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Thermodynamic quantities for the Klein–Gordon equation with a linear plus inverse-linear potential: Biconfluent Heun functions

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Abstract.   We study some thermodynamic quantities for the Klein–Gordon equation with a linear plus inverse-linear, scalar potential. We obtain the energy eigenvalues with the help of the quantization rule from the biconfluent Heun's equation. We use a method based on the Euler–MacLaurin formula to analytically compute the thermal functions by considering only the contribution of positive part of the spectrum to the partition function.

1. Introduction

If the one-dimensional linear potential having a form proportional to $|x|$ is considered as the time-like component of a Lorentz vector, then this potential will be related to the Coulomb potential ([1] and references therein). As per quantum theory, the linear potential is also a basic ground for confinement of the particles having an odd half-integer spin. If we consider the linear potential as a Lorentz scalar, then it becomes important for the structure of quarkonium. So, the one-dimensional linear potential has received lot of attention. The solutions of the Dirac equation and the non-relativistic limit for a linear scalar potential have been studied in ref. [2]. The bound state solutions of the Dirac equation have been analysed for the one-dimensional linear potential with Lorentz scalar and vector couplings [1]. Some other relativistic equations such as the Duffin–Kemmer–Petiau [3] and the Klein–Gordon (KG) equation [4] have also been studied for the linear potential. For the non-relativistic case, namely the Schrödinger equation,

it is well known that the analytical solutions are obtained in terms of Airy functions [4].

If the potential is written as inversely linear ($\sim |x|^{-1}$) it denotes another interesting interaction because this potential represents hydrogen atom in one-dimensional space [5]. The non-relativistic results for this potential show that the ground-state solution has an infinite energy with an eigenfunction written in terms of delta function near the origin [5]. Analysing this potential for the Klein–Gordon equation with the help of continuous dimensionality technique [6] gives unacceptable solutions. As a result, it could be interesting to solve the Klein–Gordon equation for the combination of the above potentials as

$$V(x) = a_1 + a_2|x| + \frac{a_3}{|x|},$$

to find statistical quantities for the whole system.

The study of thermodynamics quantities for quantum systems in different potentials has been received special attention in the last few decades. In ref. [7], the one-dimensional Dirac-oscillator has been analysed in

a thermal bath, and then the three-dimensional case has been computed in ref. [8]. The Dirac/Klein–Gordon oscillators have been analysed in thermodynamics point of view by using a different method in ref. [9]. The Dirac equation on graphene has been solved to study the thermal functions in ref. [10]. The non-commutative effects on thermodynamic quantities have also been discussed on graphene in [11,12]. The spin-one DKP oscillator has been analysed for the statistical functions by taking into account the non-commutative effects with an external magnetic field [13]. In ref. [14], the thermodynamic properties of a harmonic oscillator plus an inverse square potential have been studied within the non-relativistic region.

The paper is organized as follows. In §2, we study the bound-state solutions of the Klein–Gordon equation for the above potential with the help of the quantization condition giving the biconfluent Heun's equation. We shall see that the results reveal an energy eigenvalue equation which is independent of the potential parameter a_1. In §3, the partition function, $Z(\beta)$, is computed by using the Euler–MacLaurin formula in terms of a dimensionless parameter $\bar{m}$ by restricting ourselves to the case where only the particle–particle interactions appear. For this case, the partition function does not involve a sum over the negative energy states [15]. We search then for the other thermal quantities such as the free energy, the mean energy, and the specific heat numerically. In §4, conclusions are given.

2. The bound states

The time-independent one-dimensional Klein–Gordon equation with scalar, $V_S(x)$, and vector, $V_V(x)$, potentials reads as [16]

$$\left\{-\frac{d^2}{dx^2}+Q^2[mc^2+V_S(x)]^2-Q^2[V_V(x)-E]^2\right\}\psi(x)=0,$$

(1)

where $Q=1/\hbar c$, c is the speed of light, m is the rest mass, and E is the energy. Here, the vector potential $V_V(x)=0$, and the scalar part is given as above. So, we have

$$\frac{d^2\psi(x)}{dx^2}-Q^2\left[(mc^2+a_1)^2+2a_2a_3+2a_3(mc^2+a_1)\frac{1}{|x|}\right.$$
$$\left.+\frac{a_3^2}{x^2}+2a_2(mc^2+a_1)|x|-a_2^2x^2\right]\psi(x)$$
$$=-Q^2E^2\psi(x).$$

(2)

By defining a new variable $y=\sqrt{Qa_2}\,|x|$, and using the abbreviations

$$\varepsilon_1=\frac{Q}{a_2}[E^2-(mc^2+a_1)^2-2a_2a_3],$$

$$A_1=-2Qa_3(mc^2+a_1)\sqrt{\frac{Q}{a_2}},$$

$$A_2=-Q^2a_3^2,\quad A_3=-2\sqrt{\frac{Q}{a_2}}(mc^2+a_1),$$

(3)

we have

$$\frac{d^2\psi(y)}{dy^2}+\left(\varepsilon_1+\frac{A_1}{y}+\frac{A_2}{y^2}+A_3y-y^2\right)\psi(y)=0.$$

(4)

In order to get a more suitable form for eq. (4) we write the wave function as

$$\psi(y)=|y|^P e^{-qy^2-ry}\phi(y),$$

(5)

with

$$p=\frac{1}{2}+\frac{1}{2}\sqrt{1-4A_2}.$$

(6)

Now substituting eq. (5) into eq. (4), the resulting equation reads as

$$y\phi''(y)+(2p-2ry-4qy^2)\phi'(y)$$
$$+\left[(-4pq+r^2+\varepsilon_1)y-(2pr-a_1)\right]\phi(y)=0.$$

(7)

This equation is the biconfluent Heun's differential equation having a general form [17]

$$\xi u''(\xi)+(1+c_1-c_2\xi-2\xi^2)u'(\xi)$$
$$+\left\{(c_3-c_1-2)\xi-\frac{1}{2}[c_4+c_2(1+c_1)]\right\}u(\xi)=0,$$

(8)

with solutions the so-called biconfluent Heun functions, HB

$$\phi(y)\sim HB\left(\sqrt{1-4A_2},2Q\sqrt{a_2}(mc^2+a_1),\right.$$

$$\left.1+\gamma^2+\varepsilon_1,\frac{4Q^2a_3}{\sqrt{Qa_2}}(mc^2+a_1),y\right).$$

(9)

The biconfluent Heun's equation has many applications in different fields to find the quantization condition and the wave functions for the system under consideration [18–21]. The general solution of this equation can be computed by using the Frobenius

methods, and the biconfluent Heun series results in a polynomial form of degree n when [18]

$$\varepsilon_1 + \frac{1}{4}A_3^2 - 2p = 2n, \qquad (10)$$

with $n = 0, 1, 2, \ldots$. By using eq. (3), we obtain the bound states of the system

$$E_n^2 = 2a_2a_3 + \frac{a_2}{Q}(2n + 1 + \sqrt{1 + 4Q^2a_3^2}), \qquad (11)$$

with the eigenfunctions

$$\psi(y) \sim |y|^{\frac{1}{2} + \frac{1}{2}\sqrt{1-4A_2}} e^{\frac{1}{2}(A_3y - y^2)}$$

$$\times HB\left(\sqrt{1 - 4A_2}, 2Q\sqrt{a_2}(mc^2 + a_1),\right.$$

$$\left.1 + \gamma^2 + \varepsilon_1, \frac{4Q^2a_3}{\sqrt{Qa_2}}(mc^2 + a_1), y\right). \qquad (12)$$

In figure 1, we present plots of some eigenfunctions for different quantum numbers. In addition, last two equations make it possible to handle the single-particle level density defined basically as the number of energy levels in the energy interval dE [22], that is,

$$\rho(E) = \frac{dE}{dn}, \qquad (13)$$

which gives, for the system under consideration,

$$\rho(E) = \frac{Q}{a_2}\sqrt{E}, \qquad (14)$$

where it is clearly seen that the level density depends on the strength of linear and inverse-linear parts of the potential.

In order to have an equation with the same dimensions in the left- and right-hand sides in (11), let us denote the quantity a_2a_3 as ε^2 in the rest of the computation which makes it possible to write eq. (11) more clearly as
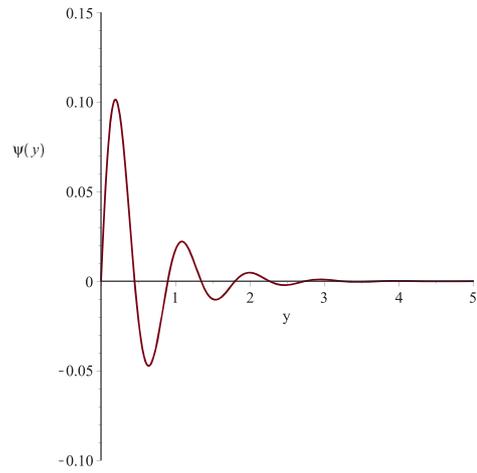
$$E_n = \mp\varepsilon\sqrt{2 + q^{-1}(2n + 1 + \sqrt{1 + 4q^2})} \qquad (15)$$

where the dimensionless parameter $q = Qa_3$. In the next section, we compute the thermal functions in terms of a dimensionless parameter $\bar{m}$ written with the help of ε.
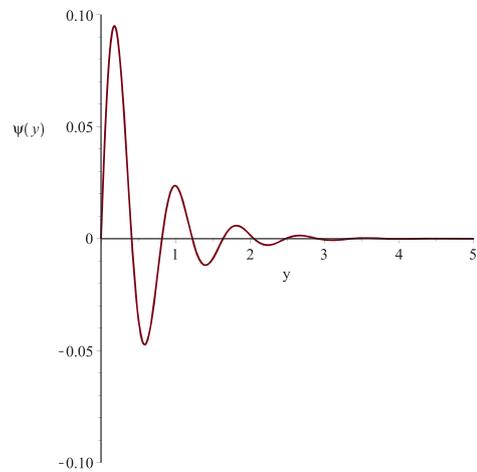
3. The thermodynamic quantities

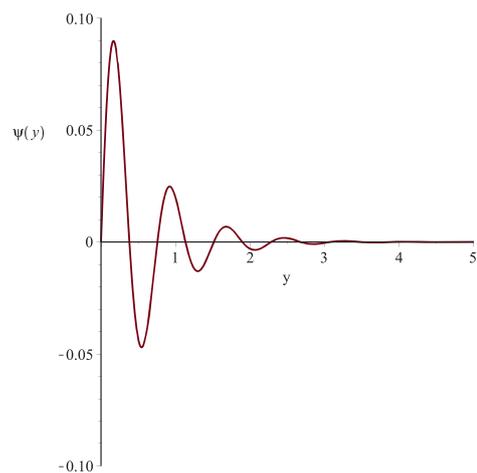The partition function given as a summation over all the quantum states can be written as [7]

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-(E_n - E_0)\beta} = e^{\beta E_0}\sum_{n=0}^{\infty} e^{-\beta\varepsilon\sqrt{\sigma_1 n + \sigma_2}}, \qquad (16)$$

Figure 1. Some eigenfunctions given in (12) when $a_1 = a_2 = a_3 = 0.1$, $m = 0.5$ ($\hbar = c = 1$). (**a**) The eigenfunction for $n = 0$, (**b**) the eigenfunction for $n = 5$ and (**c**) the eigenfunction for $n = 10$.

where $\beta = 1/k_{\mathrm{B}}T$, k_{B} being the Boltzmann constant, T is the temperature in Kelvin with the constants $\sigma_1 = 2/q$ and $\sigma_2 = 2 + (1/q)(1 + \sqrt{1 + 4q^2})$. We tend to compute the following thermal quantities such as the free energy, the mean energy, and the specific heat written in terms of the partition function:

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta),$$

$$U(\beta) = -\frac{\partial}{\partial \beta} \ln Z(\beta),$$

$$C(\beta) = -k_{\mathrm{B}}\beta^2 \frac{\partial}{\partial \beta} U(\beta). \tag{17}$$

The following integral equation [7,8]

$$\int_0^\infty e^{-\beta_1 \sqrt{\beta_2 n + \beta_3}}\, \mathrm{d}n = \frac{2}{\beta_1^2 \beta_2} e^{-\beta_1 \sqrt{\beta_3}} (1 + \beta_1 \sqrt{\beta_3}), \tag{18}$$

shows that the partition function in eq. (16) is convergent. The result in eq. (18) makes it possible to compute the partition function with the help of the Euler–MacLaurin formula

$$\sum_{n=0}^\infty f(n) = \frac{1}{2} f(0) + \int_0^\infty f(x)\mathrm{d}x$$
$$- \sum_{i=1}^\infty \frac{1}{(2i)!} B_{2i} f^{(2i-1)}(0), \tag{19}$$

where B_{2i} are the Benoulli numbers, $B_2 = 1/6$, $B_4 = -1/30, \ldots$ [7,8]. Up to $i = 2$, eq. (16) with the help of

(13) gives the partition function of the system written in a dimensionless parameter $\beta\varepsilon = 1/\bar{m}$ as

$$Z(\bar{m}) = \frac{1}{2} + \frac{2\bar{m}^2}{\sigma_1}\left(1 + \frac{\sqrt{\sigma_2}}{\bar{m}}\right) + \frac{\sigma_1}{24\bar{m}\sqrt{\sigma_2}}$$
$$- \frac{\sigma_1^3}{5760\bar{m}\sigma_2^{5/2}}\left(3 + 3\frac{\sqrt{\sigma_2}}{\bar{m}} + \frac{\sigma_2}{\bar{m}^2}\right). \tag{20}$$

We observe that the thermodynamic quantities in eq. (17) depend on the parameter q including the potential parameter. So, we give our numerical results as the variation of thermodynamic quantities vs. temperature for three different values of q, namely, $q = 0.5$, $q = 1.0$ and $q = 1.5$, in figures 2–4. Figure 2 shows that the Helmholtz free energy increases with increasing values of q. In figure 3, we see that the effect of q on the mean energy is more apparent for nearly low temperatures. On the other hand, the plots for different q-values for the mean energy are close to each other. We give the variation of specific heat according to the temperature in figure 4 where it has an upper value when the temperature increases.

Now we give results briefly for the thermal functions for high temperatures which correspond to $\beta \ll 1$. For this case, eq. (20) gives the results

$$Z(\bar{m}) \sim \frac{2\bar{m}^2}{\sigma_1} \sim \bar{m}^2 Q a_3,$$
$$U(\bar{m}) \sim 2\bar{m},$$
$$C(\bar{m}) \sim 2, \tag{21}$$

where the upper limit for the specific heat can be seen clearly in figure 4.
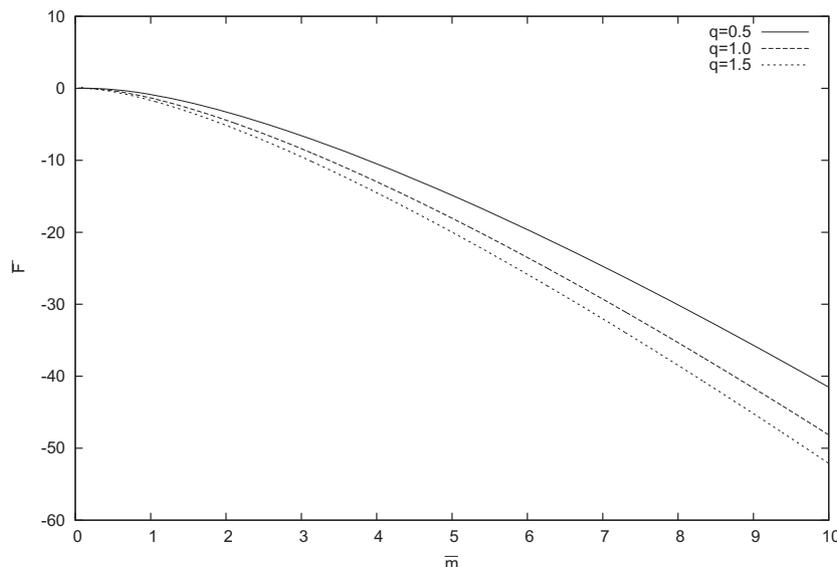

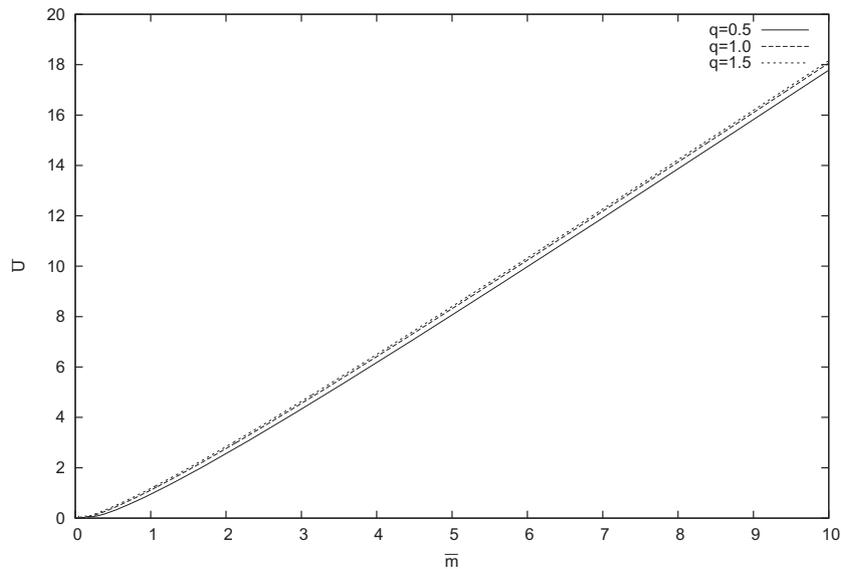**Figure 2.** The variation of free energy for the present potential vs. $\bar{m}$.

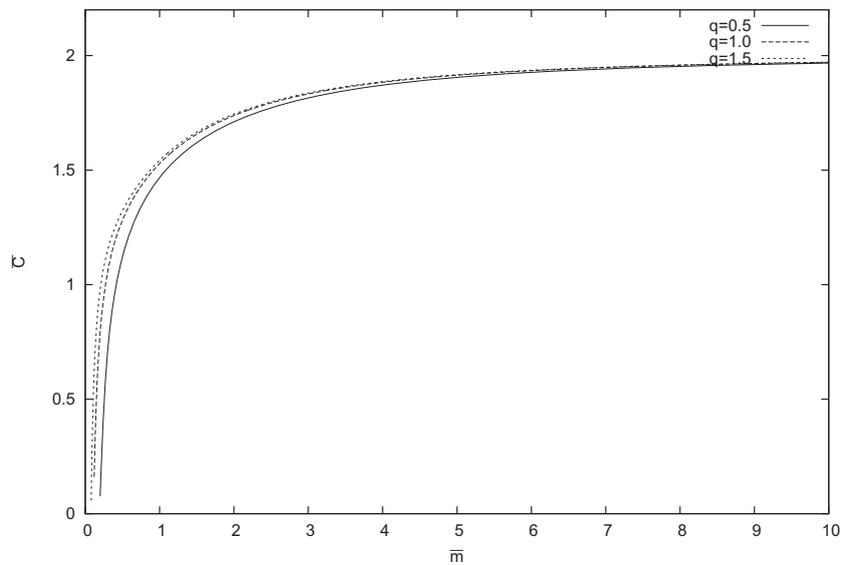**Figure 3.** The variation of mean energy for the present potential vs. $\bar{m}$.

Figure 4. The variation of specific heat for the present potential vs. $\bar{m}$.

Studying the partition function in eq. (20) according to the potential parameters shows that the descent contribution, which is inverse-linear, comes from the part of the potential proportional to $|x|$. The other part of the potential proportional to $1/|x|$ gives a weaker contribution, which is linear in some terms and inverse-linear in others. On the other hand, eq. (21) shows that only the potential parameter a_2 gives an inverse-linear contribution to partition function for low temperature while both parameters a_2 and a_3 give an inverse-squared contribution to the mean energy for high temperature.

4. Conclusions

We have obtained the thermodynamic quantities for the Klein–Gordon equation with a linear plus inverse-linear potential by using the quantization condition appeared in biconfluent Heun's equation. The variation of a few eigenfunctions vs. y has been given in a figure, and the single-particle level density analysed briefly. The thermodynamic quantities such as the free energy, the mean energy, and the specific heat have been computed by a method based on the Euler–MacLaurin formula. We have obtained the variation of

thermal functions with temperature, and also discussed the results for high temperatures.

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