Pramana – J. Phys. (2017) 88: 25
DOI 10.1007/s12043-016-1326-3

Asymptotic iteration method for the modified Pöschl–Teller potential and trigonometric Scarf II non-central potential in the Dirac equation spin symmetry

BETA NUR PRATIWI*, A SUPARMI, C CARI and ANDRI SOFYAN HUSEIN

Physics Department, Graduate Program, Sebelas Maret University, Jl.Ir. Sutami 36A Kentingan Surakarta 57126, Indonesia
*Corresponding author. E-mail: namakubetanurpratiwi@gmail.com

MS received 7 October 2015; revised 13 April 2016; accepted 17 June 2016; published online 4 January 2017

Abstract. Analytical solution of the Dirac equation for the modified Pöschl–Teller potential and trigonometric Scarf II non-central potential for spin symmetry is studied using asymptotic iteration method. One-dimensional Dirac equation consisting of the radial and angular parts can be obtained by the separation of variables. By using asymptotic iteration method, the relativistic energy equation and orbital quantum number (l) equation can be obtained, where both are interrelated. Relativistic energy equation is calculated numerically by the Matlab software. The increase in the radial quantum number n_{r} causes a decrease in the energy value, and the wave functions of the radial and the angular parts are expressed in terms of hypergeometric functions. Some thermodynamical properties of the system can be determined by reducing the relativistic energy equation to the non-relativistic energy equation. Thermodynamical properties such as vibrational partition function, vibrational specific heat function and vibrational mean energy function are expressed in terms of error function.

Keywords. Dirac equation spin symmetry; modified Pöschl–Teller potential; trigonometric Scarf II non-central potential; asymptotic iteration method.

PACS Nos 03; 03.65.−w

1. Introduction

In quantum mechanics, Schrödinger equation is the basic equation that is used to describe the behaviour of microscopic particles in the study of non-relativistic energy equation, while in the study of relativistic energy equation, Dirac equation is used for spin-1/2 particles or Klein–Gordon equation for spin-0 particles [1,2]. Dirac equation consisting of two special cases, spin symmetry and pseudospin symmetry, has been applied to solve many problems in high-energy physics and nuclear physics [3,4]. Spin symmetry occurs when the scalar potential $S(r)$ is equal to the vector potential $V(r)$ and pseudospin (p-spin) symmetry occurs when $S(r) = -V(r)$ [4]. Spin symmetry concept is used to explain the spectra of mesons and antinucleons [5,6] while pseudospin symmetry concept is used to explain the quasidegeneracy of the nucleon doublets [5,7], exotic nuclei [8], superdeformation in nuclei [9], and to establish an affective nuclear shell-model scheme [10,11].

Researchers have solved the Dirac equation using various potentials, such as the generalized Pöschl–Teller potential plus trigonometric Pöschl–Teller non-central potential [12], q-deformed Scarf II potential plus Coulomb-type tensor [13], the modified Pöschl–Teller non-central potential [14], the Pöschl–Teller potential plus Manning Rosen potential [15], the Deng–Fan potential and the Coulomb potential [16], the Woods–Saxon potential [17], the Hulthen potential and Coulomb-type tensor potential [18], the Pöschl–Teller trigonometric potential and tensor Coulomb potential [3], q-deformed hyperbolic Pöschl–Teller potential and trigonometric Scarf II non-central potential [2], Scarf potential with new tensor coupling potential [19], the Deng–Fan and Eckart potentials with Coulomb-like and Yukawa-like tensor interactions [20], tensor coupling on the Mie-type potential [21], etc. Various methods such as asymptotic iteration method [21,22], hypergeometric method [15], SUSY quantum mechanics [12], Romanovski polynomials [19], the Nikiforov–Uvarov method [2,14,20], transformation method [23], and others have been used.

In this study, we shall solve the Dirac equation in the case of spin symmetry for modified Pöschl–Teller potential plus trigonometric Scarf II potential using asymptotic iteration method. The basic strategy to obtain the solutions is to reduce the Dirac equation to the hypergeometric-type equation with suitable changes of variables. Then the eigenvalue and eigenfunction can be obtained using the asymptotic iteration method. Some thermodynamical properties can be obtained from non-relativistic energy equation [24].

2. Asymptotic iteration method

Asymptotic iteration method (AIM) is used to solve the second-order differential equation

$$y_n''(x) - \lambda_0(x)\, y_n'(x) - s_0(x)\, y_n(x) = 0, \tag{1}$$

where $\lambda_0(x) \neq 0$ and $s_0(x)$ are coefficients of the differential equation and they are well-defined as well as sufficiently differentiable functions.

The one-dimensional Dirac equation can be reduced into hypergeometric or confluent hypergeometric-type differential equation by suitable changes of variables, and then, hypergeometric or confluent hypergeometric-type differential equation is reduced into the AIM-type differential equation which has the form in eq. (1).

The solution of eq. (1) can be obtained by using iteration of λ_i and s_i,

$$\lambda_i(x) = \lambda_{i-1}' + \lambda_{i-1}\lambda_0 + s_{i-1}$$
$$s_i(x) = s_{i-1}' + s_0\lambda_{i-1}; \quad i = 1, 2, 3, \ldots. \tag{2}$$

Eigenvalues from (1) can be obtained using eq. (3) [25]:

$$\lambda_i(x)s_{i-1}(x) - \lambda_{i-1}(x)s_i(x) = 0 = \Delta_i,$$
$$i = 1, 2, 3 \ldots. \tag{3}$$

On the other hand, eq. (1) can be written in the following form:

$$y''(x) = 2\left(\frac{tx^{N+1}}{1-bx^{N+2}} - \frac{c+1}{x}\right)y'(x) - \frac{Wx^N}{1-bx^{N+2}}y. \tag{4}$$

Equation (4) is AIM-type differential equation which is solved by using eq. (5) [26,27]:

$$y_n(x) = (-1)^n C'(N+2)^n\, {}_2F_1(-n, p+n, \sigma, bx^{N+2}), \tag{5}$$

where

$$(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} \tag{6}$$

$$\sigma = \frac{2c+N+3}{N+2} \tag{7}$$

$$p = \frac{(2c+1)b+2t}{(N+2)b}. \tag{8}$$

Here C' is the normalization constant and ${}_2F_1$ is the hypergeometric function. Equation (5) is the eigenfunction of the AIM-type differential equation (4). Equation (5) is used to obtain wave functions of the Dirac equation. The relativistic energy equation can be formulated by equating eigenvalue equation using eq. (3).

3. The Dirac equation in spin symmetry with modified Pöschl-Teller potential and trigonometric Scarf II non-central potential

The Dirac equation for a single particle with mass M in a scalar potential $S(r)$ and vector potential $V(r)$ can be given as ($\hbar = 1, c = 1$),

$$\{\vec{\alpha}\cdot\vec{p} + \beta(M + S(\vec{r}))\}\psi(\vec{r}) = \{E - V(\vec{r})\}\psi(\vec{r}), \tag{9}$$

where E is the relativistic energy of the system and $\vec{p}$ is the momentum operator ($\vec{p} = -i\nabla$), while $\vec{\alpha}$ and β are:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \tag{10}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{11}$$

where I is the identity matrix 2×2 and $\vec{\sigma}$ is the Pauli matrix,

$$\vec{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \vec{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

In the Pauli–Dirac representation, let [26]

$$\psi_{nk}(\vec{r}) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix}. \tag{13}$$

We have

$$\vec{\sigma}\cdot\vec{p}g_{nk}(\vec{r}) = [E - V(\vec{r}) - M - S(\vec{r})]f_{nk}(\vec{r}), \tag{14a}$$

$$\vec{\sigma}\cdot\vec{p}f_{nk}(\vec{r}) = [E - V(\vec{r}) + M + S(\vec{r})]g_{nk}(\vec{r}). \tag{14b}$$

For spin symmetry, eq. (14) becomes

$$\vec{\sigma}\cdot\vec{p}g_{nk}(\vec{r}) = [E - 2V(\vec{r}) - M]f_{nk}(\vec{r}) \tag{15a}$$

$$g_{nk}(\vec{r}) = \frac{\vec{\sigma}\cdot\vec{p}\,f_{nk}(\vec{r})}{[E+M]}. \tag{15b}$$

Substituting eq. (15b) into eq. (15a) yields

$$[p^2+2(E+M)V(r)]f_{nk}(\vec{r}) = [E^2-M^2]f_{nk}(\vec{r}). \tag{16}$$

In spherical coordinates, modified Pöschl–Teller potential combined with trigonometric Scarf II non-central potential is defined as

$$V(r,\theta)=\mu^2\left(\frac{\kappa(\kappa-1)}{\sinh^2\mu r}-\frac{\eta(\eta+1)}{\cosh^2\mu r}\right) + \frac{1}{r^2}\left(\frac{b^2+a(a-1)}{\sin^2\theta}-\frac{2b\left(a-\frac{1}{2}\right)\cos\theta}{\sin^2\theta}\right), \tag{17}$$

where κ, η, $b^2+a(a-1)$, $b(2a-1)$ are positive real numbers and μ shows the reach of Pöschl–Teller potential. The Pöschl–Teller potential is used to explain spectrum vibration and interaction of atomic systems [28]. The Scarf potential is applied to explain the atomic or molecular force [29].

Putting (17) into (16) and simplifying the equation, and if

$$f_{nk} = \frac{U(r)}{r}\Theta(\theta)\phi(\varphi) \tag{18}$$

we have

$$\left[\left(\frac{r^2}{U(r)}\frac{d^2U(r)}{dr^2}+\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right) + \frac{1}{\phi(\varphi)}\frac{1}{\sin^2\theta}\frac{\partial^2\phi(\varphi)}{\partial\phi^2}\right) - \left(r^2\mu^2\left(\frac{\kappa(\kappa-1)}{\sinh^2\mu r} - \frac{\eta(\eta+1)}{\cosh^2\mu r}\right)+\left(\frac{b^2+a(a-1)}{\sin^2\theta} - \frac{2b\left(a-\frac{1}{2}\right)\cos\theta}{\sin^2\theta}\right)\right)(E+M)\right] = -[E^2-M^2]r^2. \tag{19}$$

Separating the variables in (19), we obtain

$$\frac{r^2}{U(r)}\frac{d^2U(r)}{dr^2}-(E+M)r^2\mu^2\left(\frac{\kappa(\kappa-1)}{\sinh^2\mu r}-\frac{\eta(\eta+1)}{\cosh^2\mu r}\right) + [E^2-M^2]r^2-l(l+1)=0, \tag{20a}$$

$$\frac{\sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)-(E+M)\sin^2\theta \times\left(\frac{b^2+a(a-1)}{\sin^2\theta}-\frac{2b\left(a-\frac{1}{2}\right)\cos\theta}{\sin^2\theta}\right) + l(l+1)\sin^2\theta-m^2=0, \tag{20b}$$

$$-\frac{1}{\phi(\varphi)}\frac{\partial^2\phi(\varphi)}{\partial\phi^2}=m^2. \tag{20c}$$

Equation (20c) is well known with its solution

$$\phi_m=\frac{1}{\sqrt{2\pi}}e^{im\varphi},\quad m=0,\pm1,\pm2,\ldots. \tag{21}$$

4. Analytical solution of radial and angular parts of the Dirac equation and some thermodynamical properties

4.1 *Solution of the radial part*

The radial part of the Dirac equation (eq. (20a)) can be solved with approximation,

$$\frac{1}{r^2}\approx\frac{\mu^2}{\sinh^2\mu r}. \tag{22}$$

Substituting eq. (22) into eq. (20a) and simplifying the equation by substituting variable $\cosh^2\mu r = w$, we have

$$w(1-w)\frac{d^2U(r)}{dw^2}+\left(\frac{1}{2}-w\right)\frac{dU(r)}{dw} - \left\{(E+M)\left(\frac{\kappa(\kappa-1)}{4(1-w)}+\frac{\eta(\eta+1)}{4w}\right) + \frac{[E^2-M^2]}{4\mu^2}+\frac{l(l+1)}{4(1-w)}\right\}U(r)=0. \tag{23}$$

Substituting,

$$U(r)=w^{\delta}(1-w)^{\gamma}p(w) \tag{24}$$

into eq. (23), yields

$$w(1-w)p''+p'\left(\left(2\delta+\frac{1}{2}\right)-(2\delta+2\gamma+1)w\right) + p\left(-\frac{(E^2-M^2)}{4\mu^2}-(\delta+\gamma)^2\right)=0, \tag{25}$$

where

$$\delta=\frac{1}{4}-\frac{1}{2}\sqrt{(E+M)(\eta(\eta+1))+1/4}, \tag{26}$$

$$\gamma=\frac{1}{4}+\frac{1}{2}\sqrt{(E+M)(\kappa(\kappa-1))+l(l+1)+1/4}. \tag{27}$$

Equation (25) can be transformed to the differential equation of type AIM,

$$p''+p'\left(\frac{\left(\left(2\delta+\frac{1}{2}\right)-(2\delta+2\gamma+1)w\right)}{w(1-w)}\right) + p\left(\frac{\left(-\frac{(E^2-M^2)}{4\mu^2}-(\delta+\gamma)^2\right)}{w(1-w)}\right)=0. \tag{28}$$

From (28), we have

$$\lambda_0 = \frac{\left((2\delta+2\gamma+1)w - \left(2\delta+\frac{1}{2}\right)\right)}{w(1-w)} \tag{29}$$

$$s_0 = \frac{\left((\delta+\gamma)^2 + \frac{(E^2-M^2)}{4\mu^2}\right)}{w(1-w)}. \tag{30}$$

To have eigenvalue of this equation, we have done the iteration for $\lambda(i)$ and $s(i)$, where i is the iteration number using eq. (3). By using Matlab software, energy eigenvalue can be obtained, with $\varepsilon = (M^2 - E^2)/4\mu^2$,

Δ_0 $\varepsilon_0 = (\delta+\gamma)^2$
Δ_1 $\varepsilon_1 = (2\delta+2\gamma+1) + (\delta+\gamma)^2 = (\delta+\gamma+1)^2$,
Δ_2 $\varepsilon_2 = (4\delta+4\gamma+4) + (\delta+\gamma)^2 = (\delta+\gamma+2)^2$,
Δ_3 $\varepsilon_3 = (6\delta+6\gamma+9) + (\delta+\gamma)^2 = (\delta+\gamma+3)^2$,
Δ_i ....

Relativistic energy eigenvalue can be obtained by generalized Δ_i, which yields

$$\varepsilon = (\delta+\gamma+n_r)^2 = \left(-\frac{1}{2}\sqrt{(E+M)\left(\eta(\eta+1)+\frac{1}{4}\right)} + \frac{1}{2}\sqrt{(E+M)\left(\kappa(\kappa-1)+l(l+1)+\frac{1}{4}\right)} + n_r + \frac{1}{2}\right)^2, \tag{31}$$

where n_r is the radial quantum number ($n_r = 0,1,2,\ldots$) and l is the orbital quantum number which is calculated in §4.2.

In non-relativistic limit $E - M \to E_{NR}$, E_{NR} is the non-relativistic energy, $E + M \to 2W$ where W is the non-relativistic mass, and eq. (31) reduces to

$$E_{NR} = -\frac{1}{2W}\left(-\frac{1}{2}\sqrt{2W\left(\eta(\eta+1)+\frac{1}{4}\right)} + \frac{1}{2}\sqrt{2W\left(\kappa(\kappa-1)+l(l+1)+\frac{1}{4}\right)} + n_r + \frac{1}{2}\right)^2. \tag{32}$$

Equation (32) can be written as

$$E_{NR} = -\frac{1}{2W}(\tau - n_r)^2, \tag{33}$$

where

$$\tau = \left(\frac{1}{2}\sqrt{2W\left(\eta(\eta+1)+\frac{1}{4}\right)} - \frac{1}{2}\sqrt{2W\left(\kappa(\kappa-1)+l(l+1)+\frac{1}{4}\right)} - \frac{1}{2}\right). \tag{34}$$

From non-relativistic energy equation (eq. (33)), some of the thermodynamical properties, such as the vibrational partition function, the vibrational specific heat and the vibrational mean energy can be obtained. The vibrational partition function is defined as [24]

$$Z(\tau,\beta) = \sum_{n=0}^{\tau} e^{\beta\left(\frac{1}{2W}(\tau-n)^2\right)}, \quad \beta = \frac{1}{kT}, \tag{35}$$

where k is the Boltzmann constant, and by setting $(1/\delta^2) = (\beta/2W)$, from (33) and (35) we have

$$Z(\tau,\beta) = \sum_{n=0}^{\tau/\delta} e^{-\{(n-\tau)/\delta\}^2}. \tag{36}$$

In the classical regime when the temperature, T, is high enough, the value of τ is high, and β is small. Therefore, (36) can be written in the integral form as

$$Z(\tau,\beta) = \sum_{n=0}^{\tau/\delta} e^{-y^2} = \delta\int^{\tau/\delta} e^{-y^2}\,dy = \delta\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\frac{\tau}{\delta}\right), \tag{37}$$

where $y = (n-\tau)/\delta$ and erf is the error function given as [33]

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{38}$$

The vibrational specific heat and the vibrational mean energy are defined as [24]

$$U(\beta,\tau) = -\frac{\partial}{\partial\beta}\ln Z(\tau,\beta)$$

and

$$C = -\frac{\partial}{\partial T}U = -k\beta^2\frac{\partial}{\partial\beta}U. \tag{39}$$

By using eqs (37)–(39), we obtain the vibrational mean energy equation as

$$U(\beta,\xi) = -\frac{1}{\beta}\left(-\frac{1}{2} + \left(\left(\frac{1}{\sqrt{\pi}}\frac{\sqrt{\beta}}{\sqrt{2W}} \times \exp\left(-\frac{\tau^2\beta}{2W}\right)\right)\Big/\mathrm{erf}\left(\frac{\tau\sqrt{\beta}}{\sqrt{2W}}\right)\right)\right) \tag{40}$$

and the vibrational specific heat which is obtained from (37)–(40) as

$$C(\xi,\beta) = k\left(\frac{1}{2} - \frac{\left((\sqrt{\beta}/\sqrt{2W\pi})\exp(-\tau^2\beta/2W)\right)}{\mathrm{erf}\left(\tau\sqrt{\beta}/\sqrt{2W}\right)} \times \left(\frac{1}{2}+\frac{\tau^2\beta}{2W}\right) - \frac{\left((\beta\sqrt{\beta}/\sqrt{2W\pi})\exp(-\tau^2\beta/W)\right)}{\left(\mathrm{erf}\left(\tau\sqrt{\beta}/2W\right)\right)^2}\right). \tag{41}$$

Thermodynamical properties are very usefull for the characterizations or fabrications of materials.

Radial wave function can be obtained using eqs (5)–(8). From eqs (6)–(8), we obtain

$$c = \frac{2\delta - 3/2}{2}, \quad N = -1,$$
$$t = \frac{2\gamma + 1/2}{2}, \quad b = 1.$$

So,

$$\sigma = \frac{2c + N + 3}{N + 2} = 2\delta + \frac{1}{2},$$
$$p = \frac{(2c + 1)b + 2t}{(N + 2)b} = 2\delta + 2\gamma.$$

And then from (5), we have

$$p(w) = (-1)^{n_\mathrm{r}} C_2 (1)^{n_\mathrm{r}} \left(2\delta + \frac{1}{2}\right)_{n_\mathrm{r}} \times {}_2F_1\left(-n_\mathrm{r}, 2\delta + 2\gamma + n_\mathrm{r}, 2\delta + \frac{1}{2}, w\right). \quad (42)$$

By substituting (42) into (24), we have the radial wave function,

$$U(w) = w^{\delta}(1 - w)^{\gamma}(-1)^{n_\mathrm{r}} C(n_\mathrm{r})(1)^{n_\mathrm{r}} \left(2\delta + \frac{1}{2}\right)_{n_\mathrm{r}} \times {}_2F_1\left(-n_\mathrm{r}, 2\delta + 2\gamma + n_\mathrm{r}, 2\delta + \frac{1}{2}, w\right), \quad (43)$$

where $w = \cosh^2 \mu r$. So

$$U(r) = (\cosh^2(\mu r))^{\delta}(-\sinh^2(\mu r))^{\gamma} \times (-1)^{n_\mathrm{r}} C(n_\mathrm{r})(1)^{n_\mathrm{r}} \left(2\delta + \frac{1}{2}\right)_{n_\mathrm{r}} \times {}_2F_1\left(-n_\mathrm{r}, 2\delta + 2\gamma + n_\mathrm{r}, 2\delta + \frac{1}{2}, \cosh^2(\mu r)\right), \quad (44)$$

where $C(n_\mathrm{r})$ is the radial normalization constant, ${}_2F_1$ is the hypergeometric function and $(2\delta + 1/2)_{n_\mathrm{r}}$ is the Pochamer symbol. Equation (44) is the unnormalized radial wave function.

4.2 *Solution of the angular part*

From the angular part in (20b), we can obtain orbital quantum number. By multiplying (20b) with $\Theta(\theta)/\sin\theta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right) - (E + M)\sin\theta \times \left(\frac{b^2 + a(a-1)}{\sin^2\theta} - \frac{2b\left(a - \frac{1}{2}\right)\cos\theta}{\sin^2\theta}\right) \times \Theta(\theta) + l(l+1)\sin\theta\,\Theta(\theta) - m^2 \frac{\Theta(\theta)}{\sin\theta} = 0. \quad (45)$$

Equation (45) must be simplified by using the parameter $\Theta = H(\theta)/\sqrt{\sin\theta}$, $\cos\theta = 1 - 2u$ and by simplying it, we get

$$u(1-u)\frac{\mathrm{d}^2 H(\theta)}{\mathrm{d}u^2} + \left(\frac{1}{2} - u\right)\frac{\mathrm{d}H(\theta)}{\mathrm{d}u} - \left\{(E + M) \times \left(\frac{b^2 + a(a-1)}{4u(1-u)} - \frac{2b\left(a - \frac{1}{2}\right)(1 - 2u)}{4u(1-u)}\right) + \frac{m^2 - \frac{1}{4}}{4u(1-u)}\right\} H(\theta) + \left(l(l+1) + \frac{1}{4}\right) H(\theta) = 0 \quad (46)$$

By using

$$H(\theta) = u^{\alpha}(1-u)^{\beta} q(u) \quad (47)$$

and simplying it, eq. (46) can be transformed to the hypergeometric differential equation:

$$u(1-u)q'' + q'\left(\left(2\alpha + \frac{1}{2}\right) - (2\alpha + 2\beta + 1)u\right) + q\left(\left(l + \frac{1}{2}\right)^2 - (\alpha + \beta)^2\right) = 0, \quad (48)$$

where

$$\alpha = \frac{1}{4} + \frac{1}{2}\sqrt{(E + M)\left(\left(\left(a - \tfrac{1}{2}\right) - b\right)^2 - \tfrac{1}{4}\right) + m^2}, \quad (49)$$

$$\beta = \frac{1}{4} + \frac{1}{2}\sqrt{(E + M)\left(\left(\left(a - \tfrac{1}{2}\right) + b\right)^2 - \tfrac{1}{4}\right) + m^2}. \quad (50)$$

m is the magnetic quantum number.

Equation (48) is the hypergeometric differential equation and so we must transform it to AIM-type equation by dividing (48) with $u(1-u)$, to get

$$q'' + q'\frac{\left(\left(2\alpha + \frac{1}{2}\right) - (2\alpha + 2\beta + 1)u\right)}{u(1-u)} + q\frac{\left(\left(l + \frac{1}{2}\right)^2 - (\alpha + \beta)^2\right)}{u(1-u)} = 0. \quad (51)$$

From (51), we have

$$\lambda_0 = \frac{((2\alpha+2\beta+1)u) - \left(2\alpha+\frac{1}{2}\right)}{u(1-u)} \tag{52}$$

$$s_0 = \frac{\left((\alpha+\beta)^2 - \left(l+\frac{1}{2}\right)^2\right)}{u(1-u)}. \tag{53}$$

To get eigenvalue of this equation, we have done the iteration for $\lambda(i)$ and $s(i)$, where i is the iteration number using eq. (3). By using Matlab software, eigenvalues can be obtained:

$$\Delta_0 = s_0\lambda_1 - s_1\lambda_0 = 0,\ \text{yields } (\alpha+\beta)^2 = \left(l+\frac{1}{2}\right)^2$$

$$\Delta_1 = s_1\lambda_2 - s_2\lambda_1 = 0,\ \text{yields } (\alpha+\beta+1)^2 = \left(l+\frac{1}{2}\right)^2$$

$$\Delta_2 = s_2\lambda_3 - s_3\lambda_2 = 0,\ \text{yields } (\alpha+\beta+2)^2 = \left(l+\frac{1}{2}\right)^2$$

$$\Delta_i \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

Eigenvalue can be obtained by the generalized Δ_i, which yields

$$l = \left(\alpha+\beta+n_l-\frac{1}{2}\right), \tag{54}$$

where l is the orbital quantum number and n_l is the angular quantum number. Then, angular wave function can be obtained by using eqs (5)–(8). We have,

$$c = \frac{2\alpha-\frac{3}{2}}{2}, \quad N=-1, \quad t=\frac{2\beta+\frac{1}{2}}{2}, \quad b=1.$$

So

$$\sigma = \frac{2c+N+3}{N+2} = 2\alpha+\frac{1}{2},$$
$$p = \frac{(2c+1)b+2t}{(N+2)b} = 2\alpha+2\beta.$$

From (5), we have

$$q(u) = (-1)^{n_l} C_2 (1)^{n_l} \left(2\alpha+\frac{1}{2}\right)_{n_l} \times {}_2F_1\left(-n_l, 2\alpha+2\beta+n_l, 2\alpha+\frac{1}{2}, u\right). \tag{55}$$

By substituting eq. (55) into eq. (47), we obtain

$$H(u) = u^{\alpha}(1-u)^{\beta}(-1)^{n_l} C_2(1)^{n_l}\left(2\alpha+\frac{1}{2}\right)_{n_l} \times {}_2F_1\left(-n_l, 2\alpha+2\beta+n_l, 2\alpha+\frac{1}{2}, u\right),$$

where $u = -\frac{1}{2}(\cos\theta - 1)$, and so the angular part of the unnormalized wave function can be obtained as follows:

$$H(\theta) = \left(\frac{1}{2}-\frac{1}{2}\cos\theta\right)^{\alpha}\left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)^{\beta} \times (-1)^{n_l} C_{n_l}(1)^{n_l}\left(2\alpha+\frac{1}{2}\right)_{n_l} \times {}_2F_1\left(-n_l, 2\alpha+2\beta+n_l, 2\alpha+\frac{1}{2}, \left(\frac{1}{2}-\frac{1}{2}\cos\theta\right)\right). \tag{56}$$

C_{n_l} is the angular normalization constant.

5. Results and discussion

In this section, we discuss results obtained in the previous section. From relativistic energy equation (eq. (31)) and orbital quantum number equation (eq. (54)), and by using Matlab software we have numeric solutions of relativistic energy which are listed in tables 1 and 2 with parameters $\kappa = 2$, $\eta = 1.5$, $a = 0.35$, $b = 0.65$ and $M = 5\ \text{fm}^{-1}$. Table 1 gives the relativistic

Table 1. Relativistic energy corresponding to several states of a particle under the influence of modified Pöschl–Teller potential and trigonometric Scarf II potential.

			$E_{n_r n_l m}$ (fm^{-1})		
n_r	n_l	m	$\mu = 0.3$ fm^{-1}	$\mu = 0.6$ fm^{-1}	$\mu = 0.9$ fm^{-1}
1	0	0	4.9777	4.9098	4.7922
1	1	0	4.9467	4.8560	4.6639
1	1	1	4.9519	4.8022	4.5318
2	0	0	4.8834	4.5072	3.7549
2	1	0	4.8549	4.3772	3.3521
2	1	1	4.8293	4.2555	2.9080

Table 2. Relativistic energy corresponding to several states of a particle under the influence of modified Pöschl–Teller potential.

			$E_{n_r n_l m}$(fm^{-1})		
n_r	n_l	m	$\mu = 0.3$ fm^{-1}	$\mu = 0.6$ fm^{-1}	$\mu = 0.9$ fm^{-1}
1	0	0	4.9834	4.9328	4.8459
1	1	0	4.9776	4.9091	4.7902
1	1	1	4.9645	4.8545	4.6587
2	0	0	4.8971	4.5670	3.9205
2	1	0	4.8830	4.5036	3.7331
2	1	1	4.8542	4.3697	3.2940

energy values with non-central potential and table 2 gives relativistic energy values without non-central potential. Tables 1 and 2 show that the relativistic energy values are always positive with and without the non-central potential. So, the non-central potential does not have any effect on the energy spectra in the spin symmetry of the Dirac equation. The positive values of the energy are taken due to the spin symmetry limit and the values agree with the values obtained by Falaye and Oyewumi [31] and Aydogdu *et al* [32].

By inspecting tables 1 and 2, we can see that increase of μ and n_{r} causes a decrease in energy eigenvalue. This agrees with the research by Aysiah [30] and Kurniawan [2]. In the non-relativistic system, the relativistic energy equation (31) is reduced to the non-relativistic energy equation (32) and for particles which are governed by modified Pöschl–Teller potential, the energy decreases with increasing radial quantum number n_{r}. The sign of non-relativistic energy in (32) is negative, and then if n_{r} is increased, there will be greater energy reduction.

The dimension of the parameter μ is inverse of the distance in space that describes the reach of Pöschl–Teller potential. If μ is increased, it means that the potential reach is smaller in space and it causes a decrease in energy.

Table 3 presents the relativistic energy with variation potential constants κ, η, a and b. By inspecting table 3, we can see that increase of potential constants κ and η causes increase in energy at E_1 and E_2, while increase in potential constants a and b, causes decrease in energy.

By varying the potential constants, we obtained some radial wave functions which are shown in figure 1. Unnormalized radial wave functions for particle under the influence of modified Pöschl–Teller potential and Scarf II potential are affected by potential constants κ, η, a, b and μ. The dimension of μ is inverse of the distance in space that describes the reach of Pöschl–Teller potential. If μ is enlarged, physically it means that the potential reach is smaller in space. By inspecting figure 1, we can see that due to the increase in the value of μ, particle moves further away from the nucleus and also that changes in radial wave functions are affected by the potential constants κ, η, a and b.

From figure 1, we can conclude that when n_r increases, the distance (r) of the particles increases from the core, i.e., radial wave functions diverge as r increases.

For the angular part of the unnormalized wave function in (56), we compare two spinor wave functions with the same angular quantum number $n_l = 5$ but different values of α and β corresponding to the parameters a, b and m in figure 2. Two- and three-dimensional plots are displayed. In figure 2, $\alpha < \beta$, where α is a constant, but the value of β in figures 2c and 2d is greater than value of β in figures 2a and 2b. So figure 2c has greater amplitude than figure 2a. For the same value of n_l, the waveform remains the same.

Angular wave functions for different n_l are shown in figure 3. From (56), with $n_{\mathrm{r}} = 1$, $m = 2$, $\mu = 0.9\ \mathrm{fm}^{-1}$, $a = 0.25$, $b = 0.75$, $\kappa = 2$, $\eta = 1.5$, we have angular wave function in figure 3a for $n_l = 0$, figure 3b for $n_l = 1$ and figure 3c for $n_l = 2$. By inspecting figure 3, we can see that the variation of n_l influences the shape of the orbital distribution probability in space. Figure 3 indicates that the increase in n_l causes an increase in the amplitude of the wave.

Table 3. Relativistic energy in fm^{-1} with $n_l = 2$, $m = 2$, $M = 5\ \mathrm{fm}^{-1}$, $\mu = 0.3\ \mathrm{fm}^{-1}$, for particles under the influence of modified Pöschl-Teller potential and trigonometric Scarf II potential for different values of κ, η, a and b.

κ	η	a	b	E_0	E_1	E_2
2	1.5	0	0	4.9870	4.9061	4.7467
2	1.5	1	1	4.9520	4.8296	4.6250
2	1.5	2	2	4.7736	4.4554	4.2201
1	1	0.25	0.75	4.9739	4.8741	4.6933
2	2	0.25	0.75	4.9999	4.9601	4.8445
3	3	0.25	0.75	4.9963	4.9832	4.8960

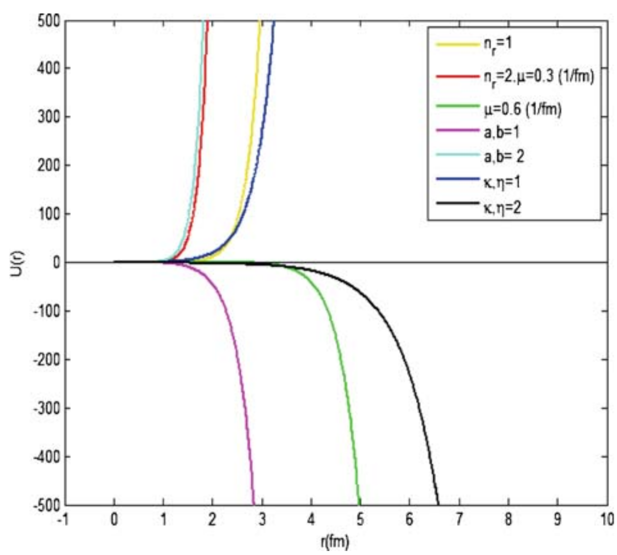

Figure 1. Unnormalized wave functions with $n_l = 1$, $m = 1$, $M = 5\ \mathrm{fm}^{-1}$ for particles under the influence of modified Pöschl–Teller potential and trigonometric Scarf II potential, under different values of n_{r}, μ, κ, η, a and b.

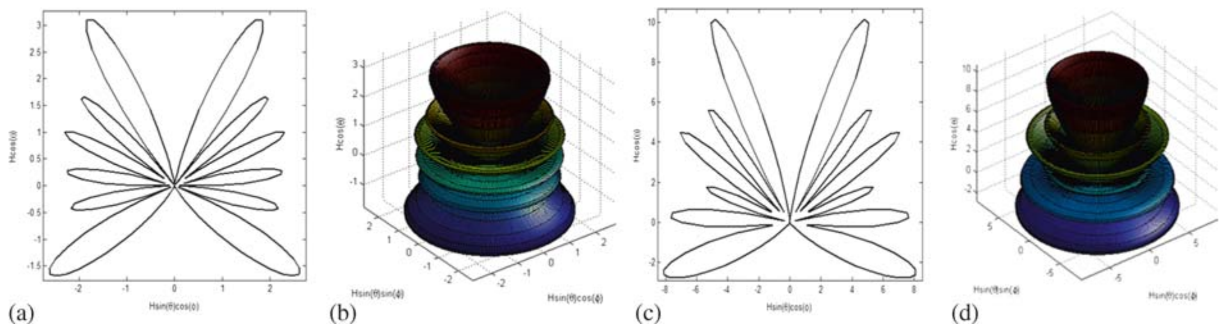

Figure 2. (**a**), (**c**) Two-dimensional and (**b**), (**d**) three-dimensional angular wave function unnormalization for a state $n_l = 5$. In panels (**a**) and (**b**), $H(\theta) = \frac{(-81.83(1-\cos\theta)^{2.75}(1+\cos\theta)^{5.7796}(1-9.19(1-\cos\theta)+30.28(1-\cos\theta)^2-45.53(1-\cos\theta)^3+31.69(1-\cos\theta)^4-8.26(1-\cos\theta)^5))}{\sqrt{\sin\theta}}$; in panels (**c**) and (**d**), $H(\theta) = \frac{(-53.43(1-\cos\theta)^{2.75}(1+\cos\theta)^{9.1445}(1-11.99(1-\cos\theta)+51.05(1-\cos\theta)^2-98.23(1-\cos\theta)^3+86.74(1-\cos\theta)^4-28.44(1-\cos\theta)^5))}{\sqrt{\sin\theta}}$.

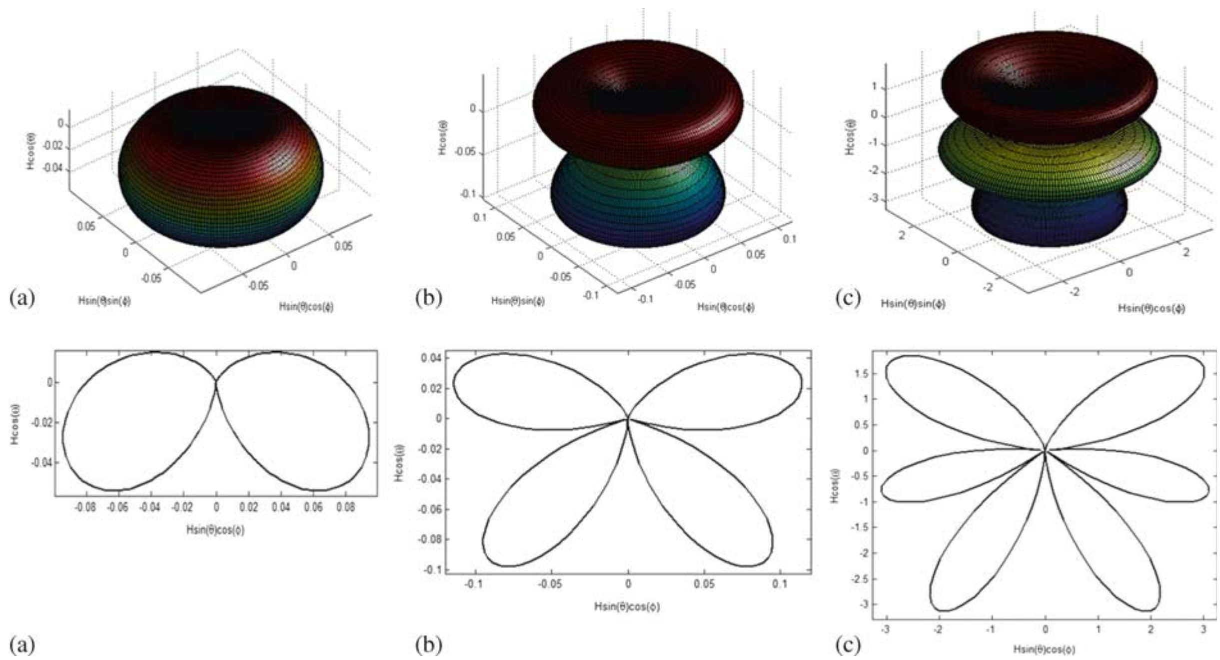

Figure 3. Three-dimensional and two-dimensional (coordinate spherical) angular wave function unnormalization for (**a**) $n_l = 0$, (**b**) $n_l = 1$, (**c**) $n_l = 2$.

Pöschl–Teller potential can be used to obtain some thermodynamical properties such as the vibrational partition function (eq. (37)), the vibrational mean energy (eq. (40)) and the vibrational specific heat (eq. (41)). We can get thermodynamical properties of some diatomic molecules from that equations.

6. Concluding remarks

In this paper, we study the Dirac equation for spin-$1/2$ particle in the modified Pöschl–Teller potential and the trigonometric Scarf II non-central potential under conditions of spin symmetry. The radial part of the unnormalized spinor wave function is obtained approximately from (44) and the angular part in (56). The results show that the disturbance of the modified Pöschl–Teller potential and trigonometric Scarf II potential changes the wave functions of the radial and the angular parts. Relativistic energy eigenvalue can be obtained via AIM in eq. (31) and equation of orbital quantum number l in (54). In both equations, the quantum numbers are interrelated. Energy spectrum also is solved numerically using the Matlab software, where the increase in the radial quantum number n_r causes decrease in the energy spectrum. The positive value of energy is taken due to the spin symmetry limit. Some thermodynamical properties are obtained from non-relativistic energy equation using erf functions.

Acknowledgements

This research was partly supported by Higher Education Project Grant with Contract No. 698/UN27/PN/2015.

References

[1] W Greiner, *Relativistic quantum mechanics, wave equations* 3rd edn (Springer-Verlag, Berlin, 2000)
[2] A Kurniawan, A Suparmi and C Cari, *Chin. Phys. B* **24**, 030302 (2015)
[3] M Hamzavi and A A Rajabi, *Advances in high energy physics* (Hindawi Publishing Corporation, 2013) Vol. 2013, Article ID: 196986
[4] J N Ginocchio, A Leviatan, J Meng and S G Zhou, *Phys. Rev. C* **69**, 034303 (2004)
[5] J N Ginocchio, *Phys. Rep.* **414**, 165 (2005)
[6] J N Ginocchio, *Phys. Rep. C* **69**, 034313 (2004)
[7] J N Ginocchio, *Phys. Rev. Lett.* **78**, 436 (1997)
[8] J Meng, K Sugawara-Tanabe, S Yamaji and A Arima, *Phys. Rev. C* **59**, 154 (1999)
[9] J Dudek, W Nazarewicz, Z Szymanski and G A Leander, *Phys. Rev. Lett.* **59**, 1405 (1987)
[10] D Troltenier, C Bahri and J P Draayer, *Nucl. Phys. A* **586**, 53 (1995)
[11] H Liang, J Meng and S G Zhou, *Phys. Rep.* **570**, 1 (2015)
[12] A Suparmi and C Cari, *J. Math. Fund. Sci.* **46**, 205 (2014)
[13] N Fitriani and C Cari, *Prosiding Seminar Nasional dan Pendidikan Sains IX, Fakultas Sains dan Matematika, UKSW*, June 21, 2014, Salatiga, p. 201
[14] I Saraswati, A Suparmi, C Cari and Ihtiari, *Seminar Nasional 2nd Lontar Physics Forum 2013*, Lontar, p. 1357-1 (2013)
[15] V D Aryanthy, A Suparmi, C Cari and I Saraswati, *Seminar Nasional 2nd Lontar Physics Forum 2013*, Lontar, p. 1340-1 (2013)
[16] A U Ortakaya, S H Hassanabadi and B H Yazarloo, *Chin. Phys. B* **23**, 3 (2014)
[17] O Aydogdu and R Sever, *Eur. Phys. J. A* **43**, 73 (2010)
[18] S G Zhou, J Meng and P Ring, *Phys. Rev. Lett.* **91**, 26 (2003)
[19] A Suparmi, C Cari and U A Deta, *Chin. Phys. B* **23**, 090304 (2014)
[20] A N Ikot, S Zarrinkamar, B H Yazarloo and H Hassanabadi, *Chin. Phys. B* **23**, 100306 (2014)
[21] H Ciftci, R I Hall and N Saad, arXiv:0505069v1 [math ph] (2005)
[22] K J Oyewumi, B J Falaye, C A Onate, O J Oluwadare and W A Yahya, arXiv:1402.2181v1 [quant-ph] (2014)
[23] M Eshghi and S M Ikhdair, *Chin. Phys. B* **23**, 120304 (2014)
[24] S M Ikhdair, arXiv:1308.0155v1 [quant-ph] (1 Aug. 2013)
[25] A Rostami and H Motavali, *Progr. Electromagn. Res. B* **4**, 171 (2008)
[26] A Soylu, O Bayrak and I Boztosun, *J. Phys. A* **41**, 065308 (2008)
[27] B J Falaye, M Hamzavi and S M Ikhdair, arXiv:1207.1218v [nucl-th] (2012)
[28] A Suparmi, C Cari, H Yuliani and D Yuniati, *J. Fisika Indonesia NO: 51* XVII, 41 (2013)
[29] Nurhayati, Suparmi, V I Variani, C Cari and Wahyudi, *Prosiding Pertemuan Ilmiah XXVI HFI Jateng & DIY*, April 14, 2012 Purworejo.p.229 (2012)
[30] M Abramowitz and I A Stegun (eds) *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 9th edn (Dover, New York, 1972) p. 295 and 297
[31] B J Falaye and K J Oyewumi, arXiv:1111.6501v2 [quant-ph] (2011)
[32] O Aydogdu, G Maghsoodi and H Hassanabadi, *Chin. Phys. B* **22**, 010302 (2013)
[33] D Aysiah, *Penyelesaian persamaan Dirac untuk potensial non-sentral Poschl-Teller hiperbolik termodifikasi-q plus Manning-Rosen untuk simetri spin dengan metode Nikivorof-Uvarof* (S.Si. Skripsi) (Surakarta: Sebelas Maret University) (in Indonesia) (2014)