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Dynamics of higher-dimensional FRW cosmology in $R^p \exp(\lambda R)$ gravity

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Abstract. We study the cosmological dynamics for $R^p \exp(\lambda R)$ gravity theory in the metric formalism, using dynamical systems approach. Considering higher-dimensional FRW geometries in case of an imperfect fluid which has two different scale factors in the normal and extra dimensions, we find the exact solutions, and study its behaviour and stability for both vacuum and matter cases. It is found that stable solutions corresponding to accelerated expansion at late times exist, which can describe the inflationary era of the Universe. We also study the evolution of scale factors both in the normal and extra dimensions for different values of anisotropy parameter and the number of extra dimensions for such a scenario.

Keywords. Dynamical systems analysis; higher-dimensional FRW cosmology; $R^p \exp(\lambda R)$ gravity.

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1. Introduction

$f(R)$ gravity theory [1–8] was introduced in cosmology as an alternative to dark energy for explaining the observed late-time accelerated expansion of the Universe [9–11]. In these theories, the Hilbert–Einstein action of general relativity (GR) is generalized by replacing the Ricci scalar R with a non-linear function. The first reason for choosing $f(R)$ gravity theory is its relative simplicity compared to other higher-order theories of gravity (HOTG), which makes it an excellent candidate for toy-theories. In other words, $f(R)$ gravity is used to understand the principles and limitations of modified gravity [1]. Secondly, $f(R)$ gravity seems to be the only case that can avoid the Ostrogradski instability [12]. Some of the viable forms of $f(R)$ gravity theories used in literature are: R^n, $R+\alpha R^{-n}$, $\exp(\lambda R)$, $R^p \exp(\lambda R)$, $R \ln R$ etc. [13–15]. One of the conditions for viability of $f(R)$ is $f'(R) > 0$, $f''(R) > 0$. In our analysis, we have chosen $f(R) = R^p \exp(\lambda R)$, where the parameters λ and p are arbitrary real numbers. This type of Lagrangian is of considerable interest as the exponential can be expanded in powers of the Ricci scalar. So, by studying an exponential Lagrangian we can investigate what could happen if a Lagrangian is made up of a combination of different powers of Ricci scalar, in a relatively easy way [16]. As p is a real number, it can have both positive and negative values. If p is negative then, $f(R)$ will contain both positive and negative powers of curvature R. The terms with positive powers of curvature (like R^n) are dominant in the early Universe, when the curvature is bigger; but becomes negligible at late times when the curvature is smaller. So, these terms are suitable to explain the inflationary expansion of the Universe. On the other hand, the terms with negative powers of curvature (like $1/R^n$) are negligible compared to R at early Universe, but becomes dominant at late times. So, these terms are used to explain the late-time expansion of the Universe. In this way, such $f(R)$ theories can explain both inflationary expansion and late-time expansion of the Universe.

At the present time, we live in an ordinary $(1+3)$-dimensional space–time. However, in earlier epochs, our Universe was much smaller and the energy of the Universe was typically high enough so that the present four-dimensional space–time could have been preceded by a higher dimensional one. The main feature of theories with higher dimensions is that there are more spatial dimensions, other than well-known four-dimensional space–time. The scale factor of extra dimensions will in general be different from that of the normal dimensions. So, the dynamics of the Universe with the extra dimensions could have been different as compared to the normal four-dimensional one. Higher-dimensional cosmological model could have been a promising mechanism to explain the late-time accelerated expansion of the Universe. In recent years, many important solutions of Einstein's equations dealing with higher-dimensional model have been obtained [17–23]. However, the dynamics of higher-dimensional models with $f(R)$ gravity have not been much explored using dynamical system analysis (DSA). Detailed analysis of the dynamics of $f(R)$ gravity has been performed using DSA in the case of normal four-dimensional space–time [14,15,24–30]. In a recent paper, we analysed the cosmological dynamics of homogeneous and anisotropic Bianchi I geometries in $f(R) = \exp(\lambda R)$ gravity [31].

The main aim of this paper is to investigate the phase-space and stability analysis of higher-dimensional FRW geometries, in a Universe governed by exponential gravity of the form $f(R) = R^p \exp(\lambda R)$, considering the matter content as a perfect fluid. We also examine the possible cosmological behaviours. The major focus is on the late-time stable solutions. We analyse the possibility of accelerating expansion at late times. The kinematical quantities such as the deceleration parameter, scale factor and shear scalar have also been studied. We examine how the dynamics of the Universe evolves with different values of anisotropy parameters as well as the number of extra dimensions.

The paper is organized as follows: In §2 we present the field equations and the evolution equations for higher-dimensional FRW metric in $f(R)$ gravity model. In §3 we use DSA to the system. In §4 we find the exact solutions and their stability for both the vacuum and matter cases. In this section, we also discuss the physical behaviour of the scenario. Finally, §5 contains a summary of the conclusions.

2. Field equations

Here we consider a $(1+3+D)$-dimensional space–time metric which has D extra spatial dimensions, in addition to one temporal and three normal spatial dimensions. The line element for this space–time is described by [22]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + B^2(t)\delta_{IJ}\mathrm{d}X^I\mathrm{d}X^J, \quad (1)$$

where $i, j = 1, 2, 3$ denote normal spatial dimensions and $I, J = 4, 5, ..., (D+3)$ represent extra spatial dimensions. Here D is a parameter which takes integral values and hence, $n = D + 3$ represents total spatial dimensions. $A(t)$ and $B(t)$ denote the scale factors in normal dimensions and extra dimensions respectively. We consider the whole $(1+3+D)$-dimensional Universe to be homogeneous and hence the expansion scale factors $A(t)$ and $B(t)$ are functions only of time.

The Universe is assumed to be filled with distribution of matter described by the energy–momentum tensor of a general imperfect fluid [19,22]

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, P_A, P_A, P_A, P_B, ..., P_B), \quad (2)$$

where ρ is the energy density, P_A $(=\rho w_A)$ is the pressure exerted in normal dimensions and P_B $(=\rho w_B)$ is the pressure in extra dimensions. There is an istropy in pressure within the subspaces in normal dimensions and also within the subspaces in extra dimensions. However, the pressures in the two subspaces are different.

The Friedmann's field equation in $(1+3+D)$ dimensions using the metric (1) can be written as

$$\frac{(n-1)}{2n}\Theta^2 - \sigma^2 + \Theta\frac{\dot{f}'}{f'} - \frac{1}{2f}(Rf' - f) - \frac{\rho}{f'} = 0, \quad (3)$$

where

$$\Theta = n\frac{\dot{a}}{a} = 3\frac{\dot{A}}{A} + D\frac{\dot{B}}{B}, \quad (4)$$

$$\sigma = \sqrt{\frac{3D}{2n}}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right), \quad (5)$$

$$\begin{aligned} R &= 6\frac{\ddot{A}}{A} + 2D\frac{\ddot{B}}{B} + 6\frac{\dot{A}^2}{A^2} + 6D\frac{\dot{A}\dot{B}}{AB} + D(D-1)\frac{\dot{B}^2}{B^2} \\ &= 2\dot{\Theta} + \frac{n+1}{n}\Theta^2 + 2\sigma^2. \end{aligned} \quad (6)$$

Here, Θ, σ and R represent the volume expansion scalar, the shear scalar and the Ricci scalar respectively [32–35]. $H = \dot{a}/a$ is the Hubble parameter and a is the average scale factor. The dot denotes derivative with respect to time. The measure of total volume of the whole higher-dimensional Universe is given as

$$a(t)^n = A(t)^3 B(t)^D. \quad (7)$$

We may obtain solutions for the scale factors directly from the Einstein equations as [22,36]

$$\begin{aligned} A(t) &= a(t)\exp(\Sigma_1 W(t)), \\ B(t) &= a(t)\exp(\Sigma_2 W(t)), \end{aligned} \quad (8)$$

where $W(t)$ is defined as

$$W(t) = \int \frac{\mathrm{d}t}{a(t)^n} \quad (9)$$

and the constants Σ_1 and Σ_2 satisfy the relation

$$3\Sigma_1 + D\Sigma_2 = 0. \quad (10)$$

The trace-free Gauss–Codazzi equation for the $f(R)$ gravity model for local rotational symmetry (LRS) [26] is given by

$$\dot{\sigma} = -\left(\Theta + \frac{f''\dot{R}}{f'}\right)\sigma. \tag{11}$$

The conservation equation for the total energy–momentum tensor [19,22] is given by

$$\dot{\rho} = -3\frac{\dot{A}}{A}\rho(1 + w_A) - D\frac{\dot{B}}{B}\rho(1 + w_B) \tag{12}$$

which can also be written as [35]

$$\dot{\rho} = -\theta\rho\left(1 + \frac{3w_A + Dw_B}{n}\right) - \sqrt{\frac{6D}{n}}\rho\sigma(w_B - w_A). \tag{13}$$

3. Dynamical system approach

In order to perform a systematic analysis of the phase-space, as well as for doing the stability analysis of the cosmological models, it is beneficial to transform cosmological equations into a system of autonomous first-order differential equations, using DSA. The first step in the implementation of DSA is the definition of the variables. Let us introduce the set of expansion-normalized dimensionless variables as

$$\Sigma = \sqrt{\frac{2n}{n-1}}\frac{\sigma}{\Theta}$$

$$x = \frac{2n}{n-1}\frac{\dot{f}'}{f'\Theta}$$

$$y = \frac{n}{n-1}\frac{R}{\Theta^2},$$

$$z = \frac{n}{n-1}\frac{f}{f'\Theta^2}$$

$$\Omega = \frac{2n}{n-1}\frac{\rho}{f'\Theta^2}. \tag{14}$$

Using the above-mentioned variables (14) and eliminating x using the constraint equation (15) for the dynamical variables

$$1 - \Sigma^2 + x - y + z - \Omega = 0 \tag{15}$$

the Friedmann equation (3) can be written as

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} = \frac{(n-1)}{2}\Sigma[-2y + z - \Omega],$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{(n-1)}{2}y\left[\frac{2(n+1)}{n-1} + 2\Sigma^2 - 2y + (-1 + \Sigma^2 + y - z + \Omega)Q\right],$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{(n-1)}{2}z\left[\frac{3n+1}{n-1} + \Sigma^2 - 3y + z - \Omega + \frac{y}{z}(-1 + \Sigma^2 + y - z + \Omega)Q\right],$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = \frac{(n-1)}{2}\Omega\left[\frac{1 + 3(1 - 2w_A) + D(1 - 2w_B)}{(n-1)} - 2\sqrt{\frac{2D}{(n-1)}}\Sigma(w_B - w_A) + \Sigma^2 - 3y + z - \Omega\right], \tag{16}$$

where the new time variable τ is defined as $\tau = \ln a$.

In our case,

$$f(R) = R^p\exp(\lambda R). \tag{17}$$

For this model the value of the function Q defined as in [14] will be

$$Q(y, z) = \frac{f'}{Rf''} = \frac{yz}{y^2 - pz^2}. \tag{18}$$

Substituting this function into eq. (16) we get

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\tau} = \frac{(n-1)}{2}\Sigma[-2y + z - \Omega]$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{(n-1)}{2}y\left[\frac{2(n+1)}{n-1} + 2\Sigma^2 - 2y + \frac{(-1 + \Sigma^2 + y - z + \Omega)yz}{y^2 - pz^2}\right]$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{(n-1)}{2}z\left[\frac{3n+1}{n-1} + \Sigma^2 - 3y + z - \Omega + \frac{(-1 + \Sigma^2 + y - z + \Omega)y^2}{y^2 - pz^2}\right]$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = \frac{(n-1)}{2}\Omega\left[\frac{1 + 3(1 - 2w_A) + D(1 - 2w_B)}{(n-1)} - 2\sqrt{\frac{2D}{(n-1)}}\Sigma(w_B - w_A) + \Sigma^2 - 3y + z - \Omega\right]. \tag{19}$$

4. Results and discussion

4.1 *The vacuum case*

For the vacuum case, $\Omega = 0$ ($\rho = 0$), eq. (19), becomes

$$\frac{d\Sigma}{d\tau} = \frac{(n-1)}{2}\Sigma[-2y+z]$$
$$\frac{dy}{d\tau} = \frac{(n-1)}{2}y\left[\frac{2(n+1)}{n-1} + 2\Sigma^2 - 2y + \frac{(-1+\Sigma^2+y-z)yz}{y^2-pz^2}\right]$$
$$\frac{dz}{d\tau} = \frac{(n-1)}{2}z\left[\frac{3n+1}{n-1} + \Sigma^2 - 3y + z - \Omega + \frac{(-1+\Sigma^2+y-z)y^2}{y^2-pz^2}\right]. \quad (20)$$

4.1.1 *Exact solutions corresponding to the fixed points.* In order to find the fixed points, we need to set eq. (20) equal to zero. The fixed points for the vacuum case are presented in table 1, which include five isotropic (since $\Sigma = 0$, shear $\sigma = 0$) fixed points A_v, B_v, C_v, D_v, E_v and a line of anisotropic fixed points $L_v(\Sigma_*, 0, 0)$. The point $\Sigma_* = 0$ on line L_v would merge with A_v which represents another isotropic fixed point. For the point E_v, $n \neq 2p-1$. When $n = 2p-1$, it will become C_v.

Let us now present a formalism based on [26,31] for obtaining basic observable quantities like the deceleration parameter, scale factor and shear for a fixed point. Firstly, for a fixed point we obtain first-order differential equations for Θ and σ to find scale factor and shear respectively.

In order to find scale factors at the fixed points, we use the equation

$$\dot{\Theta} = -\left(\frac{n+1}{n-1} + \Sigma_i^2 - y_i\right)\frac{(n-1)\Theta^2}{2n} = -(1+q_i)\frac{\Theta^2}{n}, \quad (21)$$

where $q_i = ((n-1)/2)(1+\Sigma_i^2 - y_i)$ represents the deceleration parameter.

For fixed points A_v, D_v, E_v and L_v, where $q \neq -1$, i.e., $((n-1)/2)(1+\Sigma_i^2 - y_i) \neq 0$, eq. (21) is integrated and setting the Big Bang time $t_0 = 0$ we obtain the scale factor as

$$a = a_0 t^{\alpha_i}, \quad (22)$$

where

$$\alpha_i = (1+q_i)^{-1} = \left(\frac{(n-1)}{2}\left(\frac{n+1}{n-1} + \Sigma_i^2 - y_i\right)\right)^{-1}.$$

Subscript i denotes the evaluation at a specific fixed point.

For the points B_v and C_v, where $q = -1$, i.e., $((n-1)/2)(1+\Sigma_i^2 - y_i) = 0$, $\dot{\Theta} = 0$. Solutions for such points correspond to de Sitter solutions given by

$$a = a_0 e^{(1/n)\Theta_0 t}. \quad (23)$$

So they represent an exponential evolution.

The fixed points A_v, B_v, C_v, D_v, E_v and the point at $\Sigma_* = 0$ on the fixed line L_v are isotropic points, because they lie on the plane $\Sigma = 0$. All the other points on L_v have non-zero shear.

In order to find the evolution of shear we use the trace-free Gauss–Codazzi eq. (11) which can be written in terms of the dynamical variables (14) as

$$\frac{\dot{\sigma}}{\sigma} = -\frac{(n-1)}{2n}\left(\frac{n+1}{n-1} + \Sigma_i^2 + y_i - z_i\right)\Theta. \quad (24)$$

For the points with non-zero shear on L_v, the evolution of shear will be

$$\sigma = \sigma_0 a^{-(1/2)(n+1+\Sigma_*^2(n-1))} = \sigma_0 a_0^{-(1/2)(n+1+\Sigma_*^2(n-1))} t^{-1}. \quad (25)$$

Thus, the shear evolves inversely with time.

The solutions, i.e., deceleration parameter, scale factor and shear for each fixed point in vacuum case are listed in table 1.

4.1.2 *Stability analysis of the fixed points.* Eigenvalues and stability results for the fixed points are summarized in table 2. The fixed points B_v and C_v (when $n = 2p-1$) are non-hyperbolic with 2D stable manifold. Centre manifold theorem is used for these two points, to find their stability, because only these two points have the probability of being a solution.

4.2 *The matter case*

For matter case the equations of the system in terms of the four variables Σ, y, z and Ω are given by eq. (19).

4.2.1 *Exact solutions corresponding to the fixed points.* In order to find the fixed points, we need to set eq. (19) equal to zero as done in vacuum case. These fixed points are listed in table 3, which contains seven isotropic (as $\Sigma = 0$, shear $\sigma = 0$) fixed points $A_m, B_m, C_m, D_m, E_m, F_m, G_m$ and a line of anisotropic fixed points $L_m(\Sigma_*, 0, 0, 0)$. The point $\Sigma_* = 0$ on line L_m would merge with A_m representing another isotropic fixed point. For the point E_m, $n \neq 2p-1$. When $n = 2p-1$, it will become C_m.

Proceeding in a similar manner as in the vacuum case, we can find the scale factor and shear for a fixed

Table 1. Fixed points and their solutions for deceleration parameter, scale factor and shear associated with fixed points in vacuum case. We use the notation $\sigma_* = \sigma_0 a^{-(1/2)[n+1+\Sigma_*^2(n+1)]}$.

Points/ line	Fixed points (Σ, y, z)	Deceleration parameter (q)	Scale factor (a)	Shear (σ)	Existence
A_v	$(0, 0, 0)$	$\frac{n-1}{2}$	$a_0 t^{\frac{2}{n+1}}$	0	Always
B_v	$\left(0, \frac{n+1}{n-1}, 0\right)$	-1	$a_0 e^{\frac{1}{n}\Theta_0 t}$	0	Always
C_v	$\left(0, \frac{n+1}{n-1}, \frac{2}{n-1}\right)$	-1	$a_0 e^{\frac{1}{n}\Theta_0 t}$	0	Always
D_v	$\left(0, 0, -\frac{3n+1}{n-1}\right)$	$\frac{n-1}{2}$	$a_0 t^{\frac{2}{n+1}}$	0	$p \neq 0$
E_v	$\left(0, \frac{p(2np-3n+2p-1)}{(n-1)(p-1)(2p-1)}, \frac{(2np-3n+2p-1)}{(n-1)(p-1)(2p-1)}\right)$	$\frac{n-4p^2+4p-1}{2(p-1)(2p-1)}$	$a_0 t^{\frac{2(p-1)(2p-1)}{n-2p+1}}$	0	$p \neq 1, \frac{1}{2}$; $n \neq 2p-1$
L_v	$(\Sigma_*, 0, 0)$	$\frac{n-1}{2}(\Sigma_*^2+1)$	$a_0 t^{\frac{2}{n+1+\Sigma_*^2(n+1)}}$	σ_*	$\Sigma_*^2 \geq 0$

Table 2. Eigenvalues and stability for the fixed points in vacuum case.

Points	Eigenvalues	Stability
A_v	$\left[0, n+1, \frac{1}{2}(3n+1)\right]$, for $p \neq 0$; $[0, n+1, n+1]$, for $p = 0$	Non-hyperbolic with 2D unstable manifold
B_v	$[0, -(n+1), -2(n+1)]$	Non-hyperbolic with 2D stable manifold
C_v	$\left[-n, \frac{1}{2}\left(-n \pm \sqrt{n^2 - \frac{8(n+1)(2p-n-1)}{(n+1)^2-4p}}\right)\right]$	Non-hyperbolic with 2D stable manifold for $n = 2p-1$; Stable for $\frac{8(n+1)(2p-n-1)}{(n+1)^2-4p} > 0$ for $n \neq 2p-1$
D_v	$\left[-\frac{1}{2}(3n+1), n+1, -\frac{1}{2}(3n+1)\right]$	Saddle with 2D stable manifold
E_v	$\left[-\frac{2np-3n+2p-1}{2(p-1)}, -\frac{2np-3n+2p-1}{2(p-1)}, -\frac{n-2p+1}{(p-1)(2p-1)}\right]$	Stable for $\frac{2np-3n+2p-1}{(p-1)} > 0$, $\frac{n-2p+1}{(p-1)(2p-1)} > 0$
L_v	$\left[0, n+1+\Sigma_*^2(n-1), \frac{1}{2}(3n+1+\Sigma_*^2(n-1))\right]$, for $p \neq 0$; $\left[0, n+1+\Sigma_*^2(n-1), n+1+\Sigma_*^2(n-1)\right]$, for $p = 0$	Non-hyperbolic with 2D unstable manifold

Table 3. Fixed points and their solutions for deceleration parameter, scale factor and shear associated with fixed points in matter case. We use the notation $\sigma_* = \sigma_0 a^{-(1/2)[n+1+\Sigma_*^2(n+1)]}$.

Points/ line	Fixed points (Σ, y, z, Ω)	Deceleration parameter (q)	Scale factor (a)	Shear (σ)	Existence
A_m	$(0, 0, 0, 0)$	$\frac{n-1}{2}$	$a_0 t^{\frac{2}{n+1}}$	0	Always
B_m	$\left(0, \frac{n+1}{n-1}, 0, 0\right)$	-1	$a_0 e^{\frac{1}{n}\Theta_0 t}$	0	Always
C_m	$\left(0, \frac{n+1}{n-1}, \frac{2}{n-1}, 0\right)$	-1	$a_0 e^{\frac{1}{n}\Theta_0 t}$	0	Always
D_m	$\left(0, 0, -\frac{3n+1}{n-1}, 0\right)$	$\frac{n-1}{2}$	$a_0 t^{\frac{2}{n+1}}$	0	$p \neq 0$
E_m	$\left(0, \frac{p(2np-3n+2p-1)}{(n-1)(p-1)(2p-1)}, \frac{(2np-3n+2p-1)}{(n-1)(p-1)(2p-1)}, 0\right)$	$\frac{n-4p^2+4p-1}{2(p-1)(2p-1)}$	$a_0 t^{\frac{2(p-1)(2p-1)}{n-2p+1}}$	0	$p \neq 1, \frac{1}{2}$; $n \neq 2p-1$
F_m	$\left(0, 0, 0, \frac{1+3(1-2w_A)+D(1-2w_B)}{n-1}\right)$	$\frac{n-1}{2}$	$a_0 t^{\frac{2}{n+1}}$	0	Always
G_m	$\left(0, \frac{n+1}{n-1} 0, -\frac{2(1+3(1+w_A)+D(1+w_B))}{n-1}\right)$	-1	$a_0 e^{\frac{1}{n}\Theta_0 t}$	0	Always
L_m	$(\Sigma_*, 0, 0, 0)$	$\frac{n-1}{2}(\Sigma_*^2+1)$	$a_0 t^{\frac{2}{n+1+\Sigma_*^2(n+1)}}$	σ_*	$\Sigma_*^2 \geq 0$

point in the matter case, using first-order differential equations of Θ and σ respectively.

The fixed points A_m, D_m, E_m, F_m and L_m are found to represent power-law solution, while for points B_m, C_m and G_m, we have $\dot{\Theta} = 0$, implying such points correspond to de Sitter solutions.

In order to find shear for the anisoropic fixed points in the matter case, the form of Gauss–Codazzi equation is given by

$$\frac{\dot{\sigma}}{\sigma} = -\frac{(n-1)}{2n}\left(\frac{n+1}{n-1} + \Sigma_i^2 + y_i - z_i + \Omega_i\right)\Theta. \tag{26}$$

For the points with non-zero shear on L_m, eq. (26) reduces to

$$\sigma = \sigma_0 a^{-(1/2)(n+1+\Sigma_*^2(n-1))} = \sigma_0 a_0^{-(1/2)(n+1+\Sigma_*^2(n-1))} t^{-1}. \tag{27}$$

The above solution is the same as obtained in the vacuum case. The solutions, i.e., deceleration parameter, scale factor and shear for each fixed point in the vacuum case are listed in table 3.

4.2.2 *Stability analysis of the fixed points*. In radiation-dominated era, the Universe is filled with radiation in normal subspace and with dust matter in extra-dimensional subspace. Thus, in radiation domination we have $P_A = \rho/3$, $P_B = 0$, while in matter-dominated era, the Universe is filled with dust matter in normal subspace and with radiation in extra-dimensional subspace. In this case, we have $P_A = 0$, $P_B = \rho/D$ [37]. The eigenvalues and stability results for these two cases at each fixed point are summarized in tables 4 and 5 respectively. Centre manifold theorem is applied for the two fixed points B_m and C_m, to find their stabilities.

4.3 *Physical implications*

In this section, we discuss the physical implications of the fixed points in more details.

The fixed point A_v corresponds to an isotropic expanding Universe in which expansion is decelerating. This solution is a power-law solution. It is a non-hyperbolic point with 2D unstable manifold. Thus, it cannot be the late-time state of the Universe.

The fixed point B_v corresponds to an isotropic expanding Universe that is accelerating. It is again a non-hyperbolic point, but it possesses 2D stable manifold. Using the centre manifold theory, it is found that

Table 4. Eigenvalues of the fixed points in matter case.

Points	Eigenvalues
A_m	$\left[0, n+1, \frac{1}{2}(3n+1), \frac{1}{2}\left(1+3(1-2w_A)+(1-2w_B)\right)\right]$, for $p \neq 0$
	$\left[0, n+1, n+1, \frac{1}{2}\left(1+3(1-2w_A)+(1-2w_B)\right)\right]$, for $p = 0$
B_m	$[0, -(n+1), -(n+1), -\left(1+3(1+w_A)+D(1+w_B)\right)]$
C_m	$\left[-n, \frac{1}{2}\left(-n \pm \sqrt{n^2 - \frac{8(n+1)(2p-n-1)}{(n+1)^2-4p}}\right), -3(1+w_A)-D(1+w_B)\right]$
D_m	$\left[-\frac{1}{2}(3n+1), n+1, -\frac{1}{2}(3n+1), -3(1+w_A)-D(1+w_B)\right]$
E_m	$\left[-\frac{2np-3n+2p-1}{2(p-1)}, -\frac{2np-3n+2p-1}{2(p-1)}, -\frac{n-2p+1}{(p-1)(2p-1)}, -\frac{2p^2-np-p+(p-1)(2p-1)\left(3(1+w_A)+D(1+w_B)\right)}{(p-1)(2p-1)}\right]$
F_m	$\left[-\frac{1}{2}\left(1+3(1-2w_A)+D(1-2w_B)\right), n+1, 3(1+w_A)+D(1+w_B), -\frac{1}{2}\left(1+3(1-2w_A)+D(1-2w_B)\right)\right]$, for $p \neq 0$
	$\left[-\frac{1}{2}\left(1+3(1-2w_A)+D(1-2w_B)\right), n+1, n+1, -\frac{1}{2}\left(1+3(1-2w_A)+D(1-2w_B)\right)\right]$, for $p = 0$
G_m	$[3w_A + Dw_B, -(n+1), 0, 1+3(1+w_A)+D(1+w_B)]$
L_m	$\left[0, n+1+\Sigma_*^2(n-1), \frac{1}{2}(3n+1+\Sigma_*^2(n-1)) \; \frac{1}{2}(1+3(1-2w_A)+D(1-2w_B)+\Sigma_*^2(n-1))\right]$, for $p \neq 0$
	$\left[0, n+1+\Sigma_*^2(n-1), n+1+\Sigma_*^2(n-1) \; \frac{1}{2}(1+3(1-2w_A)+D(1-2w_B)+\Sigma_*^2(n-1))\right]$, for $p = 0$

the point B_v represents a saddle node, i.e., it acts like a saddle or a stable node (attractor). Thus, it can be a solution of the Universe at late times corresponding to a de-Sitter expansion.

The fixed point C_v corresponds to an isotropic expanding Universe and this is also an accelerated expansion which correspond to a de-Sitter solution. It is a non-hyperbolic point which possesses a 2D stable manifold for $n = 2p - 1$. Centre manifold theorem is used and the point is found to be a saddle point. For $n \neq 2p - 1$, it is stable when

$$\frac{8(n+1)(2p-n-1)}{(n+1)^2-4p} > 0.$$

Thus, it has the probability to be a late-time solution for the Universe.

The fixed point D_v which represents isotropic expanding Universe in which expansion is decelerating, is a saddle with 2D stable manifold. This solution

Table 5. Stability of the fixed points in matter case.

Points/ line	Stability $w_A = \frac{1}{3}, w_B = 0$	Stability $w_A = 0, w_B = \frac{1}{D}$
A_m	Non-hyperbolic 3D unstable manifold	Non-hyperbolic 3D unstable manifold
B_m	Non-hyperbolic 3D stable manifold	Non-hyperbolic 3D stable manifold
C_m	Non-hyperbolic 3D stable manifold for $n = 2p - 1$ Stable for $\frac{8(n+1)(2p-n-1)}{(n+1)^2 - 4p} > 0$, $n \neq 2p - 1$	Non-hyperbolic 3D stable manifold for $n = 2p - 1$ Stable for $\frac{8(n+1)(2p-n-1)}{(n+1)^2 - 4p} > 0$, $n \neq 2p - 1$
D_m	Saddle with 2D stable manifold	Saddle with 2D stable manifold
E_m	Stable for $\frac{n(2p-3)+2p-1}{p-1} > 0$, $\frac{n-2p+1}{(p-1)(2p-1)} > 0$, $\frac{n(2p^2-4p+1)+4p^2-2p+1}{(p-1)(2p-1)} > 0$	Stable for $\frac{n(2p-3)+2p-1}{p-1} > 0$, $\frac{n-2p+1}{(p-1)(2p-1)} > 0$, $\frac{n(2p^2-4p+1)+4p^2-2p+1}{(p-1)(2p-1)} > 0$
F_m	Saddle	Saddle
G_m	Non-hyperbolic saddle with 2D unstable manifold	Non-hyperbolic saddle with 2D unstable manifold
L_m	Non-hyperbolic 3D unstable manifold	Non-hyperbolic 3D unstable manifold

corresponds to a power-law solution. Thus, it cannot represent the late-time acceleration of the Universe.

The fixed point E_v corresponds to an isotropic Universe which shows power-law solution. The nature of the solutions for this point are as follows:

- Accelerated expansion for $[(4p^2 - n - 4p + 1)/(n - 2p + 1)] > 0$ and $n > 2p - 1$.
- Decelerated expansion for $[(4p^2 - n - 4p + 1)/(n - 2p + 1)] < 0$ and $n > 2p - 1$.
- Accelerated contraction for $[(4p^2 - n - 4p + 1)/(n - 2p + 1)] < 0$ and $n < 2p - 1$.
- Decelerated contraction for $[(4p^2 - n - 4p + 1)/(n - 2p + 1)] > 0$ and $n < 2p - 1$.

The conditions for this fixed point to be a stable (attractor) point are:

$$\frac{2np - 3n + 2p - 1}{p - 1} > 0, \quad \frac{n - 2p + 1}{(p - 1)(2p - 1)} > 0.$$

The line of fixed points L_v corresponds to an anisotropic expanding Universe where the expansion is decelerating. This solution is a power-law solution. It is a non-hyperbolic point with 2D unstable manifold. Thus, it cannot be the late-time solution for the Universe.

The fixed point A_m corresponds to an isotropic Universe which is undergoing decelerated expansion. This solution is a power-law solution. It is a non-hyperbolic point which possesses a 3D unstable manifold. Therefore, it cannot be the late-time solution of the Universe.

The fixed point B_m corresponds to an isotropic expanding Universe which is accelerating. This solution is a de-Sitter solution. It is a non-hyperbolic point, but possesses 3D stable manifold. Using centre manifold theory we found that the point B_m represents a saddle node. Thus, it can be a solution of the accelerating Universe.

The fixed point C_m corresponding to an isotropic expanding Universe in which expansion is accelerating represents a de-Sitter solution. It is a non-hyperbolic point with 3D stable manifold when $n = 2p - 1$. Centre manifold theorem is used and the point is found to be a saddle point. When $n \neq 2p - 1$, it is stable, if

$$\frac{8(n+1)(2p-n-1)}{((n+1)^2 - 4p)} > 0.$$

Thus, it can be the late-time solution for the Universe.

The fixed point D_m corresponds to an isotropic Universe undergoing decelerated expansion. This solution is a power-law solution. It is a saddle point. Thus, it cannot predict the cosmic speed up of the Universe.

The fixed point E_m is an isotropic point corresponding to a power-law solution. The behaviours of the solutions for this point are the same as the point E_v in the vacuum case. The conditions for this fixed point to be a stable (attractor) point are listed in table 5.

The fixed point F_m corresponds to an isotropic expanding Universe where expansion is decelerating. For the two cases of matter (radiation and dust matter), it behaves as a saddle. Thus, it cannot be a solution for predicting late-time acceleration of the Universe. But it can correspond to the radiation-dominated era and matter-dominated era, depending upon the equation of state parameter present in normal and extra dimensions.

The fixed point G_m corresponds to a Universe undergoing accelerated expansion. It is a non-hyperbolic saddle with 2D unstable manifold.

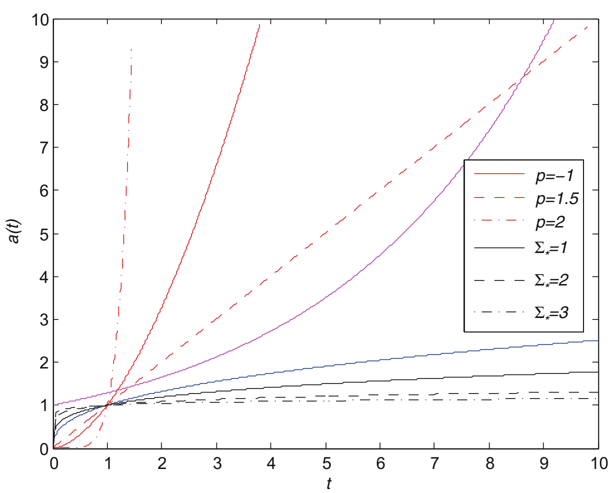

Figure 1. Evolution of average scale factor $a(t)$ for $n = 4$. Blue, magenta, red and black curves correspond to the fixed points showing isotropic power-law solution, exponential expansion, E_v (E_m) and anisotropic power-law solution respectively.

The line of fixed points L_m corresponds to an anisotropic expanding Universe (decelerated). This solution is a power-law solution. It is a non-hyperbolic point with a 2D unstable manifold. Thus, it cannot be a solution for the accelerating Universe.

Variation of the average scale factor $a(t)$ and expansion scalar $\Theta(t)$ with respect to cosmic time t for the fixed points are plotted in figures 1–4 for $n = 4, 6$ respectively. In figure 5 we plotted shear with respect to time for the anisotropic fixed point which shows that shear decreases with time.

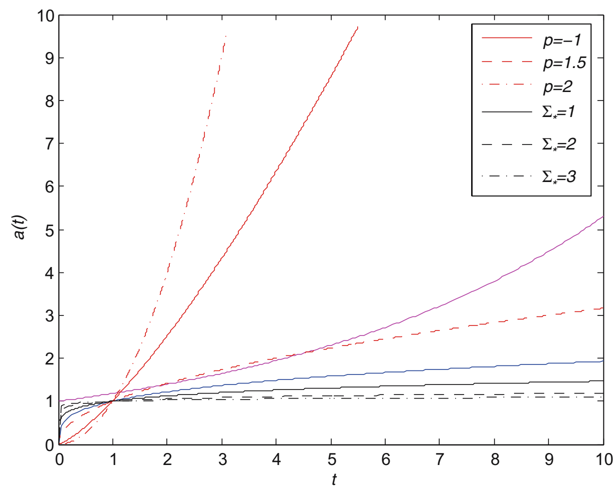

Figure 2. Evolution of average scale factor $a(t)$ for $n = 6$. Blue, magenta, red and black curves correspond to the fixed points showing isotropic power-law solution, exponential expansion, E_v (E_m) and anisotropic power-law solution respectively.

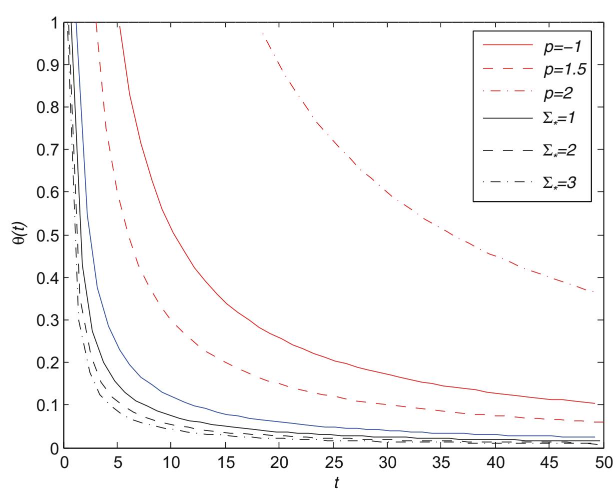

Figure 3. Evolution of expansion scalar $\Theta(t)$ for $n = 4$. Blue, red and black curves correspond to the fixed points showing isotropic power-law solution, E_v (E_m) and anisotropic power-law solution respectively.

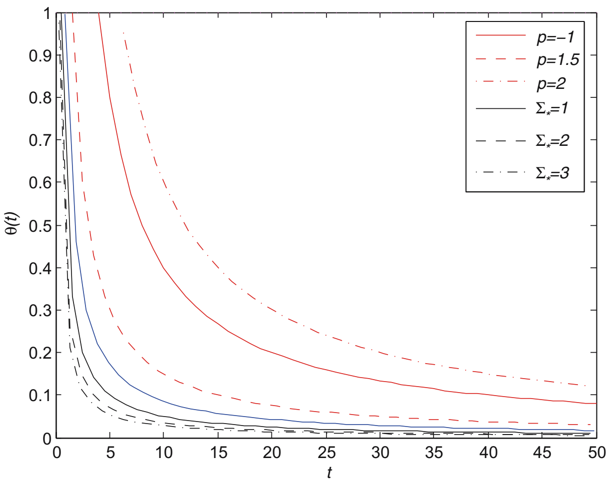

Figure 4. Evolution of expansion scalar $\Theta(t)$ for $n = 6$. Blue, red and black curves correspond to the fixed points showing isotropic power-law solution, E_v (E_m) and anisotropic power-law solution respectively.

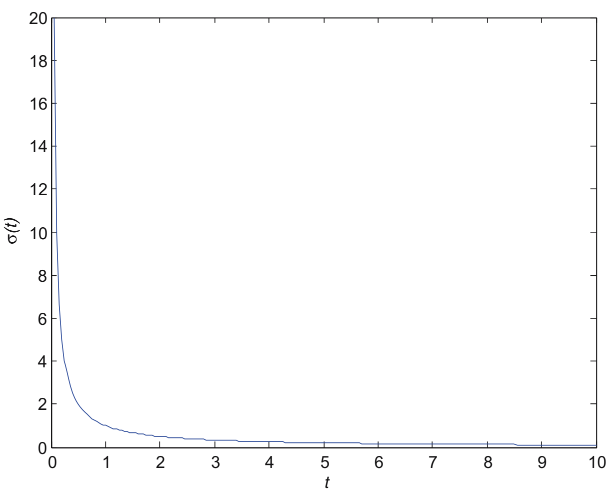

Figure 5. Variation of shear with respect to time for line of anisotropic fixed points L_v (L_m).

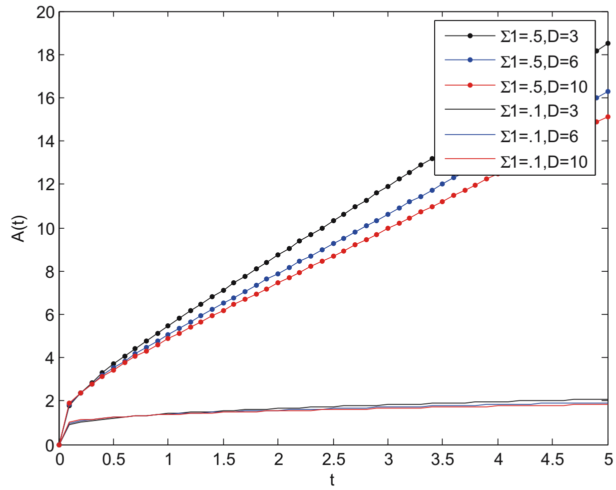

Figure 6. Evolution of scale factor $A(t)$ for different values of Σ_1 and D.

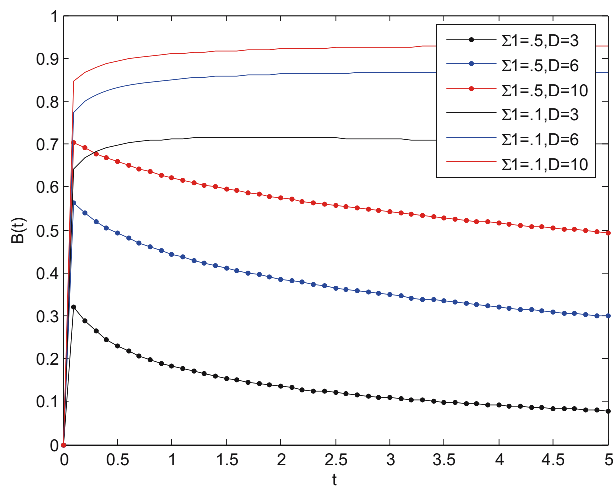

Figure 7. Evolution of scale factor $B(t)$ for different values of Σ_1 and D.

In order to investigate how the dynamics of the Universe evolves, we plot the solution of scale factors $A(t)$ and $B(t)$ with respect to time for the anisotropic fixed point taking different values of anisotropy parameter Σ_1 and the number of extra dimensions D in figures 6 and 7. Figure 6 shows that the expansion in scale factor $A(t)$ in the normal dimension varies too fast if we increase the value of anisotropy parameter Σ_1, which implies that shear helps in expansion. For the same Σ_1, the rate of expansion of scale factor $A(t)$ is reduced if the number of extra dimension D is increased. It indicates that increase in the number of extra dimensions reduces the effect of shear and hence reduces expansion. In figure 7, the scale factor $B(t)$ in extra dimension flips its behaviour from contracting to expanding phase if we decrease Σ_1 but it becomes constant asymptotically. These results are found to be consistent with the results in [22].

5. Conclusions

In this work, we have studied the cosmological behaviour of $f(R) = R^p \exp(\lambda R)$ gravity model in the case of a higher-dimensional FRW metric, where the Universe has two different scale factors in the normal and extra spatial dimensions. Here, we have considered a general imperfect fluid exerting different pressures in normal and extra dimensions. Using the dynamical system approach we have found the exact solutions along with their stability for both vacuum and matter cases. The form of $f(R) = R^p \exp(\lambda R)$ includes both positive and negative powers of curvature R, when p is negative. So, such $f(R)$ can explain both the inflationary expansion and the late-time expansion of the Universe. We have incorporated the evolution of the average scale factors, expansion scalars and shear with respect to time for the fixed points. We have also investigated the dynamical evolutions of expansion scalars in normal and extra-dimensional subspace along with the evolution of shear.

In the vacuum case, we have found that the stable de-Sitter solutions present, correspond to isotropic fixed points B_v and C_v as shown by the magenta curves in figures 1 and 2, which can allow the accelerated expansion of the present Universe. These solutions can also describe the inflationary era of the earlier Universe. Also E_v describes accelerated or decelerated expansion, or accelerated or decelerated contraction depending on the conditions given in §4.3, which are designated by the red curves in figures 1 and 2. The solutions associated with the isotropic fixed points A_v, D_v, E_v as shown by the blue curves and line of anisotropic fixed points L_v shown by the black curves in figures 1 and 2 correspond to decelerated expansion, allowing structure formation, but they are not stable. These points are required to be unstable in order to obtain a cosmology evolving towards a dark energy era.

Addition of matter into this system makes it harder to visualize by increasing the dimensionality of the phase-space. In this case, the solutions related to the isotropic fixed points B_m, C_m and G_m represent de-Sitter expansions as shown by the magenta curves in figures 1 and 2, describing accelerated expansion. Thus, this cosmological model can describe both the inflationary era and the recent accelerated expansion of the Universe. E_m represents power-law solution, as in the vacuum case as shown by the red curves in figures 1 and 2. The solutions related to the isotropic fixed points A_m, D_m, F_m shown by the blue curves and the line of anisotropic fixed points L_m shown by the black curves in figures 1 and 2, possessing unstable manifold correspond to decelerated expansion of the Universe.

Our study for the evolution of scale factors with different number of extra spatial dimensions D and different values of anisotropy parameter Σ_1 shows that shear helps in expansion in the normal space–time, while increase in the number of extra dimension reduces the effect of shear. The scale factor $B(t)$ in extra dimension flips its behaviour from contracting to expanding phase if we decrease Σ_1, but it becomes constant asymptotically.

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