Pramana – J. Phys. (2017) 88: 22
DOI 10.1007/s12043-016-1330-7

Minimally coupled scalar field cosmology in anisotropic cosmological model

C P SINGH* and MILAN SRIVASTAVA

Department of Applied Mathematics, Delhi Technological University (Formerly Delhi College of Engineering), Bawana Road, Delhi 110 042, India
*Corresponding author. E-mail: cpsphd@rediffmail.com

MS received 23 January 2016; revised 9 May 2016; accepted 17 June 2016; published online 3 January 2017

Abstract. We study a spatially homogeneous and anisotropic cosmological model in the Einstein gravitational theory with a minimally coupled scalar field. We consider a non-interacting combination of scalar field and perfect fluid as the source of matter components which are separately conserved. The dynamics of cosmic scalar fields with a zero rest mass and an exponential potential are studied, respectively. We find that both assumptions of potential along with the average scale factor as an exponential function of scalar field lead to the logarithmic form of scalar field in each case which further gives power-law form of the average scale factor. Using these forms of the average scale factor, exact solutions of the field equations are obtained to the metric functions which represent a power-law and a hybrid expansion, respectively. We find that the zero-rest-mass model expands with decelerated rate and behaves like a stiff matter. In the case of exponential potential function, the model decelerates, accelerates or shows the transition depending on the parameters. The isotropization is observed at late-time evolution of the Universe in the exponential potential model.

Keywords. Cosmology; anisotropic models; scalar field.

PACS Nos 98.80.−Jk; 98.80.−cq; 04.20.−Jb

1. Introduction

Usually, it is assumed that the Universe is filled with a perfect fluid. But observations suggest that cosmological dynamics cannot be fully explained by this standard matter. The observational results lead to the search of some kinds of exotic matter which would generate sufficient negative pressure to drive the late-time cosmic acceleration. One such exotic matter is the scalar field which provides the necessary negative pressure causing acceleration [1,2]. Thus, the cosmological models with scalar field play a vital role in the current modern cosmology to explain the early inflation and the late-time acceleration. The recent discovery of cosmic acceleration [3–7] has stimulated the interest to study cosmological models based on scalar fields. The cosmological models based on scalar fields have been discussed by many researchers for explaining the possible early inflationary scenarios [8] as well as for describing the dark energy [9].

Inflationary models are important in solving several outstanding problems in cosmology like homogeneity, flatness of the observed Universe etc. The dynamics of the evolution of the Universe is often realized by scalar field with a proper scalar potential. The self-interacting potential can act as an effective cosmological constant which drives a period of inflation. It depends on the specific form of the potential as a function of scalar field. Many researchers [10–19] have studied the scalar field cosmology in Friedmann–Robertson–Walker (FRW) model with different forms of scalar potentials like flat, constant and exponential potentials.

A physical entity which provides an accelerated expansion at late-time is commonly known as dark energy. The simplest field having this property is a canonical scalar field with a scalar potential. Nowadays, the scalar field cosmology has acquired a great popularity to analyse the dark energy of the Universe. So far, a large class of scalar field dark energy models have been studied including quintessence [20–23], a phantom field [24,25], quintom [26,27], k-essence [28], tachyon [29] and so forth.

It is well known that the evolution of the Universe admits a scenario of anisotropic expansion and gains a lot of interest, under the light of the recently announced Planck Probe results [7]. The Bianchi models, which describe homogeneous but anisotropic space–times, have been discussed to explain the significance of anisotropy in the cosmic microwave background (CMB) and large scale structures (LSS) [30–32]. Therefore, motivated by the anomalies found in the CMB anisotropies [33], which violate the statistical isotropy [34], and on the increasing interest on Bianchi cosmologies [35,36], we are interested to investigate the dynamics of a perfect fluid anisotropic Bianchi-I (B-I) cosmological model with a scalar field minimally coupled with gravity. Demianski *et al* [37] have studied the dynamics of anisotropic model filled with scalar field minimally coupled to gravity. Saha and Boyadjiev [38] have considered a self-consistent system of interacting spinor and scalar fields within the framework of a B-I cosmological model filled with perfect fluid. Do *et al* [39] have studied anisotropic power-law inflation for a two-scalar fields model. Sharif and Zubair [40] have studied the behaviour of perfect fluid and massless scalar field for homogeneous and anisotropic B-I Universe model in $f(R, T)$ gravity.

The exact solutions of the field equations play very important roles in cosmology, because they allow us to analyse the qualitative and quantitative behaviour of the Universe as a whole. The number of exact solutions based on scalar field with various forms of scalar potential is limited. It has been observed that in a cosmological model the scalar potential has very important role as it radically affects the cosmological behaviour. Therefore, in this paper, our motivation is to find exact cosmological solutions for a totally homogeneous and anisotropic perfect fluid B-I model with scalar field for various forms of scalar potential. We calculate various observable cosmological parameters like expansion scalar, deceleration parameter (DP), anisotropy parameter, shear scalar, equation of state parameters of scalar field and perfect fluid matter, and the isotropization measure to analyse the dynamics of the evolution of the Universe. We observe that the perfect fluid anisotropic models behave like an isotropic one with the inclusion of exotic matter like scalar field at late-time evolution. The results found in this paper are new and very interesting to discuss the deceleration, acceleration and their transitions. Fadragas *et al* [41] have analysed the anisotropic locally-rotationally-symmetric Bianchi models with scalar field for a wide range of potentials. We extend this work for a totally anisotropic and homogeneous B-I model by assuming two different forms of scalar potential.

The paper is organized as follows: In §2 we present the action of scalar field cosmology in anisotropic space–time metric and its field equations. The various kinematical and physical cosmological parameters are also mentioned. In §3 we discuss the solution of the field equations with two different forms of scalar potential. Section 3 is further divided into two subsections. We obtain the solution of field equations with zero potential and an exponential potential, respectively in §3.1 and 3.2 and discuss the behaviour of each model. Finally, §4 contains the conclusion of the work.

2. Action and the field equations

The action of a Universe constituted with matter and a scalar field minimally coupled to gravity, in the system of units $8\pi G = 1 = c$, reads [20] as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \mathcal{L}_\phi + \mathcal{L}_m\right], \tag{1}$$

where g is the determinant of the metric, R is the Ricci scalar, $\mathcal{L}_m$ is the total matter content in the Universe (including cold dark matter (CDM)) and $\mathcal{L}_\phi$ is the scalar field Lagrangian which is defined as

$$\mathcal{L}_\phi = -\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi), \tag{2}$$

where ϕ is the scalar field and $V(\phi)$ is the self-interacting scalar potential. The fluid energy density ρ_m and scalar field ϕ are functions of a time-like coordinate t, and therefore, the variation of S with respect to metric g_{ij} leads to the following gravitational field equations:

$$R_{ij} - \frac{1}{2}g_{ij}R = T_{ij}^{(m)} + T_{ij}^{(\phi)}, \tag{3}$$

where $T_{ij}^{(m)}$ is the energy–momentum tensor of matter which is assumed to be a perfect fluid

$$T_{ij}^{(m)} = (p_m + \rho_m)u_i u_j + p_m g_{ij}, \tag{4}$$

with energy density ρ_m, pressure p_m and its equation of state parameter $w_m = p_m/\rho_m$. In eq. (3), $T_{ij}^{(\phi)}$ is the energy–momentum tensor associated with the scalar field ϕ, which is given by

$$T_{ij}^{(\phi)} = \nabla_i\phi\nabla_j\phi - \left[\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi + V(\phi)\right]g_{ij}. \tag{5}$$

The scalar field itself obeys the wave equation

$$g^{ij}\nabla_i\nabla_j\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0. \quad (6)$$

In what follows we consider a spatially homogeneous and anisotropic B-I space–time described by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + B^2(t)\mathrm{d}y^2 + C^2(t)\mathrm{d}z^2, \quad (7)$$

where $A(t)$, $B(t)$ and $C(t)$ are directional scale factors.

With the above assumptions, the Einstein's field equations (3) give the following equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -(p_m + p_\phi), \quad (8)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -(p_m + p_\phi), \quad (9)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(p_m + p_\phi), \quad (10)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = (\rho_m + \rho_\phi), \quad (11)$$

where an overdot indicates a derivative with respect to time t. Here, ρ_ϕ and p_ϕ are respectively the energy density and pressure for the canonical scalar field, which are given by

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \quad (12)$$

and its equation of state parameter is given by

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \quad (13)$$

The quintessence cosmological model accommodates a late-time cosmic acceleration when $\omega_\phi < -1/3$ which implies that $\dot{\phi}^2 < V(\phi)$. On the other hand, if the kinetic term of the scalar field is negligible with respect to the potential energy, i.e., $\dot{\phi}^2 \ll 2V(\phi)$, then the EoS is $\omega_\phi = -1$.

It is important to notice that the usual relation $T^{ij}_{\;;j} = 0$, establishing the conservation laws satisfied by the matter fields, holds true. This follows from the assumption that all matter fields are minimally coupled to the metric g_{ij}, which means that the principle of equivalence is guaranteed. Assuming negligible interaction between the matter and the scalar field, the conservation equation for a perfect fluid leads to

$$\dot{\rho_m} + 3H(1 + \omega_m)\rho_m = 0 \quad (14)$$

and the evolution equation of the scalar field (6) gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0 \quad (15)$$

which is equivalently written as $\dot{\rho_\phi} + 3H(1+\omega_\phi)\rho_\phi = 0$.

We now define the average scale factor for the anisotropic model [42,43] as

$$a(t) = (ABC)^{1/3}. \quad (16)$$

The dynamics of the cosmological evolution is characterized by the Hubble parameter. Therefore, in analogy with Hubble parameter in FRW model, we introduce a generalized mean Hubble parameter H as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \quad (17)$$

The directional Hubble parameters which determine the expansion rate of the Universe along x, y and z-axes are given by [42,43]

$$H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B}, \quad H_3 = \frac{\dot{C}}{C}. \quad (18)$$

Now, we introduce the kinematical parameters of observational interest in cosmology such as the expansion scalar θ, the deceleration parameter q, the measure of anisotropy parameter A_p and the shear scalar σ^2. We can straightforwardly define these parameters as

$$\theta = u^i_{;\,i} = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \quad (19)$$

$$q = -1 - \frac{\dot{H}}{H^2}, \quad (20)$$

$$A_p = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \quad (21)$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{\theta^2}{6}. \quad (22)$$

The shear tensor σ^{ij} is defined as

$$\sigma^{ij} = \frac{1}{2}(u_{\mu;\alpha}h^\alpha_\nu + u_{\nu;\alpha}h^\alpha_\mu) - \frac{1}{3}\theta h_{\mu\nu},$$

where $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the projection tensor. The behaviour of the Universe models is determined by the sign of q. The positive value of deceleration parameter suggests a decelerating model while the negative value indicates inflation.

Using eqs (16)–(22), the field equations (8)–(11) can be rewritten in terms of H, q and σ^2 as

$$\rho_m = 3H^2 - \sigma^2 - \rho_\phi, \quad (23)$$

$$p_m = H^2(2q - 1) - \sigma^2 - p_\phi. \quad (24)$$

In the following section we shall try to find the solutions of the above field equations by assuming suitable physical assumptions.

3. Solution of field equations

Following the method described in [44,45], eqs (8) and (9) give

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{x_1}{ABC}, \tag{25}$$

where x_1 is a constant of integration. The solution of eq. (25) can be written as

$$\frac{A}{B} = d_1 \exp\left(x_1 \int (ABC)^{-1} \mathrm{d}t\right), \tag{26}$$

where d_1 is another constant of integration. Analogously, from eqs (8) and (10), and eqs (9) and (10), we get

$$\frac{A}{C} = d_2 \exp\left(x_2 \int (ABC)^{-1} \mathrm{d}t\right), \tag{27}$$

$$\frac{B}{C} = d_3 \exp\left(x_3 \int (ABC)^{-1} \mathrm{d}t\right), \tag{28}$$

where d_2, d_3 and x_2, x_3 are constants of integration. Using eq. (16) into eqs (26)–(28), we get the metric functions as

$$A(t) = l_1\, a(t) \exp\left(m_1 \int a^{-3} \mathrm{d}t\right), \tag{29}$$

$$B(t) = l_2\, a(t) \exp\left(m_2 \int a^{-3} \mathrm{d}t\right), \tag{30}$$

$$C(t) = l_3\, a(t) \exp\left(m_3 \int a^{-3} \mathrm{d}t\right), \tag{31}$$

where

$$l_1 = \sqrt[3]{d_1 d_2}, \quad l_2 = \sqrt[3]{d_1^{-1} d_3}, \quad l_3 = \sqrt[3]{d_2 d_3^{-1}}$$

and

$$m_1 = \frac{x_1 + x_2}{3}, \quad m_2 = \frac{x_3 - x_1}{3}, \quad m_3 = -\frac{x_3 + x_2}{3}.$$

The constants $l_1,\ l_2,\ l_3$, and $m_1,\ m_2,\ m_3$ satisfy the following relations:

$$l_1 l_2 l_3 = 1 \quad \text{and} \quad m_1 + m_2 + m_3 = 0. \tag{32}$$

The unknown quantities of the problem are A, B, C, p_m, ρ_m, ϕ and $V(\phi)$ whereas we have only five equations available, namely eqs (8)–(11) and (15). Therefore, to solve the system of differential equations we need to assume two more relations among the unknown quantities. Thus, for any arbitrary average scale factor a and scalar field potential $V(\phi)$, eq. (15) gives $\phi(t)$ and eqs (29)–(31) give $A(t)$, $B(t)$ and $C(t)$. In this paper we assume a relationship between the average scale factor a and scalar field ϕ of the form [46,47]

$$a = a_0 \mathrm{e}^{\alpha\phi(t)}, \tag{33}$$

where a_0 represents the average scale factor at the present time and α is an arbitrary positive constant. Thus, the Hubble parameter simply gives

$$H = \alpha\dot{\phi}(t). \tag{34}$$

To illustrate our analysis, we assume a specific form of $V(\phi)$ in the following subsection. In the literature, due to the unknown nature of dark energy, there are various forms of this potential (for a detailed review, see [48–55]) which describe the physical features of the scalar field cosmology. Therefore, as long as potential $V(\phi)$ is given, we can solve other geometrical and physical parameters correspondingly.

3.1 *Solution with zero potential*

In this section, we consider the case where the scalar potential is zero [53,54], that is,

$$V(\phi) = 0. \tag{35}$$

Using eqs (34) and (35), eq. (15) reduces to

$$\ddot{\phi} + 3\alpha\dot{\phi}^2 = 0. \tag{36}$$

The solution of eq. (36) is given by

$$\phi = \frac{1}{3\alpha} \ln[3\alpha(c_1 t + t_0)], \tag{37}$$

where c_1 and t_0 are positive constants of integration. We observe that at the beginning, i.e., at $t = -(t_0/c_1)$, $\phi \to -\infty$. It increases with time and tends to $+\infty$ at $t \to \infty$. Thus, the kinetic energy vanishes at the end of the evolution (an infinite expansion).

Using eq. (37) into eq. (33), the average scale factor in terms of t for an expanding Universe is given by

$$a = a_0[3\alpha(c_1 t + t_0)]^{1/3}. \tag{38}$$

This is of the power-law form $a \propto t^r$ which represents a generalized inflation for $r > 1$. But, in our case we have $r = 1/3 < 1$. Therefore, the model will expand with decelerated rate.

Using eq. (38) into the metric functions (29)–(31), we get

$$A(t) = a_0 l_1 [3\alpha(c_1 t + t_0)]^{\frac{1}{3}\left(1 + \frac{m_1}{\alpha c_1 a_0^3}\right)}, \tag{39}$$

$$B(t) = a_0 l_2 [3\alpha(c_1 t + t_0)]^{\frac{1}{3}\left(1 + \frac{m_2}{\alpha c_1 a_0^3}\right)}, \tag{40}$$

$$C(t) = a_0 l_3 [3\alpha(c_1 t + t_0)]^{\frac{1}{3}\left(1+\frac{m_3}{\alpha c_1 a_0^3}\right)}. \quad (41)$$

Equations (39)–(41) show that the directional scale factors have power-law expansion form. These three spatial scale factors are zero at $t = -t_0/c_1$ and all tend to infinity at $t \to \infty$. The model has a point singularity at $t = -t_0/c_1$. The directional Hubble parameters have the expressions:

$$H_i = \alpha c_1 \left(1 + \frac{m_i}{\alpha c_1 a_0^3}\right) [3\alpha(c_1 t + t_0)]^{-1}, \quad i = 1, 2, 3. \quad (42)$$

Using the constraint $m_1 + m_2 + m_3 = 0$, the generalized mean Hubble parameter is given by

$$H = \frac{c_1}{3(c_1 t + t_0)}. \quad (43)$$

As H is a function of time, the model is not a steady-state model. The DP, defined in eq. (20), gives

$$q = 2, \quad (44)$$

which shows that the model expands with decelerated rate. From eqs (19), (21) and (22), we have the following values of physical quantities:

$$\theta = c_1 (c_1 t + t_0)^{-1}, \quad (45)$$

$$A_p = \frac{2M}{3\alpha^2 c_1^2 a_0^6}, \quad (46)$$

$$\sigma^2 = \frac{M}{9\alpha^2 a_0^6} \frac{1}{(c_1 t + t_0)^2}, \quad (47)$$

where

$$M = \frac{m_1^2 + m_2^2 + m_3^2}{2}.$$

The values of ρ_ϕ and p_ϕ are given by

$$\rho_\phi = \frac{c_1^2}{2} [3\alpha(c_1 t + t_0)]^{-2} = p_\phi, \quad (48)$$

which gives $w_\phi = 1$ throughout the evolution and this is the case of stiff-matter state. We observe that θ, σ, ρ_ϕ and p_ϕ are decreasing functions of time which have infinite value at $t = -t_0/c_1$ but tend to zero in late-time evolution. The anisotropy parameter is constant, which shows that the nature of the model is always anisotropic throughout the evolution. Also, $\lim_{t\to\infty} \frac{\sigma}{\theta} = \text{const.}$, i.e., the measure of shear scalar to expansion rate is constant and continues throughout the evolution, which shows that the shear does not tend to zero faster than the expansion scalar and hence the model has anisotropy behaviour.

Now, eqs (23) and (24) give

$$\rho_m = \left(3\alpha^2 c_1^2 - \frac{M}{a_0^6} - \frac{c_1^2}{2}\right) [3\alpha(c_1 t + t_0)]^{-2} = p_m. \quad (49)$$

For energy density to be positive we must have the positive value within the first bracket, i.e., $6\alpha^2 c_1^2 a_0^6 - c_1^2 a_0^6 - 2M > 0$. From eq. (49), we get $w_m = 1$, which is again the case of stiff-matter. The effective density ($\rho_{\text{eff}} = \rho_m + \rho_\phi$) and pressure ($p_{\text{eff}} = p_m + p_\phi$) are respectively given by

$$\rho_{\text{eff}} = \left(3\alpha^2 c_1^2 - \frac{M}{a_0^6}\right) [3\alpha(c_1 t + t_0)]^{-2} = p_{\text{eff}}. \quad (50)$$

From eq. (50) we find that $w_{\text{eff}} = 1$. Thus, the Universe is filled with stiff-matter in the presence of scalar field with zero potential and gives decelerating Universe as $q > 0$.

3.2 *Solution with exponential potential*

Let us assume the exponential potential of the form [10,48,50]

$$V(\phi) = V_0 e^{-\lambda\phi}, \quad (51)$$

where V_0 and λ are non-negative constants.

Using (34) and (51), eq. (15) reduces to

$$\ddot{\phi} + 3\alpha\dot{\phi}^2 - V_0 \lambda e^{-\lambda\phi} = 0. \quad (52)$$

The solution of eq. (52) is given by

$$\phi = \frac{2}{\lambda}\left[\ln\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)\right], \quad (53)$$

where $c_2 > 0$ is a constant of integration and $D = 2V_0\lambda/(6\alpha - \lambda)$. The real solution of ϕ exists provided $D > 0$, i.e., $0 < \lambda < 6\alpha$. We shall choose the positive sign within the bracket, without loosing any generality to obtain an expanding model. We find that ϕ is time-dependent and is increasing function of cosmic time. Therefore, during the evolution the scalar field is growing and hence kinetic energy vanishes at late-time evolution.

Substituting eq. (53) into eq. (33), the solution of the average scale factor for an expanding Universe is given by

$$a = a_0 \left[\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right]^{2\alpha/\lambda}. \quad (54)$$

In this case, we again get the power-law form $a \propto t^r$. The model will accelerate or decelerate according to

$0 < \lambda \le 2\alpha$ or $2\alpha < \lambda < 6\alpha$. From eqs (51) and (53), the potential turns out to be

$$V(\phi) = V_0 \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-2}, \tag{55}$$

which shows that V is a decreasing function of time and vanishes as $t \to \infty$. By use of eq. (54) in eqs (29)–(31), the spatial scale factors can be obtained as

$$A(t) = a_0 l_1 \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{2\alpha/\lambda} \times \exp\left[-\frac{2m_1}{a_0^3\sqrt{D}(6\alpha-\lambda)} \frac{1}{\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{\left(\frac{6\alpha-\lambda}{\lambda}\right)}} \right], \tag{56}$$

$$B(t) = a_0 l_2 \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{2\alpha/\lambda} \times \exp\left[-\frac{2m_2}{a_0^3\sqrt{D}(6\alpha-\lambda)} \frac{1}{\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{\left(\frac{6\alpha-\lambda}{\lambda}\right)}} \right], \tag{57}$$

$$C(t) = a_0 l_3 \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{2\alpha/\lambda} \times \exp\left[-\frac{2m_3}{a_0^3\sqrt{D}(6\alpha-\lambda)} \frac{1}{\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{\left(\frac{6\alpha-\lambda}{\lambda}\right)}} \right]. \tag{58}$$

Equations (56)–(58) show that the scale factors have hybrid (a combination of power-law and exponential) type expansion. As we know the power-law behaviour dominates in the early phase of the cosmic evolution whereas the exponential factor dominates at late phase. Therefore, this form of scale factor describes both types of expansion depending on the dominating factor. The model has a point singularity at $t = -c_2/\sqrt{D}$. This solution describes an evolution from a point singularity to an infinite expansion. The directional Hubble parameters in this model are given by

$$H_i = \alpha\sqrt{D} \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-1} + \frac{m_i}{a_0^3} \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-6\alpha/\lambda}, \quad i = 1, 2, 3. \tag{59}$$

Now, the generalized mean Hubble parameter is obtained as

$$H = \alpha\sqrt{D} \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-1}, \quad m_1+m_2+m_3 = 0. \tag{60}$$

The deceleration parameter (DP) is given by

$$q = \frac{\lambda}{2\alpha} - 1. \tag{61}$$

Equation (61) clearly shows that q is a constant and therefore, its nature ($q < 0$, $q = 0$, or $q > 0$) depends on the values of α and λ. Therefore, for an accelerating Universe where $-1 < q < 0$, we must have $0 < \lambda < 2\alpha$, for marginal inflation where $q = 0$, we have $\lambda = 2\alpha$ and for a decelerating Universe where $q > 0$, we must have the constraints $2\alpha < \lambda < 6\alpha$. It is to be noted that $\lambda > 6\alpha$ gives the imaginary value.

The scalar field density and pressure are respectively given by

$$\rho_\phi = \frac{6\alpha V_0}{(6\alpha - \lambda)} \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-2}, \tag{62}$$

$$p_\phi = \frac{2V_0(\lambda - 3\alpha)}{(6\alpha - \lambda)} \left(\frac{\lambda}{2} c_2 + \frac{\lambda\sqrt{D}}{2} t \right)^{-2}. \tag{63}$$

We observe that ρ_ϕ is always positive as $\lambda < 6\alpha$, and both ρ_ϕ and p_ϕ are decreasing functions of time. The corresponding EoS parameter is given by

$$w_\phi = \frac{\lambda}{3\alpha} - 1 = \frac{2q - 1}{3}. \tag{64}$$

As we know, the EoS parameter to a quintessence region is $-1 < w_\phi < -(1/3)$, and this can be achieved when $0 < \lambda < 2\alpha$. Also, $2\alpha < \lambda < 6\alpha$ gives the EoS parameter $-(1/3) < w_\phi < 1$.

The parameters θ, A_p and σ^2 are respectively obtained as

$$\theta = 3\alpha\sqrt{D}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-1}, \tag{65}$$

$$A_p = \frac{2M}{3\alpha^2 D a_0^6}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{(-12\alpha/\lambda)+2}, \tag{66}$$

$$\sigma^2 = \frac{M}{a_0^6}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}. \tag{67}$$

We observe that θ, A_p and σ^2 are infinite at the point of singularity but they decrease with time and all tend to zero in late-time evolution. Thus, the Universe is expanding with the increase of time but the rate of expansion, measure of anisotropy and shear scalar decrease to zero and become isotropic in late time. It is also observed that $\lim_{t\to\infty}(\sigma/\theta) = 0$ for $\lambda < 6\alpha$ which shows that the model approaches isotropy in late time. Therefore, the anisotropy of the Universe damp out during the course of evolution which is consistent with the present observation. Now, eqs (23) and (24) give

$$\rho_m = \frac{6\alpha V_0(\alpha\lambda - 1)}{(6\alpha - \lambda)}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2} - \frac{M}{a_0^6}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}, \tag{68}$$

$$p_m = \frac{2V_0(\alpha\lambda - 1)(\lambda - 3\alpha)}{(6\alpha - \lambda)}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2} - \frac{M}{a_0^6}\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}. \tag{69}$$

The condition $\rho_m \geq 0$ gives
$t \geq \left(\frac{2c_3}{\lambda\sqrt{D}} - \frac{c_2}{\sqrt{D}}\right)$,
where
$c_3 = \left[\frac{M(6\alpha - \lambda)}{6\alpha V_0 a_0^6(\alpha\lambda - 1)}\right]^{\lambda/2(6\alpha-\lambda)}$
with $\alpha\lambda \geq 1$.

The corresponding perfect fluid EoS parameter w_m is given by

$$w_m = \frac{\dfrac{2V_0(\alpha\lambda-1)(\lambda-3\alpha)\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2}}{(6\alpha-\lambda)} - \dfrac{M\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}}{a_0^6}}{\dfrac{6\alpha V_0(\alpha\lambda-1)\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2}}{(6\alpha-\lambda)} - \dfrac{M\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}}{a_0^6}}. \tag{70}$$

Now with the EoS parameter (70) it is obvious that, as long as both ρ_m and p_m are positive, $0 < w_m < 1$ only when $3\alpha < \lambda < 6\alpha$ and as p_m is negative, $-1 \leq w_m \leq 0$ only when $0 < \lambda \leq 3\alpha$. We can also calculate $\rho_{\rm eff} = \rho_\phi + \rho_m$ and $p_{\rm eff} = p_\phi + p_m$ but we avoid here to write the expressions of these quantities. The effective EoS parameter is calculated as

$$w_{\rm eff} = \frac{\dfrac{2V_0\alpha\lambda(\lambda-3\alpha)\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2}}{(6\alpha-\lambda)} - \dfrac{M\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}}{a_0^6}}{\dfrac{6\alpha^2\lambda V_0\left(\frac{\lambda}{2}c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-2}}{(6\alpha-\lambda)} - \dfrac{M\left(\lambda 2c_2 + \frac{\lambda\sqrt{D}}{2}t\right)^{-12\alpha/\lambda}}{a_0^6}}. \tag{71}$$

In this case we can give similar interpretation. As long as both $\rho_{\rm eff}$ and $p_{\rm eff}$ are positive, $0 < w_{\rm eff} < 1$ only when $3\alpha < \lambda < 6\alpha$ and as $p_{\rm eff}$ is negative, $-1 \leq w_{\rm eff} \leq 0$ only when $0 < \lambda \leq 3\alpha$.

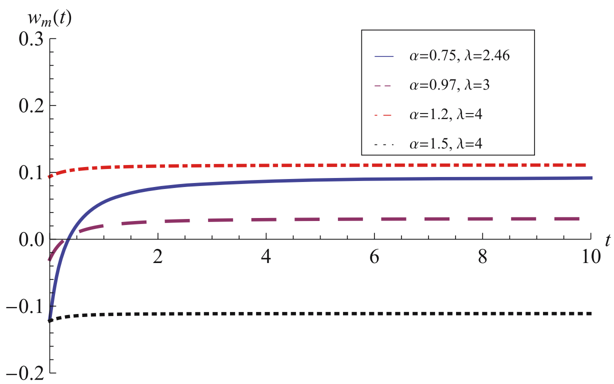

Figure 1. w_m vs. t for $c_2 = 1$, $V_0 = 1$ and some values of λ and α.

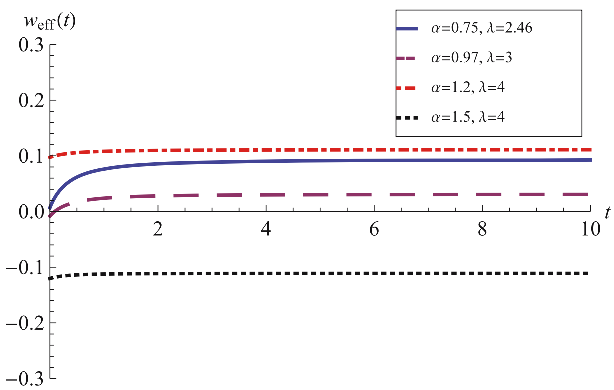

Figure 2. w_{eff} vs. t for $c_2 = 1$, $V_0 = 1$ and some values of λ and α.

Figures 1 and 2 plot the graphs of w_m and w_{eff}, respectively with respect to time t for some values of α and λ satisfying the above-mentioned constraints. It is to be noted that one can take those values of α and λ which make ρ_m and ρ_{eff} positive. We observe that for some values of parameters, e.g., $\alpha = 0.97$ and $\lambda = 3$, both w_m and w_{eff} show a transition from a negative value to a positive one. Therefore, the model shows transition from early inflationary phase to decelerated phase. There are some values like $\alpha = 0.75$ and $\lambda = 2.46$, where w_m shows transition from negative to positive but w_{eff} is always positive. However, there are some values of α and λ for which w_m and w_{eff} are only positive or negative. Thus, depending on the choices of arbitrary values of α and λ under the constraints we can observe any one of the behaviour in EoS parameters: w_m and w_{eff} may be only positive or negative throughout the evolution or may vary from negative to positive.

4. Conclusion

We have presented a detailed discussion on the evolution of a homogeneous and anisotropic B-I Universe with a perfect fluid and a scalar field. We have studied two types of models, namely, zero potential and an exponential potential. Exact solutions of the field equations for both the models have been obtained by assuming the average scale factor as an exponential function of the scalar field. Explicit forms of average scale factor (also, the spatial scale factors), energy density and pressure of the scalar field and the perfect fluid, and their respective EoS parameters have been obtained in each model. Other geometrical and physical parameters like expansion scalar, deceleration parameter, anisotropy and shear scalar have been calculated and their physical interpretations have been discussed. We have observed that these geometries are very interesting for the observed Universe isotropy as we find the late-time isotropized solutions. The solutions obtained in both models satisfy the energy conservation equation (14). We, herewith, summarize the result of both these models as follows.

In the first model, we have considered a massless scalar field by assuming the average scale factor as an exponential function of the scalar field. We have solved the field equations and found solutions for various cosmological parameters. In this case, the average scale factor as well as the directional scale factors have power-law expansion which expand with decelerated rate. The model has a point singularity at $t = -t_0/c_1$. At the beginning of the evolution, i.e., at $t = -t_0/c_1$, the scalar field $\phi \to -\infty$. During the evolution, ϕ increases and at the end of the evolution (an infinite expansion) when $t \to \infty$, it tends to $+\infty$. The kinetic energy vanishes at $t \to \infty$. The positive constant value of DP shows that the model expands but with decelerated rate. The nature of the model is always anisotropic throughout the evolution because of the constant behaviour of anisotropy parameter. All the physical parameters like energy density, pressure, expansion scalar and shear scalar are decreasing functions of time. The measure of shear scalar to expansion scale is constant throughout the evolution. Thus, the shear scalar does not tend faster than the expansion scalar. The EoS parameters of the scalar field and the perfect fluid come out to be one. Hence, we have found that the anisotropic perfect fluid model behaves like stiff-matter in the presence of scalar field.

In the second model, we have considered an exponential potential, i.e., $V(\phi) = V_0 e^{-\lambda\phi}$. We have obtained the logarithmic form of the scalar field and power-law form of the average scale factor. The directional scale factors have hybrid-type expansion which is a combination of power-law and exponential forms.

This form of scale factor describes both types of expansion depending on the dominating factor. The model has a point singularity at $t = -c_2/\sqrt{D}$. It has been observed that at the beginning of the evolution, i.e., at $t = -c_2/\sqrt{D}$, ϕ tends to $-\infty$ while $V(\phi)$ tends to $+\infty$. During the evolution ϕ increases whereas $V(\phi)$ decreases and at late-time evolution ϕ tends to $+\infty$ while $V(\phi)$ vanishes. The kinetic energy and potential energy tend to $+\infty$ at the beginning but both vanish at the end of evolution. This means that the Universe is born from the singularity at $t = -c_2/\sqrt{D}$. The DP is constant whose nature depends on the values of α and λ. The model shows the quintessence region for $0 < \lambda < 2\alpha$. The model has marginal inflation at $\lambda = 2\alpha$. The model decelerates for $2\alpha < \lambda < 6\alpha$. Physical quantities like θ, A_p, σ^2, ρ_m, p_m, ρ_ϕ, p_ϕ are decreasing functions of time and tend to zero in late-time evolution. The model approaches isotropy in late-time. The EoS parameter of matter and the effective EoS parameter of matter plus scalar field are time dependent which show that the deceleration or acceleration or transition from one phase to other depends on the constraint $0 < \lambda \leq 3\alpha$ or $3\alpha < \lambda < 6\alpha$ (shown in figures 1 and 2). We find that the physically valid range for λ is $0 < \lambda < 6\alpha$, where α is any positive constant.

Acknowledgements

The authors are thankful to the referee for the constructive suggestions to improve the manuscript. MS also expresses her thanks to UGC, India for providing financial support to carry out the research under Junior Research Fellowship (JRF).

References

[1] R R Caldwell, R Dave and P J Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998)
[2] V Sahni, *Lect. Notes Phys.* **653**, 141 (2004)
[3] A G Riess *et al*, *Astron. J.* **116**, 1009 (1998)
[4] S Perlmutter *et al*, *Astrophys. J.* **517**, 565 (1999)
[5] N Suzuki *et al*, *Astrophys. J.* **746**, 85 (2012)
[6] G Hinshaw *et al*, *Astrophys. J. Suppl.* **208**, 19 (2013)
[7] Planck Collaboration: P A R Ade *et al*, *Astron. Astrophys.* **571**, A16 (2014)
[8] A H Guth, *Phys. Rev. D* **23**, 347 (1981)
[9] J A Frieman and I Waga, *Phys. Rev. D* **57**, 4642 (1998)
[10] A B Burd and J D Barrow, *Nucl. Phys. B* **308**, 929 (1988)
[11] J D Barrow, *Phys. Lett. B* **187**, 12 (1987); *Phys. Rev. D* **49**, 3055 (1994)
[12] G F R Ellis and M S Madsen, *Class. Quantum Grav.* **8**, 667 (1991)
[13] J D Barrow and P Saich, *Class. Quantum Grav.* **10**, 279 (1993)
[14] J D Barrow and P Parsons, *Phys. Rev. D* **52**, 5576 (1995)
[15] P Parsons and J D Barrow, *Phys. Rev. D* **51**, 6757 (1995)
[16] A A Coley, J Ibanez and R J Van den Hoogen, *J. Math. Phys.* **38**, 5256 (1997)
[17] E J Copeland, A R Liddle and D Wands, *Phys. Rev. D* **57**, 4686 (1998)
[18] I P C Heard and D Wands, *Class. Quantum Grav.* **19**, 5435 (2002)
[19] E J Copeland, S Mizuno and M Shaeri, *Phys. Rev. D* **79**, 103515 (2009)
[20] B Ratra and P J E Peebles, *Phys. Rev. D* **37**, 3406 (1988)
[21] A R Liddle and R J Scherrer, *Phys. Rev. D* **59**, 023509 (1999)
[22] E J Copeland, M Sami and S Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006)
[23] S Dutta, E N Saridakis and R J Scherrer, *Phys. Rev. D* **79**, 103005 (2009)
[24] R R Caldwell, *Phys. Lett. B* **545**, 23 (2002)
[25] S Nojiri and S D Odintsov, *Phys. Lett. B* **562**, 147 (2003)
[26] Z-K Guo, Y-S Piao, X Zhang and Y-Z Zhang, *Phys. Lett. B* **608**, 177 (2005)
[27] Y-F Cai, E N Saridakis, M R Setare and J-Q Xia, *Phys. Rep.* **493**, 1 (2010)
[28] C Armendariz-Picon, V Mukhanov and P J Steinhardt, *Phys. Rev. D* **63**, 103510 (2001)
[29] A Sen, *J. High Energy Phys.* **0207**, 065 (2002)
[30] SDSS Collaboration: M Tegmark *et al*, *Phys. Rev. D* **69**, 103501 (2004)
[31] WMAP Collaboration: J Donkley *et al*, *Astrophys. J. Suppl.* **180**, 306 (2009)
[32] WMAP Collaboration: E Komatsu *et al*, *Astrophys. J. Suppl.* **192**, 18 (2011)
[33] C J Copi, D Huterer and G D Starkman, *Phys. Rev. D* **70**, 043515 (2004)
[34] A Bernui, B Mota, M J Reboucas and R Tavakol, *Int. J. Mod. Phys. D* **16**, 411 (2007)
[35] G F R Ellis and M A H MacCallum, *Commun. Math. Phys.* **12**, 108 (1969)
[36] G F R Ellis, *Gen. Rel. Gravit.* **38**, 1003 (2006)
[37] M Demianski, R De Ritis, C Rubano and P Scudellaro, *Phys. Rev. D* **46**, 1391 (1992)
[38] B Saha and T Boyadjiev, *Phys. Rev. D* **69**, 124010 (2004)
[39] T Q Do, W F Kao and I-C Lin, *Phys. Rev. D* **83**, 123002 (2011)
[40] M Sharif and M Zubair, *J. Phys. Soc. Jpn* **81**, 114005 (2012)
[41] C R Fadragas, G Leon and E N Saridakis, *Class. Quantum Grav.* **31**, 075018 (2014), arXiv:gr-qc/1308.1658
[42] Ø Grøn, *Phys. Rev. D* **32**, 2522 (1985)
[43] Ø Grøn, *Astrophys. Space Sci.* **173**, 191 (1990)
[44] B Saha and V Rikhvitsky, *Physica D* **219**, 168 (2006)

[45] S Kumar and C P Singh, *Astrophys. Space Sci.* **312**, 57 (2007)
[46] S Sen and N Banerjee, *Astrophys. Space Sci.* **254**, 133 (1997)
[47] C P Singh and V Singh, *Mod. Phys. Lett. A* **26**, 1495 (2011)
[48] L P Chimento, *Class. Quantum Grav*. **15**, 965 (1998)
[49] J W Van Holten, *Mod. Phys. Lett. A* **17**, 1383 (2002)
[50] J G Russo, *Phys. Lett. B* **600**, 185 (2004), arXiv: hep-th/0403010
[51] E J Copeland, M Sami and S Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006)
[52] R R Caldwell and M Kamionkowski, *Annu. Rev. Nucl. Part. Sci.* **59**, 397 (2009)
[53] S V Sushkov, *Phys. Rev. D* **80**, 103505 (2009)
[54] E N Saridakis and S V Sushkov, *Phys. Rev. D* **81**, 083510 (2010), arXiv:gr-qc/1002.3478
[55] V Singh and C P Singh, *Astrophys. Space Sci.* **355**, 2183 (2014)