(k, r)]^2, \quad (4)$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati–Bessel functions with

$$\hat{h}^{(1)}_\ell(x) = -\hat{\eta}_\ell(x) + i\, \hat{j}_\ell(x).$$

The scattering phase shift $\delta_\ell(k)$ is obtained by solving the equation from the origin to the asymptotic region with the initial condition $\delta_\ell(k, 0) = 0$.

3. Results and discussions

Using the parameters in table 1, we have portrayed the potentials for various partial wave states under consideration for the $\alpha-\alpha$ and $\alpha-\mathrm{He}^3$ systems in figures 1 and 2 respectively. Here we have chosen to work with [13,24] $\hbar^2/2m = 10.3675\,\mathrm{MeV\,fm^2}$, $V_0 a = 0.2758\,\mathrm{fm}^{-1}$ for $\alpha-\alpha$; $\hbar^2/2m = 24.1908\ \mathrm{MeV\,fm^2}$, $V_0 a = 0.23639\ \mathrm{fm}^{-1}$ for $\alpha-\mathrm{He}^3$ systems and $a = 50\,\mathrm{fm}$. In general, the $\alpha-\alpha$ potentials constructed from the experimental phase shifts show some common features: the $\alpha-\alpha$ potential is ℓ-dependent but is independent of the incident energy. The ranges of the inner repulsive part and the outer attractive part are of the order of 2 and 5 fm respectively. The phenomenological analysis of $\alpha-\alpha$ scattering establishes beyond doubt that a static $\alpha-\alpha$ potential common to all ℓ does not exist. Looking at figure 1 it is noticed that the S-wave potential has a repulsive core followed by a strong attractive part. The depth of the attractive part is of the order of 51 MeV. For the two-term interaction our D- and G-wave potentials follow the same pattern with increasing depth as ℓ increases. In

Table 1. Parameters for the $\alpha-\alpha$ and $\alpha-He^3$ interactions.

Systems→	$\alpha-\alpha$ system					$\alpha-He^3$ system					
						$C=1$			$C=0$		
States→		$\ell=2$		$\ell=4$							
Parameters ↓	$\ell=0$	$C=1$	$C=0$	$C=1$	$C=0$	$\left(\frac{1}{2}\right)^+$	$\left(\frac{1}{2}\right)^-$	$\left(\frac{3}{2}\right)^-$	$\left(\frac{1}{2}\right)^+$	$\left(\frac{1}{2}\right)^-$	$\left(\frac{3}{2}\right)^-$
β (fm^{-1})	2.75	2.92	2.802	2.55	2.55	0.09	1.506	1.482	−0.4	1.626	1.602
ρ (fm^{-1})	2.394	2.515	2.37	2.453	2.445	−0.6	1.231	1.246	−1.15	1.231	1.246

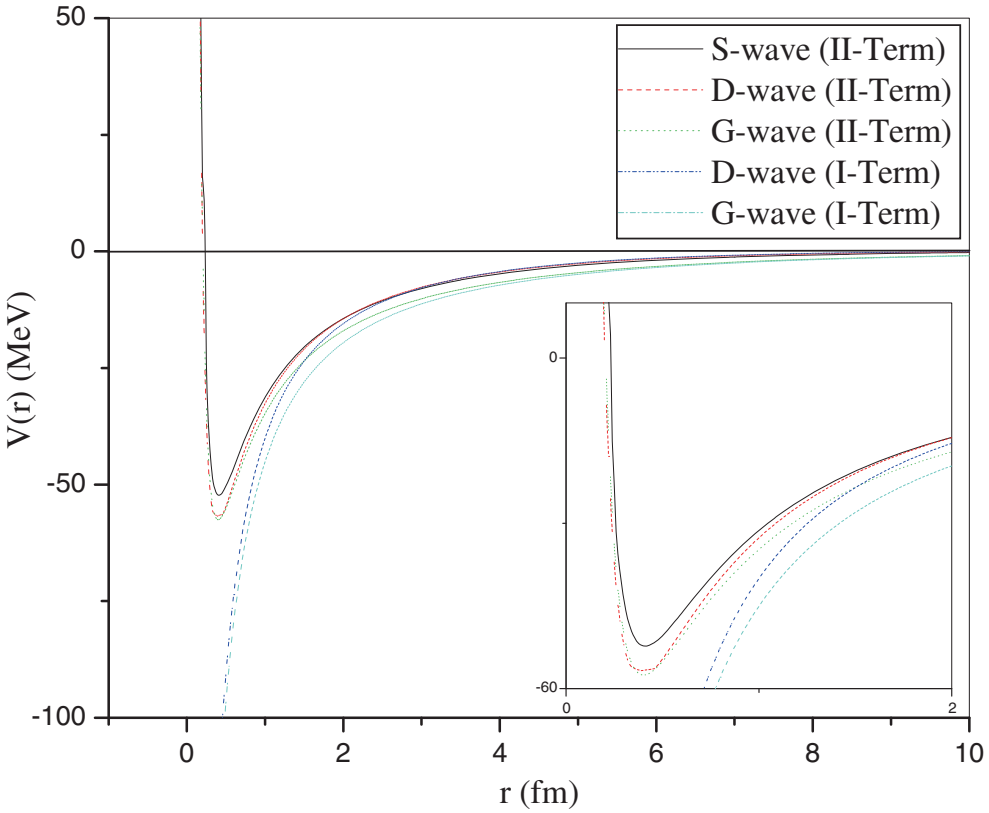

Figure 1. α–α potentials for S-, D- and G-waves.

general, with the decrease of the ℓ-value, the size of the repulsive core of the nuclear part becomes larger ensuring less and less overlap of the particles. The repulsive core associated with our S-wave potential is larger than D- and G-wave interactions but it is difficult to visualize the difference between the repulsive cores of D- and G-waves in the scale of the figure. Thus, our two-term potentials are fully consistent with the above observation [1].

For one-term potentials, the G-wave interaction is stronger than its D-wave counterpart and both possess attractive parts only. Looking closely at figure 2, it is observed that both the one- and two-term interactions for the $1/2^+$ state of the $\alpha-He^3$ system are purely repulsive in nature. However, for $1/2^-$ and $3/2^-$ states the one-term interactions are purely attractive while for two-term potential both these states posses repulsive cores followed by an attractive part. The depth of the attractive part of $3/2^-$ state is slightly greater than that of the $1/2^-$ state for both one- and two-term interactions under consideration because the $3/2^-$ state of the $\alpha-He^3$ system supports a bound state.

The phase shifts for the $\alpha-\alpha$ system are portrayed in figure 3 for the partial waves $\ell=0,\ 2,\ 4$ (two-term potential), in figure 4 for $\ell=2,\ 4$ (one-term potential) and the same for $\alpha-He^3$ system for different partial wave states under consideration in figure 5. As expected, the S-wave phase shifts as shown in figure 3 for the $\alpha-\alpha$ system are positive at low energies and become negative at high energies. The S-wave phase

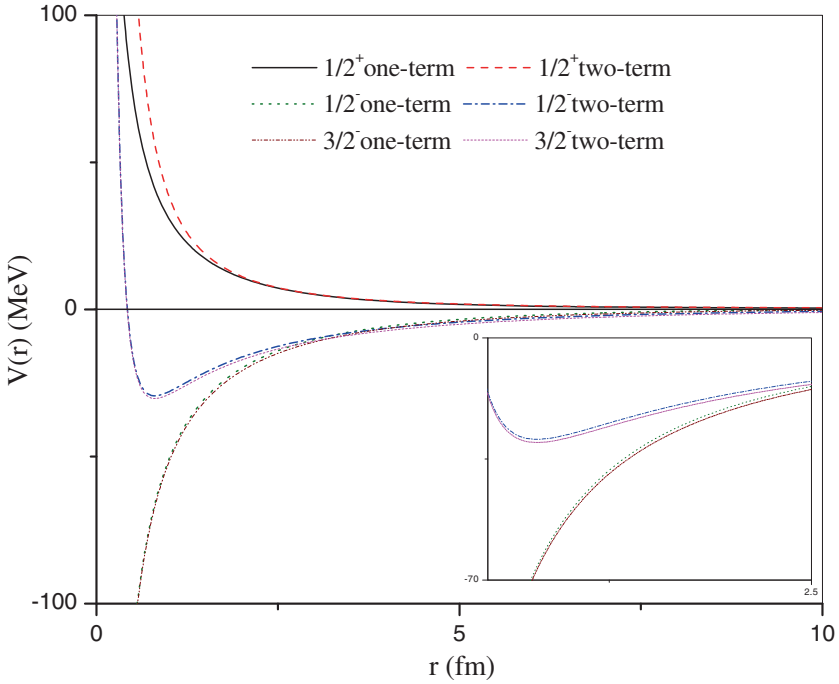

Figure 2. $\alpha-He^3$ potentials (one- and two-terms).

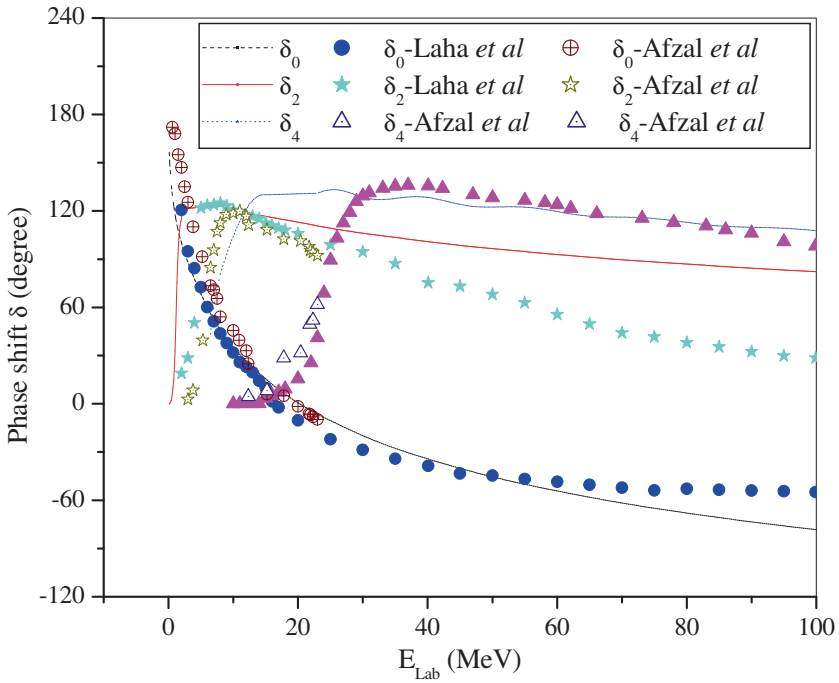

Figure 3. $\alpha-\alpha$ phase shifts for S-, D- and G-waves (two-term potential).

shift changes its sign at $E_{\text{Lab}} = 20$ MeV which is in close agreement with the works of Tombrello and Senhouse [25], Ali and Bodmer [1] and Afzal *et al* [26] but slightly deviates from the observation of Laha *et al* [24]. Except near the transition point, our S-wave phase shifts agree quite well with those of ref. [24] up to 60 MeV. However, beyond 60 MeV our phase shift gradually increases up to 100 MeV. Our D-wave phase shifts, in figures 3 and 4, reach peak values of about 123° and 125° at $E_{\text{Lab}} = 7$ MeV and $E_{\text{Lab}} = 8$ MeV for one- and two-term interactions respectively which are in close agreement with the standard results [1,24] at about 8 MeV. But our phase shift values gradually increase compared to those of refs [24–26] beyond

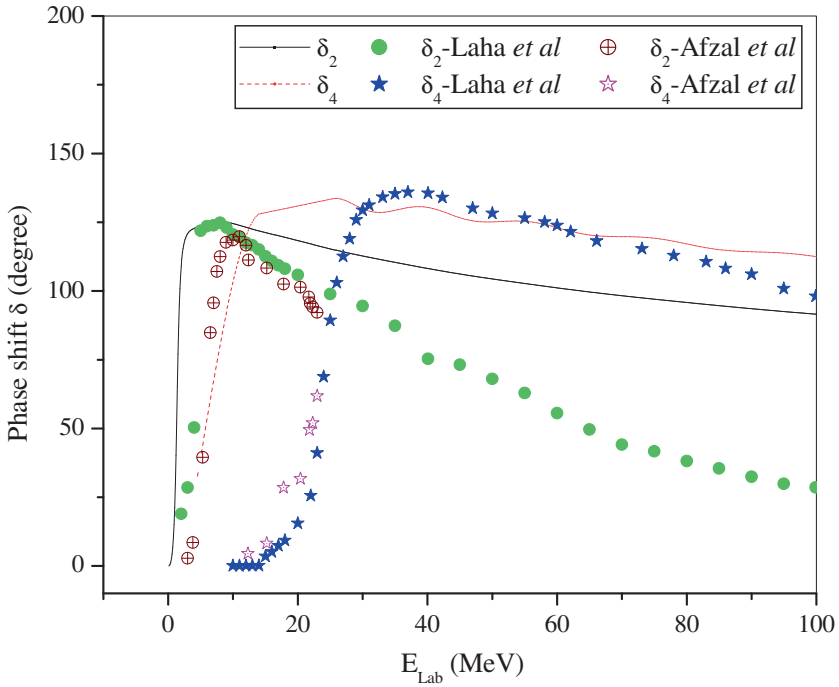

Figure 4. $\alpha-\alpha$ phase shifts for D- and G-waves (one-term potential).

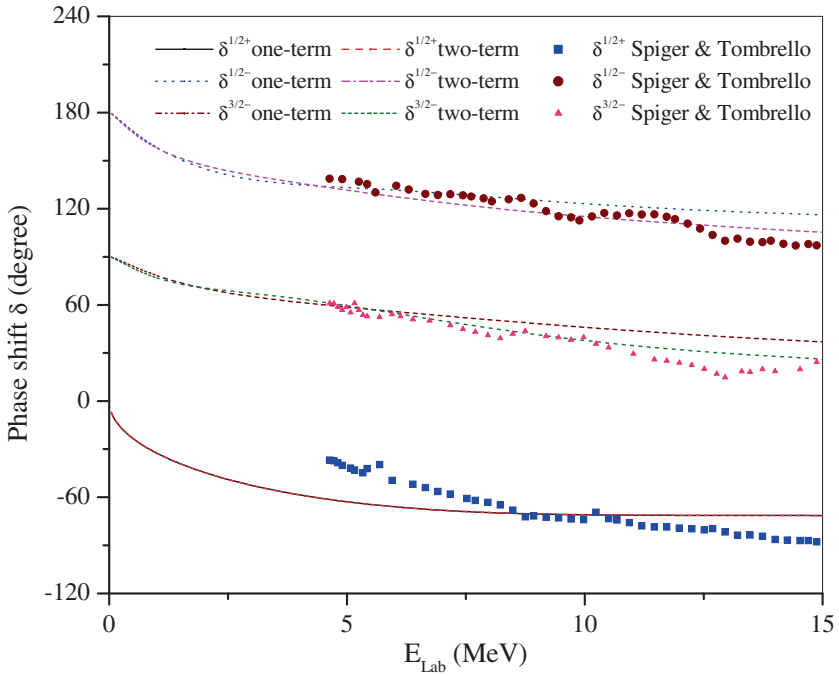

Figure 5. α–He^3 phase shifts for one- and two-term potentials. The phase shifts $\delta^{3/2-}$ are shifted by 90° for clarity of presentation.

20 MeV and 15 MeV for two- and one-term potentials respectively. Thus, one may conclude that the strength of the repulsive core in the D-wave interaction for two-term potential is weaker than the actual ones. Our G-wave phase shifts, although produce the correct nature, are in numerical disagreement with standard data [1,24,26] in the low-energy range but are in reasonable agreement with the results of ref. [24] in the high-energy range. In particular, the two-term potential produces better G-wave results than one-term potential. Furthermore, the G-wave phase shifts do not vary smoothly beyond 15 MeV. This may be attributed to the fact that the inelastic processes start in this region and hence the use of two-body potential model is doubtful.

Thus, for the G-wave case, one should restrict oneself to low laboratory energies as mentioned by Ali and Bodmer [1]. Comparing figures 3 and 4, it is also concluded that our two-term interaction is superior to its one-term counterpart. This leads to the fact that the repulsive cores are indispensible in the D- and G-wave potentials.

For the $\alpha-He^3$ system, it is noticed that the phase shifts computed for different states are in close agreement with the work of Spiger and Tombrello [27]. The phase shifts $\delta^{1/2+}$ for both the one- and two-term interactions agree quite well with ref. [27] in the energy range 5 MeV to 13 MeV while slight disagreements are observed in the very low-energy range and beyond 13 MeV. However, the same for the $1/2^-$ and $3/2^-$ states are in close agreement with the standard result [27] over the entire energy range. Also it is noticed that for the $1/2^-$ and $3/2^-$ states, the two-term potential is more effective than its one-term counterpart.

4. Conclusion

It is well known that a number of phase-equivalent potentials can be constructed with a certain number of free parameters in it to fit the scattering phase shifts. Thus, knowing the general feature of the interactions, various phenomenological potentials can be constructed which are of interest in the cluster model of light nuclei. The parametrized nuclear Hulthén potentials [13,14,17], both one- and two-term, have the ability to describe the interaction between two nucleons and the nucleus–nucleon systems quite effectively. Various $\alpha-\alpha$ potentials have been proposed earlier with the superposition of repulsive and attractive square well or Gaussian shapes. Apart from square well and Gaussian shapes, we have used here two-parameter Hulthén-type nuclear potentials, both one- and two-term, and noticed significant deviation in D-wave phase shift data beyond 20 MeV while better agreements are achieved for the partial waves $\ell = 0,\ 4$ with the standard results for the $\alpha-\alpha$ system. Also reasonable agreements in the phase shift values with the earlier calculations are obtained for the $\alpha-He^3$ system. The differences in the phase shift values in certain energy ranges for various partial wave states for the $\alpha-\alpha$ system may arise due to parametrization of the two-parameter nuclear Hulthén potential or due to the involvement of inelastic processes beyond 25 MeV. Thus, for studying $\alpha-\alpha$ and $\alpha-He^3$ elastic scattering one should restrict oneself within 25 MeV. In this low-energy range, our potential model reproduces reliable data. Within the framework of the resonating group model, although the numerical complications are considerably higher than the present method, one gets better agreement in the phase shifts with experimental results. But it seems rather difficult to reproduce the numbers of the RGM calculation by a simple minded potential as used by us. It is well known that the fundamental studies of the $\alpha-\alpha$ and $\alpha-He^3$ interactions provide a useful basis for understanding the interaction between complex nuclei. Thus, our potential model may be of considerable interest in treating the low-energy complex nucleus–nucleus scattering. It is also possible to have potentials in terms of a series of Hulthén forms with two or more parameters or other types of interactions that might produce better results, but this point needs further investigation and goes beyond the scope of this paper.

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