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On the JWKB solution of the uniformly lengthening pendulum via change of independent variable in the Bessel's equation

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Abstract. Common recipe for the lengthening pendulum (LP) involves some change of variables to give a relationship with the Bessel's equation. In this work, conventional semiclassical JWKB solution (named after Jeffreys, Wentzel, Kramers and Brillouin) of the LP is being obtained by first transforming the related Bessel's equation into the normal form 'via the suggested change of independent variable'. JWKB approximation of the first-order Bessel functions ($\nu = 1$) of both types along with their zeros are being obtained analytically with a very good accuracy as a result of the appropriately chosen associated initial values and they are extended to the neighbouring orders ($\nu = 0$ and 2) by the recursion relations. The required initial values are also being studied and a quantization rule regarding the experimental LP parameters is being determined. Although common numerical methods given in the literature require adiabatic LP systems where the lengthening rate is slow, JWKB solution presented here can safely be used for higher lengthening rates and a criterion for its validity is determined by the JWKB applicability criterion given in the literature. As a result, the semiclassical JWKB method which is normally used for the quantum mechanical and optical waveguide systems is applied to the classical LP system successfully.

Keywords. Jeffreys–Wentzel–Kramers–Brillouin; Wentzel–Kramers–Brillouin; semiclassical approximation; linear differential equations; initial value problems; the lengthening pendulum.

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1. Introduction

The JWKB method, named after the authors contributed to the theory [1–4], is a very strong and effective method used mainly in quantum mechanical and also optical waveguide systems for accurate solutions under some appropriate conditions, i.e., [5]. It is conventionally applied to quantum mechanical systems described by the time-independent Schrödinger's equation (TISE), which is a linear second-order homogeneous normal form differential equation

$$y''(\rho) + f(\rho)y(\rho) = 0, \tag{1.1a}$$

$$f(\rho) = k^2(\rho) = \frac{2m}{\hbar^2}\left[E - V(\rho)\right], \tag{1.1b}$$

where m, $\hbar$, E, and $V(\rho)$ have the usual meanings (representing mass, Planck's constant divided by 2π, total energy, and potential function) [5–10]. Although being an approximate solution, it can even give exact solutions for certain types of potentials, such as shape invariant potentials, and its reasons are widely studied in [10–14]. Moreover, JWKB solutions of the TISE for some exponential potential decorated quantum mechanical potentials are known to be accurate when compared with the exact solutions involving Bessel functions [15,16]. Similarly, differential equation describing the lengthening pendulum (LP) system is given by a second-order linear differential equation in the standard form (rather than normal form) whose solutions can also be expressed in terms of integer-order Bessel functions [17–19]. So, it can be interesting to use this quantum mechanical approximation technique to find and analyse solutions of a classical mechanical problem like the LP system where Bessel functions are common. Indeed, second-order standard form linear homogeneous differential equations can be transformed into second-order linear normal form (the TISE here) so that the conventional JWKB method involving the famous JWKB connection formulas can be applied [15,16]. In this work, a change of variable in the independent variable is suggested to transform the standard form Bessel's equation describing the LP system (after some change of variables) into a normal form differential equation to find and analyse its approximate solutions by the JWKB method as discussed in [20] and related Bessel functions are obtained experimentally by appropriately chosen initial values in a specific LP system. So, classical mechanical problem regarding the LP is being solved here by the semiclassical JWKB method, which is normally applied to quantum mechanical systems.

As a starting point, the first-order BDE ($\nu = 1$) is being studied extensively and the Bessel functions of the first-order (of both types: type-1 and type-2) and their nth zeros are being obtained analytically with a very good accuracy by appropriately chosen initial values via the semiclassical JWKB approximation. Then, by using the recursion relations, these results are being extended to obtain other neighbouring integer-order Bessel functions and their accuracy is being analysed via the JWKB accuracy (or, applicability) criteria given in [5–7]. JWKB solutions of the Bessel functions are used to describe the solution of the LP system for the given initial values, which are hereby shown to be in very good agreement with the exact solution. Although numerical solution to the uniformly lengthening pendulum is available in [21,22], they require adiabatic LP systems where the lengthening rate is slow. However, the JWKB solution presented here works very well even for high lengthening rates and a criterion for this is derived via the JWKB application criterion given in [5,6,20]. Besides being appropriate for high lengthening rates, JWKB analyses studied here can help giving us deeper understanding on the relationship between classical and quantum mechanical systems. Another advantage is that in the JWKB approximation sum of only two terms are required whereas the exact solution of the Bessel functions requires infinite terms.

2. JWKB method by formal and quantum mechanical aspect

The formal (and also mathematical) definition of the Nth order $(\mathrm{JWKB})_N$ approximation to the solution of the differential equation

$$\epsilon^2 y''(\rho) - f(\rho)y(\rho) = 0, \tag{2.1a}$$

is given in the form of the following series:

$$\tilde{y}_N(\rho) = \exp\left[\frac{1}{\delta}\sum_{n=0}^{N}\delta^n S_n(\rho)\right], \quad \begin{pmatrix} \delta \to \epsilon \\ f(\rho) \neq 0 \end{pmatrix}, \tag{2.1b}$$

where S_n represents the JWKB expansion terms given in [6,7]. Here, the smallness of ϵ controls the required term numbers ($= N + 1$) in the formal JWKB series in (2.1b) for good enough approximation and the cases where only the first two terms are enough has a special interest in quantum mechanics. Indeed, the conventional (and also traditional) JWKB approximation involves only the first two terms with indices $n = 0$ and 1 to give $(\mathrm{JWKB})_1$ where the two-valuedness of these expansion terms, namely, $S_n(\rho) \to S_{n1}(\rho)$ and $S_n(\rho) \to S_{n2}(\rho)$ give two different sets of solutions:

$$\left.\begin{aligned} \tilde{y}_1(c,\rho) &= \exp\left[\frac{S_{01}(c,\rho)}{\delta} + S_{11}(c,\rho)\right] \\ \tilde{y}_2(c,\rho) &= \exp\left[\frac{S_{02}(c,\rho)}{\delta} + S_{12}(c,\rho)\right] \end{aligned}\right\}, \quad \delta \to \epsilon, \tag{2.2}$$

where $S_{01} = -S_{02}$ and $S_{11} = S_{12}$ [6,7]. These sets constitute the complementary $(\mathrm{JWKB})_1$ functions and the general solution can be written as their linear combination:

$$\tilde{y}(\rho) = \frac{c_1}{f^{1/4}(\rho)}\exp\left[-\frac{1}{\delta}\int_{\rho_t}^{\rho}\sqrt{f(t)}\mathrm{d}t\right] + \frac{c_2}{f^{1/4}(\rho)}\exp\left[\frac{1}{\delta}\int_{\rho_t}^{\rho}\sqrt{f(t)}\mathrm{d}t\right], \quad \rho_t < \rho, \tag{2.3}$$

where ρ_t is the classical turning point (CTP) which means the solution of $f(\rho) = 0$ for ρ. Quantum mechanical systems under study might be one or more turning point systems and (1.1a) can be thought as

$$y''(\rho) - \frac{1}{i^2}f(\rho)y(\rho) = 0, \tag{2.4}$$

where $\delta \to \epsilon = i$ and (2.3) turns out to the well-known (traditional) JWKB solution:

$$\tilde{y}(\rho) = \frac{c_1}{\sqrt{k(\rho)}}\exp\left[-i\int_{\rho_t}^{\rho}k(t)\mathrm{d}t\right] + \frac{c_2}{\sqrt{k(\rho)}}\exp\left[i\int_{\rho_t}^{\rho}k(t)\mathrm{d}t\right], \quad \rho_t < \rho. \tag{2.5}$$

This solution is known to be accurate enough for slowly changing potentials in the TISE in (1.1b) and a criterion for this is given as follows [5–7]:

$$0 \le G(\rho) = \left|\frac{1}{2k^3(\rho)}\frac{\mathrm{d}^2k(\rho)}{\mathrm{d}\rho^2} - \frac{3}{4k^4(\rho)}\left[\frac{\mathrm{d}k(\rho)}{\mathrm{d}\rho}\right]^2\right| \ll 1. \tag{2.6}$$

Here we use capital G rather than g given in [6,20] to show the difference between G and the gravitational acceleration g residing in the equations. For the domain $\rho_t > \rho$, there is an exchange of upper and lower limits of the integrals in (2.3) and (2.5). Once the JWKB solution of the classically accessible region (CAR) where $f(\rho) > 0$ is solved, JWKB solution of the classically inaccessible region (CIR) where $f(\rho) < 0$ can directly be obtained by the JWKB connection formulas [5–7]. To obtain JWKB solution in the CAR from CIR is also possible but care should always be taken because the JWKB connection formulas are not equations, but valid in one direction only.

3. Exact solution of the lengthening pendulum

For the uniformly lengthening pendulum (LP), the length of the string (l) at time t changes according to

$$l(t) = l_0 + vt, \tag{3.1}$$

where l_0 is the initial length at $t = 0$ and v is a constant showing the lengthening rate of the pendulum. We have the following differential equation describing the LP [18,19,21]:

$$(l_0 + vt)\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + 2v\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + g\sin[\theta(t)] = 0 \tag{3.2a}$$

which can be written for small θ values ($\sin\theta \approx \theta$) as

$$(l_0 + vt)\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + 2v\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + g\theta(t) = 0 \tag{3.2b}$$

or, alternatively [17–19,21], by imposing t in (3.1) into (3.2b):

$$l\frac{\mathrm{d}^2\theta(l)}{\mathrm{d}l^2} + 2\frac{\mathrm{d}\theta(l)}{\mathrm{d}l} + \frac{g}{v^2}\theta(l) = 0, \tag{3.2c}$$

whose exact general solution can be written as a linear combination of the complementary solutions in both independent variables (l or t, depending on our preference to work with) as follows:

$$\begin{aligned}\theta_{\mathrm{EX}}(l) &= c_1\theta_1(l) + c_2\theta_2(l)\\ &= \theta_{\mathrm{EX}}[l(t)] =: \theta_{\mathrm{EX}}(t)\\ &= c_1\theta_1(t) + c_2\theta_2(t).\end{aligned} \tag{3.3}$$

In order to find the complementary functions, let us continue following the common recipe as in [17–19,21] by first applying a change of variable in the independent variable $l = l(t)$ in (3.2b):

$$l(t) = \frac{x^2(t)v^2}{4g} \Longrightarrow l(x) \to \frac{x^2v^2}{4g} \tag{3.4}$$

to give

$$\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2} + \frac{3}{x}\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} + \theta(x) = 0, \tag{3.5}$$

whose solution gives (3.3) in x, that is,

$$\theta_{\mathrm{EX}}(x) = c_1\theta_1(x) + c_2\theta_2(x) = \theta\left(l \to \frac{x^2v^2}{4g}\right). \tag{3.6}$$

Now, it will be useful to show its relation with the first-order Bessel's differential equation (BDE). So, let us continue following the common recipe in [17–19,21] by applying another change of variable to (3.5) (now, in the dependent variable $\theta = \theta(l) = \theta[l(t)]$) as

$$\theta_{\mathrm{EX}}(t) = y(t)/x(t) \tag{3.7}$$

to give the BDE of first-order with the dependent variable in y:

$$x^2y'' + xy' + (x^2 - 1)y = 0 \tag{3.8}$$

whose exact solution gives

$$\begin{aligned}y_{\mathrm{EX}}(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1J_1(x) + c_2Y_1(x),\end{aligned} \tag{3.9}$$

and using (3.7) we have [17–19,21]:

$$\begin{aligned}\theta_{\mathrm{EX}}(x) &= x^{-1}y_{\mathrm{EX}}(x)\\ &= x^{-1}[c_1J_1(x) + c_2Y_1(x)],\end{aligned} \tag{3.10}$$

where J_1 and Y_1 are Bessel functions of the first type and c_1 and c_2 are the constant coefficients to be determined from the initial conditions:

$$\begin{aligned}&\theta(t = t_0) = \theta_0 = \text{const.},\\ &\dot{\theta}(t = t_0) = \dot{\theta}_0 = \text{const.},\end{aligned} \tag{3.11a}$$

where dot represents derivative with respect to time, or alternatively (and more conveniently) from [22a]:

$$\begin{aligned}&\theta(x = x_0) = \theta_0 = \text{const.},\\ &[\mathrm{d}\theta(x)/\mathrm{d}x]_{x=x_0} = \theta_{x0} = \text{const.},\end{aligned} \tag{3.11b}$$

where $x_0 = x(t_0)$ is defined from (3.4) by

$$\begin{aligned}&x \to x(t) = b\sqrt{l(t)} = \frac{2\sqrt{gl(t)}}{v}\\ &\Rightarrow x_0 \to x(t_0) = \frac{2\sqrt{gl_0}}{v}.\end{aligned} \tag{3.12}$$

Applications of these initial values give

$$\begin{cases}\theta_{\rm EX}(x_0) = c_1\theta_1(x_0) + c_2\theta_2(x_0) = \theta_0 \\ \left.\dfrac{d\theta_{\rm EX}(x)}{dx}\right|_{x=x_0} = \left[c_1\dfrac{d\theta_1(x)}{dx} + c_2\dfrac{d\theta_2(x)}{dx}\right]_{x=x_0} = \theta_{x0}\end{cases} \tag{3.13}$$

and the linear solutions of these two equations for the coefficients give

$$c_1 = \frac{\theta_0 \left.\frac{d\theta_2(x)}{dx}\right|_{x=x_0} - \theta_{x0}\theta_2(x_0)}{\Delta(x_0)},$$

$$c_2 = \frac{-\theta_0 \left.\frac{d\theta_1(x)}{dx}\right|_{x=x_0} - \theta_{x0}\theta_1(x_0)}{\Delta(x_0)}, \tag{3.14}$$

where the discriminant $\Delta(x_0)$ is defined by

$$\Delta(x_0) = \begin{vmatrix} \theta_1(x_0) & \theta_2(x_0) \\ \left.\dfrac{d\theta_1(x)}{dx}\right|_{x=x_0} & \left.\dfrac{d\theta_2(x)}{dx}\right|_{x=x_0}\end{vmatrix}. \tag{3.15}$$

If the initial angular velocity at $t = 0$ is zero, then the initial conditions in (3.11a) are given by

$$\theta(t = t_0 \to 0) = \theta_0 = \text{const.},$$

$$\dot{\theta}(t = t_0 \to 0) = \dot{\theta}_0 = 0 \tag{3.16a}$$

or in the form (3.11b) which we follow here by

$$\theta(x = x_0) = \theta_0 = \text{const.},$$

$$[d\theta(x)/dx]_{x=x_0} = \theta_{x0} = 0. \tag{3.16b}$$

Then the calculations in (3.14) for these initial values give the coefficients in (3.3) or (3.10) as follows [17–19,21]:

$$c_1 = -\frac{\pi x_0^2}{2}\theta_0 Y_2(x_0), \quad c_2 = \frac{\pi x_0^2}{2}\theta_0 J_2(x_0). \tag{3.17}$$

The exact solution can finally be combined to be written as

$$\begin{aligned}\theta_{\rm EX}(x) &= x^{-1} y_{\rm EX}(x) \\ &= -\frac{\pi x_0^2}{2}\theta_0 Y_2(x_0) x^{-1} J_1(x) \\ &\quad + \frac{\pi x_0^2}{2}\theta_0 J_2(x_0) x^{-1} Y_1(x).\end{aligned} \tag{3.18}$$

Here, $J_\nu(x)$ and $Y_\nu(x)$ where $\nu = 1, 2$ are the νth-order Bessel functions of the first kind and second kind (Weber or Neumann functions), respectively. Bessel functions of integer order ($\nu \in \mathbb{Z}^+$), which have many important physical applications, can be expressed in terms of infinite series as follows [17,23–29]:

$$\begin{aligned}J_\nu(x) &= \sum_{i=0}^{\infty} \frac{(-1)^i (x/2)^{\nu+2i}}{i!\Gamma(\nu+i+1)} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i (x/2)^{\nu+2i}}{i!(\nu+i)!}, \quad \nu \in \mathbb{Z}\end{aligned} \tag{3.19a}$$

$$Y_\nu(x) = \lim_{p\to\nu} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}, \quad \nu \in \mathbb{Z}. \tag{3.19b}$$

Bessel functions can be obtained analytically via such calculations by using simple commands in conventional computer softwares such as *Mathematica* up to very reasonable accuracy, which can be called as exact [30,31]. *Mathematica* results are also used as exact analytical Bessel functions for calculating the exact general solution in (3.18) for the initial values in (3.16b) here.

4. JWKB solution of the lengthening pendulum

Formal and quantum mechanical aspect of the JWKB method has been given in §2 and now we apply this semiclassical approximation method for the traditional first-order JWKB approximation ($n = 0$ and 1 only!) to our LP system rather than infinite sum in the exact solution of the Bessel functions here. So, by saying JWKB method we mean the traditional JWKB which means $(\text{JWKB})_1$ as discussed in §2.

4.1 *General JWKB solution of the first-order BDE*

Relations between the solution of the LP and the solution of the BDE of first-order given for the exact solution in (3.9) and (3.10) (as a result of change of variables in (3.4) and (3.7)) should also be true for the JWKB solutions, i.e.,

$$\begin{aligned}\theta_{\rm JWKB}(x) &= x^{-1} y_{\rm JWKB}(x) \\ &= x^{-1}[c_1 \tilde{J}_1(x) + c_2 \tilde{Y}_1(x)],\end{aligned} \tag{4.1}$$

where $\tilde{J}_1$ and $\tilde{Y}_1$ are the JWKB approximation (with sum of only two terms) of the first-order Bessel functions of type-1 and type-2, respectively. However, as the JWKB solution to (3.8) can be written as

$$y_{\rm JWKB}(x) = \tilde{c}_1 \tilde{y}_1(x) + \tilde{c}_2 \tilde{y}_2(x), \tag{4.2}$$

it is not guaranteed that JWKB solution of (3.8) gives the same complementary functions approximately as the complementary functions residing in the exact solution (which are obviously Bessel functions) in (3.10). So, it can be written in a more convenient (or tentative) form as follows:

$$\begin{aligned}\theta_{\rm JWKB}(x) &= x^{-1} y_{\rm JWKB}(x) \\ &= x^{-1}[\tilde{c}_1 \tilde{y}_1(x) + \tilde{c}_2 \tilde{y}_2(x)] \\ &= x^{-1}[\tilde{c}_1 \tilde{J}_1(x) + \tilde{c}_2 \tilde{Y}_1(x)],\end{aligned} \tag{4.3}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants and $\tilde{y}_1$ and $\tilde{y}_2$ are the JWKB complementary solutions [31a]. However, it might be useful to note in advance here that, (4.1) and (4.3) will be equivalent ($c_1 = \tilde{c}_1$ and $c_2 = \tilde{c}_2$) by the carefully chosen appropriate initial values in our calculations discussed in §4.2. So, finding JWKB approximation of the first-order Bessel functions of either type ($\tilde{J}_1$ and $\tilde{Y}_1$) by solving the first-order BDE in (3.8) is essential for finding the JWKB solution of our LP system. Once the JWKB approximation of the first-order Bessel function of either type has been found, we can express the JWKB solution of our LP system by using eq. (4.1). In this subsection, finding the JWKB general solution in the form (4.3) is aimed and in the next subsection, $\tilde{J}_1$ and $\tilde{Y}_1$ in (4.1) will be derived from this solution by the appropriately chosen initial values.

JWKB general solution of the first-order BDE given in the standard form in (3.8) rather requires a normal form differential equation as in (1.1a) (because the JWKB technique involves the conventional JWKB connection formulas for the TISE, which is a normal form differential equation) and we can transform it into a normal form as follows:

Lemma 4.1. *Although the first-order BDE given in* (3.8) *is not in the normal form, a simple change of independent variable* ($x \to \rho$):

$$x : (-\infty, \infty) \to (0, \infty), \quad x(\rho) = \mathrm{e}^{(c-\rho)/2} \tag{4.4}$$

transforms it into the desired normal form:

$$\frac{\mathrm{d}^2 y(c, \rho)}{\mathrm{d}\rho^2} + \frac{\mathrm{e}^{c-\rho} - 1}{4} y(c, \rho) = 0 \tag{4.5}$$

as in (1.1a) *in* ρ [20].

Various change of variable applications to transform the BDE to a normal form (which is not unique) are available in [23,32–34]. The change of independent variable given in eq. (4.4) is being followed here. As a result, we have the first-order BDE in the desired normal form as in (1.1a), that is [20]:

$$\partial_\rho^2 y(c, \rho) + f(c, \rho) y(c, \rho) = 0, \tag{4.6a}$$

$$f(c, \rho) = k^2(c, \rho) = \frac{\mathrm{e}^{c-\rho} - 1}{4} \tag{4.6b}$$

from which we can see that it is now similar to the exponential potential decorated quantum mechanical system studied in [15,16]. Here we have the followings: (i) classical turning point (CTP) where $f(\rho) = 0$ occurs at $\rho_t = c$, (ii) classically inaccessible region (CIR) where $f(\rho) < 0$ lies on the right-hand side of the CTP and (iii) classically accessible region (CAR) where $f(\rho) > 0$ lies on the left-hand side of the CTP. All this information will be necessary in the application of the JWKB connection formulas.

Let us now see how well the first-order JWKB applicability criteria are satisfied by function $G(c, \rho)$, as given in (2.6) via [5–7], where the following inequality should be satisfied for the first-order JWKB method to be applicable with a good-enough accuracy:

$$0 \le G(c, \rho) = \left| \frac{1}{2k^3(c, \rho)} \frac{\partial^2 k(c, \rho)}{\partial \rho^2} - \frac{3}{4k^4(c, \rho)} \left[\frac{\partial k(c, \rho)}{\partial \rho} \right]^2 \right| \ll 1. \tag{4.7a}$$

We expect a reasonable (accurate) enough JWKB solution in consistence with the general exact solution in the region(s) where the inequality condition in (4.7a) holds. Calculation of $G(c, \rho)$ in (4.7a) gives

$$0 \le G(c, \rho) = \frac{1}{4} \mathrm{e}^{Re(\rho + c)} \left| \frac{4\mathrm{e}^\rho + \mathrm{e}^c}{\mathrm{e}^\rho - \mathrm{e}^c} \right| \ll 1 \tag{4.7b}$$

from which we have a narrow non-obedient (NO) region with

$$\text{NO: } \rho \in (c - 0.968889, c + 1.17808), \tag{4.7c}$$

which is around the CTP: $\rho_t = c$. This region is not appropriate for the conventional first-order JWKB, but the rest of it seems worth seeking the traditional first-order JWKB solutions [20]. Alternatively, eq. (4.7b) can be written by using (4.4) as

$$0 \le G(x) = G(c, \rho)\big|_{\rho \to c - 2\ln x} = \frac{\mathrm{e}^{-2Re[\ln(x)]}}{4} \left| \frac{x^4(x^2 + 4)}{(x^2 - 1)^3} \right| \ll 1 \tag{4.8a}$$

and the non-obedient domain in (4.7c) corresponds by using (4.4) and (3.12) to the following:

$$\boxed{\text{NO: } x \to x(g, l_0, v, t) = \frac{2\sqrt{g(l_0 + vt)}}{v} \in (0.554858, 1.62327)} \tag{4.8b}$$

which means that as $x(t)$ approaches zero from 0.554858 and as $x(t)$ approaches ∞ from 1.62327, then $G[x \to x(g, l_0, v, t)] \to 0$ and the traditional JWKB approximation with only first two indices in the form (2.3)–(2.5) approach exact solution. 3D graphs of x for variable lengthening rate and initial length parameters along with time as the second variable are given in figure 1 (see also green dashed curves for the graph of $G(x)$ in figure 4 for the 2D graph).

Now, as to the JWKB general solution, if we start with the CAR located at the left-hand side of the CTP

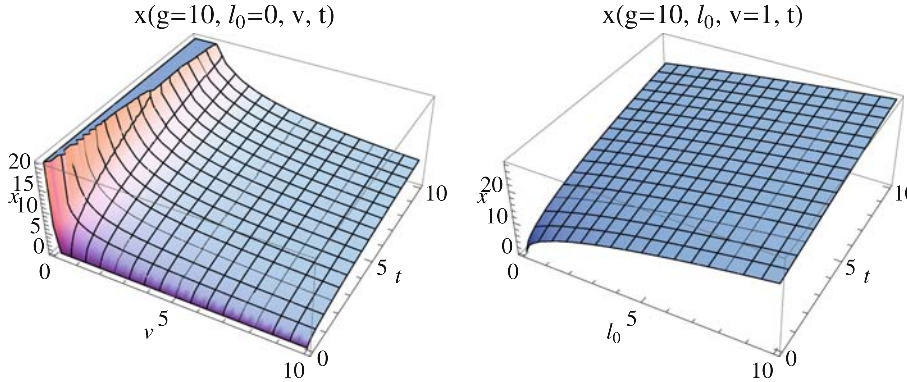

Figure 1. 3D graphs of x $(g = 10, l_0 = 0, v, t)$ and x $(g = 10, l_0, v = 1, t)$.

and connect it to the CIR by using the conventional JWKB connection formulas in the reverse direction, we find the same formulas for $y_L(c, \rho)$ and $y_R(c, \rho)$ as in Example 1 in [6] (but in (c, ρ) here rather than (c, x)) to give [20,34a]:

$$y_{\rm JWKB}(c,\rho) = \begin{cases} y_L(c,\rho), & \text{for } -\infty < \rho \le c \\ y_R(c,\rho), & \text{for } c \le \rho < \infty \end{cases}, \tag{4.9a}$$

where

$$y_L(c,\rho) = \frac{A(c)}{\sqrt{k(c,\rho)}} \sin[\eta(c,\rho) + \alpha(c)] \tag{4.9b}$$

and

$$y_R(c,\rho) = \frac{A(c)}{2\sqrt{\kappa(c,\rho)}} \cos[\alpha(c) - \pi/4] \exp[-\zeta(c,\rho)] + \frac{A(c)}{\sqrt{\kappa(c,\rho)}} \sin[\alpha(c) - \pi/4] \exp[\zeta(c,\rho)]\,. \tag{4.9c}$$

But the constituents (functions: η and ζ) in eqs (4.9b) and (4.9c) now read [20]:

$$\eta(c,\rho) = \int_\rho^c k(c,\rho)\mathrm{d}\rho = \sqrt{\mathrm{e}^{c-\rho} - 1} + i\left[\frac{\rho - c}{2} + \log(\sqrt{1 - \mathrm{e}^{c-\rho}}) + 1\right], \quad \rho < c$$

$$\zeta(c,\rho) = \int_c^\rho \kappa(c,\rho)\mathrm{d}\rho = -\frac{c}{2} - \sqrt{1 - \mathrm{e}^{c-\rho}} + \log[\mathrm{e}^{\rho/2} + \sqrt{\mathrm{e}^{\rho} - \mathrm{e}^{c}}], \quad c < \rho \tag{4.9d}$$

(where k and κ have the usual meanings: $\kappa^2 = -k^2$).

One may also express the general JWKB solution in the other form given in (4.2) where $\tilde{y}_1$ and $\tilde{y}_2$ are as in Example 1 of [6] (but in $(c, \rho$ now)), i.e., [20]:

$$y_{\rm JWKB}(c,\rho) = \tilde{c}_1(c)\tilde{y}_1(c,\rho) + \tilde{c}_2(c)\tilde{y}_2(c,\rho), \tag{4.10a}$$

$$\tilde{c}_1(c) = \frac{A(c)}{2} \cos[\alpha(c) - \pi/4] \tag{4.10b}$$

and

$$\tilde{c}_2(c) = A(c) \sin[\alpha(c) - \pi/4], \tag{4.10c}$$

$$\tilde{y}_1(c,\rho) = \begin{cases} \dfrac{2}{\sqrt{k(c,\rho)}} \sin[\eta(c,\rho) + \pi/4], \\ \quad \text{for } -\infty < \rho \le c, \\ \dfrac{1}{\sqrt{\kappa(c,\rho)}} \exp[-\zeta(c,\rho)], \\ \quad \text{for } c \le \rho < \infty \end{cases} \tag{4.10d}$$

and

$$\tilde{y}_2(c,\rho) = \begin{cases} \dfrac{1}{\sqrt{k(c,\rho)}} \sin[-\eta(c,\rho) + \pi/4], \\ \quad \text{for } -\infty < \rho \le c, \\ \dfrac{1}{\sqrt{\kappa(c,\rho)}} \exp[\zeta(c,\rho)] \\ \quad \text{for } c \le \rho < \infty. \end{cases} \tag{4.10e}$$

Here, c-dependent coefficients, $A(c)$ and $\alpha(c)$, in the JWKB general solution in (4.9a) or $\tilde{c}_1(c)$ and $\tilde{c}_2(c)$ in the other form in (4.10a) can be found by applying appropriately chosen initial values to these solutions. JWKB general solution of (3.8), which is given in (4.3), can be found easily by substituting (4.4) in the reverse direction as follows:

$$\rho\colon (0,\infty) \to (-\infty,\infty), \qquad \rho = c - 2\ln x \tag{4.11}$$

but the domains and ranges of ρ and x should be carefully treated as discussed in §4.2 (see Remark 2).

Now, we can see that application of the initial values in (3.11a) or (3.11b) to our LP system gives constant coefficients in (3.17) so that the exact solution can be written by (3.18). Since the JWKB solution in the right domain (as discussed above via the JWKB applicability criterion) should be accurate when compared with the exact solution in (3.18), it implies that: (i) we first need the JWKB approximation of the first-order Bessel functions for the JWKB complementary functions: $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ in (4.3), (ii) we also need numerical values the JWKB approximation of the second-order Bessel functions (of both types 1 and 2) at $x \to x_0$ for determining the constant coefficients: $\tilde{c}_1$ and $\tilde{c}_2$ in (4.3). Here, the $y(x)$ part in (4.3) has the initial values:

$$\begin{cases} y_{\text{JWKB}}(x)\big|_{x=x_0} = \tilde{c}_1\tilde{y}_1(x_0) + \tilde{c}_2\tilde{y}_2(x_0) = \alpha_1 \\ y'_{\text{JWKB}}(x)\big|_{x=x_0} = \tilde{c}_1\tilde{y}'_1(x_0) + \tilde{c}_2\tilde{y}'_2(x_0) = \beta_1 \end{cases} \tag{4.12}$$

corresponding to the following initial values in the $y(c, \rho)$ system:

$$\begin{cases} y_{\text{JWKB}}(c, \rho_0) = \tilde{c}_1(c)\tilde{y}_1(c, \rho_0) + \tilde{c}_2(c)\tilde{y}_2(c, \rho_0) \\ \qquad = \alpha_2(c) \\ \dfrac{\partial}{\partial\rho} y_{\text{JWKB}}(c, \rho)\,\big|_{\rho=\rho_0} \\ \quad = \left[\tilde{c}_1(c)\dfrac{\partial\tilde{y}_1(c, \rho)}{\partial\rho} + \tilde{c}_2(c)\dfrac{\partial\tilde{y}_2(c, \rho)}{\partial\rho}\right]_{\rho=\rho_0} \\ \quad = \beta_2(c) \end{cases} \tag{4.13}$$

whose linear solutions for the c-dependent coefficients give [20]

$$\tilde{c}_1(c) = \frac{\alpha_2(c)(\partial/\partial\rho)\tilde{y}_2(c, \rho)\big|_{\rho=\rho_0} - \beta_2(c)\tilde{y}_2(c, \rho_0)}{\tilde{\Delta}(c, \rho_0)} \tag{4.14a}$$

and

$$\tilde{c}_2(c) = \frac{-\alpha_2(c)(\partial/\partial\rho)\tilde{y}_1(c, \rho)\big|_{\rho=\rho_0} - \beta_2(c)\tilde{y}_1(c, \rho_0)}{\Delta(c, \rho_0)}, \tag{4.14b}$$

where the discriminant $\tilde{\Delta}(c, d)$ can be written as

$$\tilde{\Delta}(c, \rho_0) = \begin{vmatrix} \tilde{y}_1(c, \rho_0) & \tilde{y}_2(c, \rho_0) \\ \dfrac{\partial}{\partial\rho}\tilde{y}_1(c, \rho)\Big|_{\rho=\rho_0} & \dfrac{\partial}{\partial\rho}\tilde{y}_2(c, \rho)\Big|_{\rho=\rho_0} \end{vmatrix}. \tag{4.14c}$$

From (4.4) or (4.11), here we have the following relations between the points at which the initial values are chosen in two different systems:

$$\rho_0 \in (-\infty, \infty) \to x_0 = \exp\left(\frac{c - \rho_0}{2}\right)$$
$$\Longleftrightarrow x_0 \in (0, \infty) \to \rho_0 = c - 2\ln x_0, \tag{4.14d}$$

and consequently,

$$\Big[y(x) = \alpha_1\Big]_{x=x_0} = \Big[y(\rho) = \alpha_2(c)\Big]_{\rho=\rho_0}$$
$$\Rightarrow \boxed{\alpha_2(c) = \alpha_1}, \tag{4.14e}$$

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x}\bigg|_{x=x_0} = \beta_1 = \frac{\mathrm{d}y(\rho)}{\mathrm{d}\rho}\bigg|_{\rho=\rho_0} \frac{\mathrm{d}\rho(x)}{\mathrm{d}x}\bigg|_{x=x_0}$$
$$= -\left[\frac{2}{x}\beta_2(c)\right]_{x=x_0}$$
$$\Rightarrow \boxed{\beta_2(c) = -\tfrac{\beta_1 x_0}{2}}. \tag{4.14f}$$

4.2 *JWKB approximation of the first-order Bessel functions and calculations of their zeros* ($\nu = 1$)

Essential points in choosing initial values have already been discussed. They also imply that we can find the Bessel functions themselves (of order 1 for both type-1 and type-2) by the carefully chosen initial values. Our intention here is to determine two different initial values for two intentionally performed experiments whose readings are the first-order Bessel functions (of type-1 and -2) themselves for each in the general solution. This means that either of the complementary solution in (3.10) and hence approximately in (4.1) should be missing in each experimental reading by the two carefully chosen initial values with the same (or different) specific LP system. From (3.2a), (3.11a) and (3.11b) we see that LP parameters are: g, l_0, v (and the initial values: $\{\theta(x_0), \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}|_{x=x_0} = \theta_{x0}\}$ or $\{\theta(t_0), \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}|_{t=t_0} = \theta_{t0}\}$). So, our interest is to determine two different LP systems (say, $\text{LP}_1 = \text{LP}$ and $\text{LP}_2 = \text{LP}'$) which yield first-order Bessel functions of both types from (3.10):

$$\text{LP}: \{g, l_0, v, [\theta(x_0) \vee \theta(t_0)], [\theta_{x0} \vee \theta_{t0}]\}$$
$$\Rightarrow \left[\theta_{\text{EX}}(x) = c_1 x^{-1} J_1(x) \vee \theta_{\text{EX}}(t) = \theta_{\text{EX}}(x)\Big|_{x=\frac{2\sqrt{gl(t)}}{v}}\right]$$
$$\wedge \left[\theta_{\text{JWKB}}(x) = \tilde{c}_1 x^{-1}\tilde{J}_1(x) \vee \theta_{\text{JWKB}}(t) = \theta_{\text{JWKB}}(x)\Big|_{x=\frac{2\sqrt{gl(t)}}{v}}\right]$$

$$\text{LP}': \{g', l'_0, v', [\theta(x'_0) \vee \theta(t'_0)], [\theta_{x0'} \vee \theta_{t0'}]\}$$
$$\Rightarrow \Big[\theta_{\text{EX}}(x) = c_1 x^{-1} Y_1(x) \vee \theta_{\text{EX}}(t)$$

$$= \theta_{\mathrm{EX}}(x)\Big|_{x=\frac{2\sqrt{gl(t)}}{v}}\Bigg]$$
$$\wedge \Bigg[\theta_{\mathrm{JWKB}}(x) = \tilde{c}_1 x^{-1}\tilde{Y}_1(x) \vee \theta_{\mathrm{JWKB}}(t)$$
$$= \theta_{\mathrm{JWKB}}(x)\Big|_{x=\frac{2\sqrt{gl(t)}}{v}}\Bigg]. \quad (4.15)$$

Lemma 4.2. *If both* (3.10) *and* (4.1) *are* (*approximately*) *equivalent*, *so are* (3.9) *and* (4.2). *Then*:

(i) *Choosing initial values in* (4.12) *defined on a point* $x = x_0$ *in the domain satisfying* $y_{\mathrm{EX}}(x_0) = J_1(x_0) = \alpha_1$ *and* $y'_{\mathrm{EX}}(x_0) = J'_1(x_0) = \beta_1$, *one can find* $y_{\mathrm{EX}}(x) \to J_1(x) \approx y_{\mathrm{JWKB}}(x) \to \tilde{J}_1(x)$.
(ii) *Choosing initial values in* (4.12) *defined on a point* $x = x'_0$ *in the domain satisfying* $y_{\mathrm{EX}}(x'_0) = Y_1(x'_0) = \alpha'_1$ *and* $y'_{\mathrm{EX}}(x'_0) = Y'_1(x'_0) = \beta'_1$, *one can find* $y_{\mathrm{EX}}(x) \to Y_1(x) \approx y_{\mathrm{JWKB}}(x) \to \tilde{Y}_1(x)$.

Such specific initial values can be chosen from the LP experimentally via (3.10) as follows:

$$\text{(i)}: \left\{\begin{aligned} \theta(x_0) &= x_0^{-1}y(x_0) = x_0^{-1}\alpha_1 \\ \theta'(x_0) &= \theta_{x0} = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\bigg|_{x=x_0} \\ &= \frac{\mathrm{d}[x^{-1}J_1(x)]}{\mathrm{d}x}\bigg|_{x_0} \\ &= -x_0^{-2}\alpha_1 + x_0^{-1}\beta_1 \end{aligned}\right\}\Bigg|_{\substack{\alpha_1=J_1(x_0)\\ \beta_1=J'_1(x_0)}} \quad (4.16\text{a})$$

$$\text{(ii)}: \left\{\begin{aligned} \theta(x'_0) &= x_0'^{-1}y(x'_0) = x_0'^{-1}\alpha'_1 \\ \theta'(x'_0) &= \theta_{x0'} = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\bigg|_{x=x'_0} \\ &= \frac{\mathrm{d}[x^{-1}Y_1(x)]}{\mathrm{d}x}\bigg|_{x'_0} \\ &= -x_0'^{-2}\alpha'_1 + x_0'^{-1}\beta'_1 \end{aligned}\right\}\Bigg|_{\substack{\alpha'_1=Y_1(x'_0)\\ \beta'_1=Y'_1(x'_0)}} \quad (4.16\text{b})$$

where primed functions (θ', J'_1, Y'_1) represent derivatives and primed constants ($x'_0, \alpha'_1, \beta'_1$) represent different constant values in comparison with the unprimed ones. So, by choosing initial values $\theta(x_0)$ and θ_{x_0} according to (4.17d), we experimentally get $\theta(x) = x^{-1}J_1(x)$; and similarly, by choosing initial values $\theta(x'_0)$ and $\theta_{x'_0}$ according to (4.18d), we experimentally get $\theta(x) = x^{-1}Y_1(x)$ as desired. Bessel functions of the first-order (J_1 and Y_1) can then be easily determined. Here we use these initial values to determine the JWKB solutions of first-order Bessel functions ($\tilde{J}_1$ and $\tilde{Y}_1$). Our aim is to solve a specific LP system to obtain the Bessel function of first order by the semiclassical JWKB method without interfering the exact solutions. Although we make use of the exact solutions to determine these specific initial values (and hence to determine such specific LP systems), they can be thought as the experimental values by assuming it to be an ideal experiment (no resistive friction force, no measurement error, etc.). It is clear from (3.12) that, these intentionally chosen experimental initial values depend on the experimental parameters decorating any specific LP.

Remark 1. Note that, JWKB solutions in the CIR normally gives wrong results and they need asymptotical matching (or, modification) by imposing an extra asymptotical initial value for correct approximate results [6,7,20]. However, as the zeros of the first-order Bessel functions do not correspond to the CIR, which is $0 < x < 1$ (and by ignoring the non-obedient narrow region: $x \in (0.554858, 1.62327)$ – see eq. (4.8b) [34b]) we can safely use the unmodified JWKB solutions to find the zeros of the first-order Bessel functions for simplicity here. If we assume that we did not know this in advance, or if there were a scale transformation in x so that their zeros coincided with the CIR, then the correct expression in finding zeros would be calculated by using the modified JWKB solutions. Similarly, if we wished to start from zeroth-order Bessel functions, it would also be necessary to use the asymptotically modified results because the zeros of the zeroth-order Bessel functions coincide with their CIR which needs asymptotic modifications as we know already (see the next section).

(i) For finding $\tilde{J}_1(x)(\approx J_1(x))$ via Lemma 4.2, we have the following:

$$x_0 = 10.1734(68135062722...) \Rightarrow l_0 = \frac{x_0^2 v^2}{4g}$$
$$\Rightarrow \{y_{\mathrm{EX}}(x_0) = J_1(x_0) = \alpha_1 = 0,$$
$$y'_{\mathrm{EX}}(x_0) = J'_1(x_0) = \beta_1 = -0.249705\}, \quad (4.17\text{a})$$

and according to eqs (4.14d)–(4.14f), this IVP corresponds to ρ to as follows:

$$\rho_0 = c - 4.63957 \Rightarrow \alpha_2(c) = 0 \Rightarrow \beta_2(c) = 1.27018 \quad (4.17\text{b})$$

from which we can calculate the coefficients in the general JWKB solutions given in (4.9a) or in (4.10a) via (4.10b)–(4.10c) as follows:

$$\left\{\begin{array}{l} A(c) = -0.564548, \quad \alpha(c) = -8.65186 \\ \tilde{c}_1(c) = c_1 = 0.282252, \\ \tilde{c}_2(c) = c_2 = -0.00704489 \end{array}\right\} \tag{4.17c}$$

$$\Rightarrow y_{\mathrm{JWKB}}(x) = \tilde{J}_1(x)$$

whose results for the solution of $J_1(x)$ (both exact and JWKB solutions) are given in figure 2 on the left, which are consistent with the results given in [17,23,24]. Remember that in our JWKB analysis, the x domain of our transformation according to (4.11) is $x \in (0, \infty]$ (see Remark (2)). In this specific LP problem, assuming that we have ideal experiment with exact results, our experimental initial values to determine $J_1(x)$ via (4.16a)–(4.17a) should have the following experimental readings:

$$\{\theta(x_0) = 0, \theta_{x_0} = -0.0245447\}, \tag{4.17d}$$

$$\left\{\theta(t_0 = 0) = \theta\left(x_0 = \frac{2\sqrt{gl(t)}}{v}\Big|_{t=t_0=0}\right)\right.$$
$$= \frac{v}{2\sqrt{gl_0}}\alpha_1 = \frac{\alpha_1}{x_0} = \theta(x_0) = 0,$$
$$\theta_{t_0=0} = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\Big|_{x_0} \frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big|_{t=t_0=0}$$
$$= \theta_{x_0}\sqrt{\frac{g}{l_0 + vt_0}}\Big|_{t=t_0=0}$$
$$\left.= \theta_{x_0}\sqrt{\frac{g}{l_0}} = \frac{2\theta_{x_0}}{x_0 v}\right\}, \tag{4.17e}$$

which can be generalized to mean a quantization and the corresponding specific initial values in the time domain becomes

$$x_0 = \frac{2\sqrt{gl(t)}}{v}\Big|_{t=t_0=0} = n\text{th root of } J_{\nu=1}(x) = 0$$
$$\Rightarrow n\text{th quantization in initial values (experimentally):}$$
$$\left\{\theta(t_0 = 0) = 0,\right.$$
$$\theta_{t_0=0} = \theta_{x_0}\sqrt{\frac{g}{l_0}} = \frac{2\theta_{x_0}}{x_0 v}$$
$$\left.\Rightarrow \theta(t), = [x(t)]^{-1} J_1(t)\right\}. \tag{4.17f}$$

In (4.17a), $n = 3$rd zero of $J_1(x)$ is chosen without loss of generality (but remembering that; the further away from $x = 0$, the better JWKB solution, according to our $G(x)$ analyses) and it gives (4.17b)–(4.17e) via the relevant quantization rule for J_1 given in (4.17f).

(ii) For finding $\tilde{Y}_1(x)$, $(\approx Y_1(x))$ via Lemma 4.2, we have the following:

$$x'_0 = 8.59600(5868331188...) \Rightarrow l'_0 = \frac{x_0'^2 v^2}{4g}$$
$$\Rightarrow \{y_{\mathrm{EX}}(x'_0) = Y_1(x'_0) = \alpha'_1 = 0,$$
$$y'_{\mathrm{EX}}(x'_0) = Y'_1(x'_0) = \beta'_1 = 0.27146\}, \tag{4.18a}$$

and similarly this IVP corresponds via (4.14d)–(4.14f) corresponds in ρ to as follows:

$$\rho'_0 = c - 4.3026 \Rightarrow \alpha'_2(c) = 0 \Rightarrow \beta'_2(c) = -1.16674 \tag{4.18b}$$

from which we can calculate the coefficients in the general JWKB solutions given in (4.9a) or in (4.10a) via (4.10b)–(4.10c) as follows:

$$\left\{\begin{array}{l} A'(c) = 0.564701, \alpha'(c) = -7.08344 \\ \tilde{c}'_1(c) = c'_1 = -0.00419515, \tilde{c}'_2(c) = c'_2 = -0.564638 \end{array}\right\}$$
$$\Rightarrow y_{\mathrm{JWKB}}(x) = \tilde{Y}_1(x) \tag{4.18c}$$

whose results for the solution of $Y_1(x)$ (both exact and JWKB solutions) are given in figure 2 on the right. In this specific LP problem, assuming that we have ideal experiment with exact results, our experimental initial values to determine $Y_1(x)$ via (4.16b)–(4.18a) should have the following experimental readings:

$$\{\theta(x'_0) = 0, \theta_{x'_0} = 0.0315798\}, \tag{4.18d}$$

$$\left\{\theta(t'_0) = \theta\left(x'_0 = \frac{2\sqrt{gl(t)}}{v}\Big|_{t=t'_0=0}\right)\right.$$
$$= \frac{v}{2\sqrt{gl'_0}}\alpha'_1 = \frac{\alpha'_1}{x'_0} = \theta(x'_0) = 0,$$
$$\theta_{t'_0} = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\Big|_{x'_0} \frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big|_{t=t'_0=0} = \theta_{x'_0}\sqrt{\frac{g}{l'_0 + v't'_0}}\Big|_{t'_0=0}$$
$$\left.= \theta_{x'_0}\sqrt{\frac{g}{l'_0}}\Big|_{l'_0=l_0} = \frac{2\theta_{x'_0}}{x'_0 v'}\Big|_{v'=v} = 0.0998641\right\}, \tag{4.18e}$$

which can be generalized to mean a quantization and the corresponding specific initial values in the time domain becomes:

$$x'_0 = \frac{2\sqrt{gl(t)}}{v}\Big|_{t=t_0=0} = n\text{th root of } Y_{\nu=1}(x) = 0$$
$$\Rightarrow n\text{th quantization in initial values (experimentally):}$$
$$\left\{\theta(t'_0 = 0) = 0,\right.$$
$$\theta_{t'_0=0} = \theta_{x'_0}\sqrt{\frac{g}{l'_0}} = \frac{2\theta_{x'_0}}{x'_0 v'}$$
$$\left.\Rightarrow \theta(t) = [x(t)]^{-1} Y_1(t)\right\}. \tag{4.18f}$$

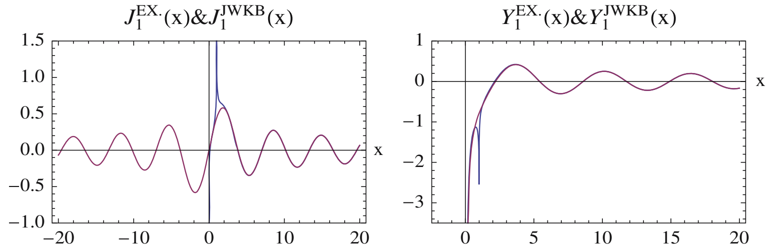

Figure 2. The graphs of the exact and JWKB-approximated Bessel functions of the first order of type-1 ($=J_1(x)$ on the left) and of the first-order of type-2 ($=Y_1(x)$ on the right).

In (4.18a), $n = 3$rd zero of $Y_1(x)$ is chosen without loss of generality (but remembering that, the further away from $x = 0$, the better is the JWKB solution, according to our $G(x)$ analyses) and it gives (4.18b)–(4.18e) via the relevant quantization rule for Y_1 given in (4.18f).

Remark 2. Note that here the domain of the JWKB solutions of the Bessel functions lies on $(0, \infty)$ as seen in (4.11) where $\rho \in (-\infty, \infty)$ and $x \in (0, \infty)$ and this agrees with the exact solution of $Y_1(x)$. However, it does not agree with the exact solution of $J_1(x)$ whose domain actually lies on $(-\infty, \infty)$. So, by the transformation in (4.4) and re-transformation in (4.11) suggested here, we are studying the Bessel functions (of first order and of first and second kind) in $x \in (0, \infty)$. However, one may extend the domain of the JWKB solutions of $J_1(x)$ to $(-\infty, 0)$ by using the following oddity/evenness relations given in the theory of Bessel functions [17,23]:

$$J_n\colon (0, \infty) \to (-\infty, \infty),$$

$$J_n(x) = \begin{cases} -J_n(-x), & \text{if } n = 1, 3, 5, \ldots\text{(odd)}, \\ J_n(-x), & \text{if } n = 0, 2, 4, \ldots\text{(even)}. \end{cases} \tag{4.19}$$

So, for the first-order Bessel functions we can write

$$J_1\colon (0, \infty) \to (-\infty, \infty), \quad J_1(x) = -J_1(-x). \tag{4.20}$$

Zeros of the unmodified JWKB solutions whose graphs are given in figure 2 are given in table 1.

5. Extensions to the other orders of Bessel functions ($\nu = 0, 2$)

General JWKB solution of the νth-order BDE:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \tag{5.1}$$

Table 1. Zeros of Bessel functions of first order.

	Roots of $J_{\nu=1}(x) = 0$ $(x_{\nu=1,n})$		Roots of $Y_{\nu=1}(x) = 0$	
n	Exact	JWKB	Exact	JWKB
1	3.83171	3.8203	2.19714	2.12349
2	7.01559	7.02221	5.42968	5.41257
3	10.1735	10.186	8.59601	8.58853
4	13.3237	13.3392	11.7492	11.7458
5	16.4706	16.488	14.8974	14.8964
6	19.6159	19.6344	18.0434	18.0439
7	22.7601	22.7795	21.1881	21.1896
8	25.9037	25.9238	24.3319	24.3342
9	29.0468	29.0675	27.4753	27.4782
10	32.1897	32.2108	30.6183	30.6216
11	35.3323	35.3537	33.7610	33.7647
12	38.4748	38.4965	36.9036	36.9076
13	41.6171	41.639	40.0459	40.0503
14	44.7593	44.7815	43.1882	43.1928
15	47.9015	47.9238	46.3304	46.3351
16	51.0435	51.066	49.4725	49.4774
17	54.1856	54.2082	52.6146	52.6196
18	57.3275	57.3503	55.7565	55.7617
19	60.4695	60.4923	58.8985	58.9038
20	63.6114	63.6343	62.0404	62.0458

can be calculated by first transforming into the normal form via an appropriate change of variable and then the related Bessel functions (J_ν and Y_ν) along with their zeros can be obtained by imposing some carefully chosen initial values as we have already done for $\nu = 1$ in the previous section. However, our starting point here is the JWKB solutions of the first-order ($\nu = 1$) Bessel functions from which we wish to seek ways to solve the LP problem. We give the following proposition that the other order JWKB solutions can be obtained by using the results of the first-order Bessel functions with $\nu = 1$ which we have already obtained successfully.

PROPOSITION 5.1

If the JWKB solutions of the $(\nu - 1)th$, $(\nu)th$, *and* $(\nu + 1)th$ *order BDEs* (*that is*, *the Bessel functions of the mentioned orders*) *and their first derivatives are really good approximations to the exact solutions*, *then the conventional Bessel recurrence relations* (*relating three adjacent Bessel orders*) *in Bessel theories*, *i.e.*, *in* [17,26,27], *given for the exact solutions by*:

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x), \tag{5.2a}$$

$$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z'_\nu(x), \tag{5.2b}$$

(*where* Z *stands for either* J *or* Y) *can also be expected to be valid for their JWKB solutions.*

COROLLARY 5.2

The following alternative recursion relation relating two adjacent Bessel orders:

$$Z_{\nu+1}(x) = \frac{\nu Z_\nu(x) - x Z'_\nu(x)}{x} \tag{5.3a}$$

$$Z_{\nu-1}(x) = \frac{\nu Z_\nu(x) + x Z'_\nu(x)}{x} \tag{5.3b}$$

obtained from (5.2a) *and* (5.2b) *can be used to find the JWKB solutions of* $Z_2 \to \{J_2(x), Y_2(x)\}$ *and* $Z_0 \to \{(J_0(x), Y_0(x)\}$ *pairs* (*by taking* $\nu = 1$) *from the JWKB solutions of* $Z_1 \to \{J_1(x), Y_1(x)\}$ *pairs which have already been found* (*and shown to be accurate*) *above.*

The JWKB graphs of the $\nu = 2$nd- and $\nu = 0$th-order Bessel functions obtained according to Corollary 5.2 from the recurrence relations (5.3a) and (5.3b) along with their exact graphs are given in figure 3. The results of calculations of their nth zeros are given in tables 2 and 3. We can see that the solutions of the $\nu = 2$nd- and $\nu = 0$th order Bessel functions are also very accurate in the JWKB applicable region and their accuracy increases asymptotically as $x \to \infty$. To check this, we again consult the first-order JWKB applicability (or accuracy) criteria given in (4.7a) with the substitution of (4.4) as we did in our intense $\nu = 1$ analysis. For the νth-order Bessel functions, we have

$$f(c, \rho, \nu) = k^2(c, \rho, \nu) = \frac{1}{4}(\mathrm{e}^{c-\rho} - \nu^2) \tag{5.4}$$

and using $k(c, \rho, \nu)$ (for $k(c, \rho)$ in (4.7a)), we get

$$0 \le G(c, \rho, \nu) = \frac{1}{4}\mathrm{e}^{Re(\rho+c)} \left| \frac{4\mathrm{e}^\rho \nu^2 + \mathrm{e}^c}{(\mathrm{e}^\rho \nu^2 - \mathrm{e}^c)^3} \right| \ll 1. \tag{5.5}$$

The graphs of f and G in both ρ and x systems for $\nu = 0$, 1, and 2 are given in figure 4. Remembering that we obtained JWKB approximations of $Z_0(x)$ and $Z_2(x)$ from the JWKB-approximated solution of $Z_1(x)$ where $Z_\nu(x)$ stands for either $J_\nu(x)$ or $Y_\nu(x)$, we see

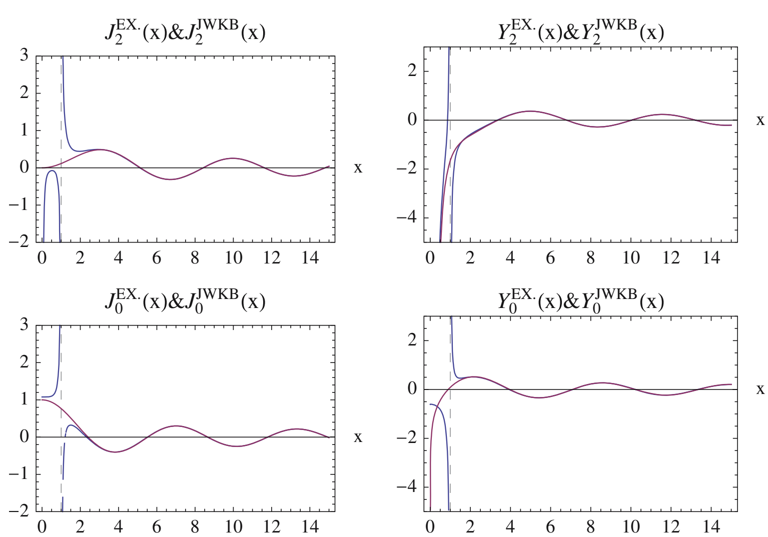

Figure 3. The graphs of JWKB-approximated and exact Bessel functions of second and zeroth order of type-1 (left column) and type-2 (right column).

Table 2. Zeros of Bessel functions of second order.

	Roots of $J_{\nu=2}(x) = 0$ ($x_{\nu=2,n}$)		Roots of $Y_{\nu=2}(x) = 0$	
n	Exact	JWKB	Exact	JWKB
1	5.13562	5.13177	3.38424	3.32906
2	8.41724	8.42671	6.79381	6.78051
3	11.6198	11.6339	10.0235	10.0178
4	14.796	14.8124	13.21	13.2077
5	17.9598	17.9778	16.379	16.3786
6	21.117	21.1361	19.539	19.54
7	24.2701	24.2899	22.694	22.6958
8	27.4206	27.441	25.8456	25.8482
9	30.5692	30.5901	28.9951	28.9982
10	33.7165	33.7378	32.143	32.1465
11	36.8629	36.8844	35.2898	35.2937
12	40.0084	40.0303	38.4357	38.4399
13	43.1535	43.1755	41.581	41.5854
14	46.298	46.3203	44.7258	44.7304
15	49.4422	49.4646	47.8701	47.8749
16	52.586	52.6086	51.0141	51.0191
17	55.7296	55.7524	54.1579	54.163
18	58.873	58.8959	57.3013	57.3066
19	62.0162	62.0392	60.4446	60.45
20	65.1593	65.1823	63.5878	63.5932

from figure 4 that for $\nu = 0, 1, 2$, the criterion in (5.5) is satisfied to give more accurate results as $x \to \infty$, which is in agreement with the graphs of the JWKB solutions given in figure 3. Note that, in order to obtain accurate results in the domain where the criterion in (5.5) is not satisfied, higher-order JWKB approximations [5–7] are needed, which is out of our scope here. For our traditional first-order analysis, we can say that

Table 3. Zeros of Bessel functions of zeroth order.

	Roots of $J_{\nu=0}(x) = 0$ ($x_{\nu=0,n}$)		Roots of $Y_{\nu=0}(x) = 0$	
n	Exact	JWKB	Exact	JWKB
1	2.40483	2.34587	0.893577	–
2	5.52008	5.52066	3.95768	3.92833
3	8.65373	8.66381	7.08605	7.07504
4	11.7915	11.8057	10.2223	10.2173
5	14.9309	14.9474	13.3611	13.3591
6	18.0711	18.0891	16.5009	16.5007
7	21.2116	21.2307	19.6413	19.6423
8	24.3525	24.3723	22.782	22.7839
9	27.4935	27.5139	25.923	25.9255
10	30.6346	30.6555	29.064	29.0671
11	33.7758	33.7971	32.2052	32.2087
12	36.9171	36.9387	35.3465	35.3503
13	40.0584	40.0803	38.4878	38.4919
14	43.1998	43.2218	41.6291	41.6335
15	46.3412	46.3634	44.7705	44.7751
16	49.4826	49.505	47.9119	47.9167
17	52.6241	52.6466	51.0533	51.0583
18	55.7655	55.7882	54.1948	54.1999
19	58.907	58.9298	57.3362	57.3415
20	62.0485	62.0714	60.4777	60.4831

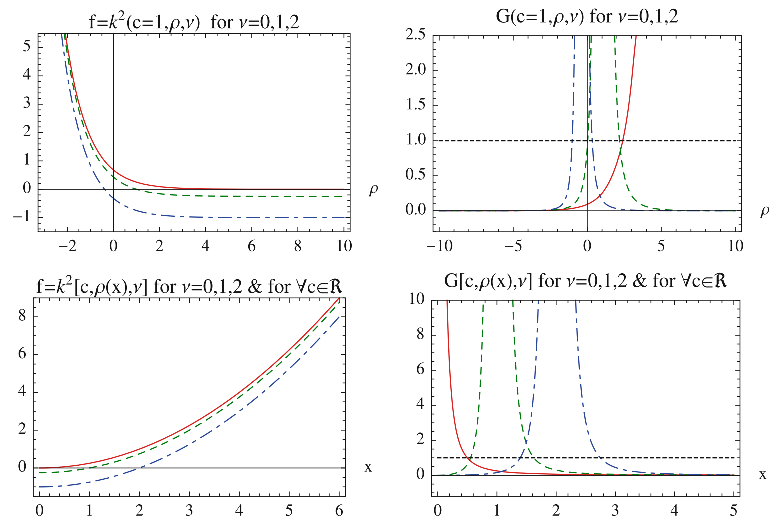

Figure 4. Graph of f and G functions for $c = 1$ and for some ν values (red curve: $\nu = 0$, green dotted curve: $\nu = 1$, blue dashed curve: $\nu = 2$).

one can obtain accurate JWKB solutions in the CAR as long as the criterion (5.5) is satisfied and asymptotic matching is required for the CIR as explained [6,7]. The domain of the accurately obtained JWKB solution for the $\nu = 1$st order from which we derive the other orders obviously has a special importance here. This also explains why we could not have obtained the $n = 1$st zero of $Y_0(x)$ in table 3 because its exact value (=0.893577) corresponds to the small region where the JWKB applicability criterion for the $\nu = 1$st order fails (see eq. (4.8b) and figure 4).

Remark 3. Graphs of $J_0(x)$ and $J_2(x)$ are plotted in figure 2 in $x \in (0, \infty)$ because it is the domain of the transformed x-system. However, as just seen in the plot of the exact solution of $J_1(x)$ in figure 2, the real domain of $J_0(x)$ and $J_2(x)$ are $x \in (-\infty, \infty)$, and by using eq. (4.19), we have even $J_0(x)$ and $J_2(x)$ functions. So, we can extend the domain of our JWKB solution through the LHS by

$$J_{\nu=0,2}\colon (0,\infty) \rightarrow (-\infty,\infty),$$

$$J_{\nu=0,2}(x) = J_{\nu=0,2}(-x) \qquad (5.6)$$

when needed.

6. A numerical example for the JWKB solution of a specific uniformly lengthening pendulum

For the given lengthening parameters (l_0 and v) in (3.1) along with the initial values given in (3.11a), one can find $\theta_{\mathrm{EX}}(t)$ from (3.18) by using $\theta_{\mathrm{EX}}(t) = \theta_{\mathrm{EX}}[x(t)]$. Similarly, JWKB solution becomes

$$\theta_{\mathrm{JWKB}}(x) = x^{-1} y_{\mathrm{JWKB}}(x) = -\frac{\pi x_0^2}{2}\theta_0 \tilde{Y}_2(x_0) x^{-1} \tilde{J}_1(x) + \frac{\pi x_0^2}{2}\theta_0 \tilde{J}_2(x_0) x^{-1} \tilde{Y}_1(x) \qquad (6.1)$$

and for the time domain, via (3.12) we have: $\theta_{\mathrm{EX}}(t) = \theta_{\mathrm{EX}}[x \rightarrow x(t)]$ and $\theta_{\mathrm{JWKB}}(t) = \theta_{\mathrm{JWKB}}[x \rightarrow x(t)]$. Here, $\tilde{J}_{1,2}(x)$ and $\tilde{Y}_{1,2}(x)$ are the JWKB-approximated Bessel functions we have already obtained from the carefully chosen initial values and we use them in (6.1) to find the JWKB solution of a specific LP system. The exact and JWKB solution of the LP problem for the following specific values:

$$\theta_0 = 5^\circ = 0.0872222 \text{ rad}, \; \theta_{x0} = \left.\frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\right|_{x=x_0} = 0^\circ \,[34c],$$

$$l_0 = 1 \text{ m}, \quad v = 1 \text{ m/s}, \quad g = 10 \text{ m/s}^2 \qquad (6.2)$$

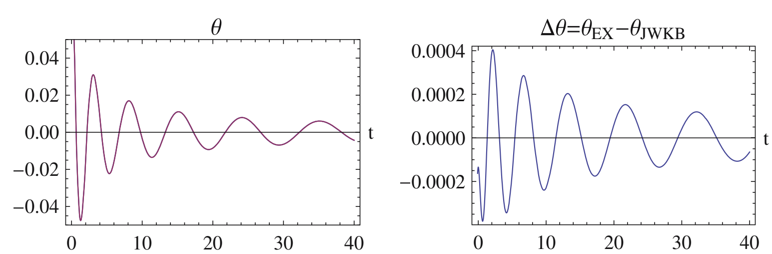

Figure 5. Graphs of exact and JWKB solutions (on the left) and error: $\Delta\theta = \theta_{\mathrm{EX}} - \theta_{\mathrm{JWKB}}$ (on the right) of the LP with some typical parameters: $\theta_0 = 5^\circ$, $\theta_{x0} = 0^\circ$, $l_0 = 1$ m, $v = 1$ m/s, $g = 10$ m/s^2.

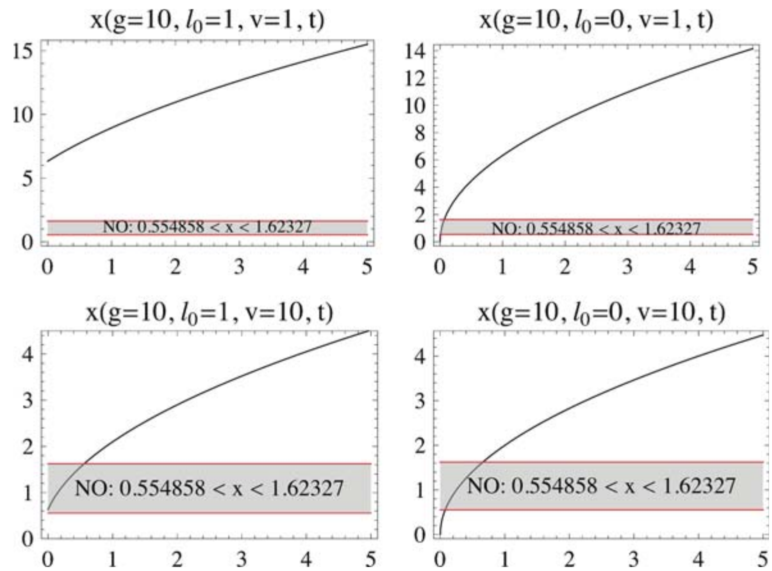

Figure 6. Graphs of $x(g, l_0, v, t)$ for some given g, l_0, v parametric values of a specific LP as a function of time.

are given in figure 5 for comparison. The validity of our JWKB solution for this specific LP can be seen in figure 6 (on the upper left graph) where the shaded rectangle shows the non-obedient (NO) region where JWKB method fails. Rest of the graphs in figure 6 are for the same LP system with only one of the parameters changed in each.

But now, let us discuss how to choose the initial values for this LP system to determine the required initial values in finding J_1 and Y_1. In this specific LP problem, according to (3.12) we have: $x_0 = 2\sqrt{10}$ and assuming that we have ideal experiment (to give exact results), our experimental initial values to determine $J_1(x) \approx \tilde{J}_1(x)$ via (4.17d) $\Rightarrow$ (4.17a)–(4.17e) and similarly; to determine $Y_1(x) \approx \tilde{Y}_1(x)$ via (4.18d) $\Rightarrow$ (4.18a)–(4.18e) we have the following:

(i) For finding $\tilde{J}_1(x)$ ($\approx J_1(x)$), results in (4.17f) with $x_0 = 10.1734(68135062722...)$ as given in (4.17a) along with the consequences in (4.17b)–(4.17e) give the following equations:

$$\boxed{\begin{array}{l} \text{Quantization for } n=3\text{: } x_0 = \left.\dfrac{2\sqrt{g(l_0+vt)}}{v}\right|_{t=t_0=0} = 10.1734(68135062722...) \\ \text{Exp. initial values for } n=3\text{: } \left\{ \begin{array}{l} \{\theta(x_0)=0,\ \theta_{x_0} = -0.0245447 \text{ rad}\} \\ \left\{\theta(t_0=0)=0,\ \theta_{t_0=0} = \theta_{x_0}\sqrt{\dfrac{g}{l_0}} = \dfrac{2\theta_{x_0}}{x_0 v} = \dfrac{-0.00482524}{v}\right\} \end{array}\right. \\ \text{Exp. result: } \theta(t) = [x(t)]^{-1} J_1(t) \approx \tilde{\theta}(t) \end{array}} \tag{6.3}$$

which means, for any given two parameters among $\{g, l_0, v\}$ in (6.2), the third one should be quantized according to (6.3), and the experimental initial values should be as in (6.3) too. Let us assume that, $g = 10\ \text{m/s}^2, l_0 = 1\ \text{m} \Rightarrow v = ?$. Then, for finding $\tilde{J}_1(x)(\approx J_1(x))$ via Lemma 4.2:

$$\begin{aligned} & x_0 = 10.1734(68135062722...) \\ & \Rightarrow \rho_0 = c - 2\ln x_0 = c - 4.63957 \\ & \Rightarrow \{y_{\text{EX}}(x_0) = J_1(x_0) = \alpha_1 = 0, \\ & \qquad y'_{\text{EX}}(x_0) = J'_1(x_0) = \beta_1 = -0.249705\} \\ & \Rightarrow \left\{\alpha_2 = \alpha_1 = 0,\ \beta_2 = \frac{-\beta_1 x_0}{2} = 1.27019\right\} \\ & \Rightarrow \text{Exp. parameters: } l_0 = 1\ \text{m},\ v = ?\ \text{m/s}, \\ & g = 10\ \text{m/s}^2 \Rightarrow v = \frac{2\sqrt{g l_0}}{x_0} = \boxed{0.621670\ \text{m/s}} \\ & \Rightarrow \left\{\theta(x_0) = \frac{\alpha_1}{x_0} = 0,\right. \\ & \left.\theta_{x_0} = -x_0^{-2}\alpha_1 + x_0^{-1}\beta_1 = -0.0245447\right\} \\ & \Rightarrow \text{Exp. initial values: } \left\{\theta(t_0=0) = \frac{\alpha_1}{x_0} = 0 \text{ rad},\right. \\ & \left.\theta_{t_0=0} = \frac{2\theta_{x_0}}{x_0 v} = 0.0245446 \text{ rad/s}\right\}. \end{aligned} \tag{6.4}$$

(ii) For finding $\tilde{Y}_1(x)$ ($\approx Y_1(x)$), results in (4.18f) with $x_0 = 8.59600(5868331188...)$ as given in (4.18a) along with the consequences in (4.18b)–(4.18e) give the following equations:

$$\begin{array}{|l|}\hline \text{Quantization for } n=3\text{: } x_0' = \left.\dfrac{2\sqrt{g'(l_0'+v't')}}{v'}\right|_{t'=t_0'=0} = 8.59600(5868331188...) \\ \text{Exp. initial values for } n=3\text{: } \left\{\begin{array}{l}\{\theta(x_0')=0,\ \theta_{x_0'}=0.0315798 \text{ rad}\} \\ \left\{\theta(t_0'=0)=0,\ \theta_{t_0'=0}=\theta_{x_0'}\sqrt{\dfrac{g'}{l_0'}}=\dfrac{2\theta_{x_0'}}{x_0'v'}=\dfrac{-0.00734755}{v'}\right\}\end{array}\right. \\ \text{Exp. result: } \theta(t)=[x(t)]^{-1}Y_1(t)\approx\tilde{\theta}(t) \\ \hline\end{array} \tag{6.5}$$

which means, for any given two parameters among $\{g,\ l_0, v\}$ in (6.2), the third one should be quantized according to (6.5), and the experimental initial values should be as in (6.5), too. Let us assume that, $g'=10\ \text{m/s}^2, l_0'=1\ \text{m} \Rightarrow v'=?$. Then, for finding $\tilde{Y}_1(x)$ $(\approx Y_1(x))$ via Lemma 4.2:

$$\begin{aligned}
&x_0' = 8.59600(5868331188...) \\
&\Rightarrow \rho_0 = c - 2\ln x_0 = c - 4.3026 \\
&\Rightarrow \{y_{\text{EX}}(x_0') = Y_1(x_0') = \alpha_1' = 0 = \alpha_1, \\
&\quad y_{\text{EX}}'(x_0') = Y_1'(x_0') = \beta_1' = 0.27146\} \\
&\Rightarrow \left\{\alpha_2' = \alpha_1' = 0,\ \beta_2' = \frac{-\beta_1' x_0'}{2} = -1.16674\right\} \\
&\Rightarrow \text{Exp. parameters: } l_0' = l_0 = 1\ \text{m},\ v' = ?\ \text{m/s}, \\
&g' = g = 10\ \text{m/s}^2 \Rightarrow v' = \frac{2\sqrt{g'l_0'}}{x_0'} = \boxed{0.735755\ \text{m/s}} \\
&\Rightarrow \left\{\theta(x_0') = \frac{\alpha_1'}{x_0'} = 0,\ \theta_{x_0'} = -x_0'^{-2}\alpha_1' + x_0'^{-1}\beta_1'\right. \\
&\qquad \left. = 0.0255168\right\} \\
&\Rightarrow \text{Exp. initial values: } \left\{\theta(t_0'=0) = \frac{\alpha_1'}{x_0'} = 0\ \text{rad},\right. \\
&\left.\theta_{t_0'=0} = \frac{2\theta_{x_0'}}{x_0' v} = 0.0734755\ \text{rad/s}\right\}.
\end{aligned} \tag{6.6}$$

Here, according to Lemma 4.2, we have chosen x_0 in such a way that $J_1(x_0) = \alpha_1 = 0$ and $J_1'(x_0) = \beta_1 \neq 0$ and used the same LP system with $g = g' = 10\ \text{m/s}^2, l_0 = l_0' = 1\ \text{m}, v \neq v'$ set (where primes represent the second experiment parameters) but in two different initial values with $x_0' \neq x_0 \Rightarrow \{\alpha_1' = \alpha_1 = 0, \beta_1' \neq \beta_1\}$ to determine both types of Bessel functions of the first order.

However, to consider (or perform) two different experiments with two different LP systems by choosing two partially or entirely different set values: $\{g, l_0, v\}$ and $\{g', l_0', v'\}$ (which means two different specific LP systems under the experiment) is also possible provided that (4.17e)–(4.17f) and (4.18e)–(4.18f) hold. An experiment with only l_0 is changed among the g, l_0, v set is given in Appendix as an example for such alternative choices.

7. Conclusion

We can see that the BDE, whose common form is given in (5.1), can be solved in the semiclassically solvable domain (according to the criterion in (4.7a)) by the traditional first-order JWKB technique via the change of independent variable as given in (4.4). We obtained the first-order Bessel functions (of the first and second kinds) by the carefully chosen initial values according to Lemma 4.2 as in (4.17a)–(4.18a). Our JWKB calculations here are purely semiclassical except for the choice of the initial values in (4.17a)–(4.18a) to find the JWKB approximation of the Bessel functions of first order because we make use of their exact solutions, and such exact-aided initial values can be used provided that care regarding the JWKB applicability criterion discussed in §4.2 has been taken. On the other hand, in a specific physical LP problem, such initial values for determining related Bessel functions (and hence the solution of the LP) can be found without consulting the exact solutions (not from the use of the computer-aided Bessel functions themselves), but from the initial values of the physical system under study experimentally as given in (4.17d)–(4.18d) and correspondingly (4.17e)–(4.17f) and (4.18e)–(4.18f). In this case, (α_1 and β_1) and (α_1' and β_1') pairs can be determined from the algebraic solutions of the experimental initial values in (4.17d)–(4.18d) for $\alpha, \beta, \alpha', \beta'$. Once these initial values are given, we experimentally get Bessel functions of the first order, namely (initial values: α_1 and β_1) $\rightarrow$ (experimentally: $J_1 = x\theta \rightarrow \approx \tilde{J}_1 = x\tilde{\theta}$) and (initial values: α_1' and β_1') $\rightarrow$ (experimentally: $Y_1 = x\theta \rightarrow \approx \tilde{Y}_1 = x\tilde{\theta}$).

Our choice for the initial values in (4.17a) and (4.18a) according to Lemma 4.2 has been given in (4.17e) and (4.18e), respectively, and the generalizations given in (4.17f) and (4.18f) show the experimental rules for the specific LP system to determine J_1 and Y_1 where quantization in one parameter of the three and the related initial values are defined. An alternative choice obeying Lemma 4.2, but not the generalizations given in (4.17f) and (4.18f) also works very well as given in the Appendix as an extra example, because Lemma 4.2 does not necessarily require x_0 at which the initial values are defined to be the nth root of the Bessel functions of first order. But we have done this in (4.17f) and (4.18f) for simplicity in our calculations and also for convenience by using the well-known nth zero values.

In our semiclassical JWKB analysis, we obtained the first-order Bessel functions (both $J_1(x)$ and $Y_1(x)$) very accurately in the classically accessible domain from the initial values chosen as (4.17a)–(4.18a) as a starting point. However, it is clear that one can start from any other order to extend it to its adjacent orders step by step in a similar fashion via using the recurrence relations in (5.3a) and (5.3b), too. The JWKB solutions of the $\nu = 0$th and $\nu = 2$nd-order Bessel functions and their zeros obtained from the results of the JWKB solutions of the $\nu = 1$st-order by the recursion relations are very accurate and in agreement with the exact results along with the results given in [23,24,29], too. Although the Bessel functions of the $\nu = 2$nd and $\nu = 0$th orders have been obtained via the asymptotically unmatched JWKB solutions of the $\nu = 1$st order, we have very accurate results except for the $n = 1$st zero of $Y_0(x)$ as explained above. Using the recurrence relations, one seem to go further to obtain the JWKB solutions of the higher integer order Bessel functions very accurately if desired.

These results are also in a very good agreement with our sample physical application, the LP problem studied here. So, the JWKB approximation method, finding conventional applications in quantum mechanical and optical waveguide systems [5], is being used here to solve the classical mechanical LP problem successfully. JWKB solution of exponential potential decorated quantum mechanical potentials where Bessel functions are common as our LP system have been studied in [15,16]. So, the success of our proposal here for the classical mechanical LP system can provide us with a deeper understanding for the upcoming researches regarding the nature of relationship between classical mechanical and quantum mechanical systems. Moreover, and also interestingly, the JWKB solution of the LP presented here is being shown to be very accurate by its only two terms instead of the sum of infinite series as in the exact solution (for the calculation of Bessel functions). Especially, JWKB solution presented here also works very well at high lengthening rates as given in our example where $v = 1$ m/s, whereas the present numerical methods such as given in [21,22] rather require adiabatic LP systems necessarily where the lengthening rate is required to be very small, i.e., about $v = 0.2048$ m/s. Our sample LP system with $v = 1$ m/s seems also working very well at even much higher v values in consistence with the non-obedient region defined in (4.8b) and with the related 3D graph in figure 1 and 2D graphs in figure 6. Our JWKB method works very well everywhere except for the non-obedient region given by (4.8b) which is a consequence of the JWKB applicability criterion. The non-obedient region given in (4.8b) with its related graphs in figures 1–6 also shows that the non-obedient region where JWKB solution fails corresponds to a very narrow time domain close to $t = 0$ (closeness depending on the LP parameters in (4.8b)) and becomes accurate as time evolves after this narrow non-obedient time interval. These graphs are also in agreement with our failure to obtain the first zero of J_0 as discussed above. In our initial value aided analyses, we also derived the necessary experimental quantization rules relating the LP parameters with the initial values to determine the Bessel functions of either type similar to the energy quantizations in quantum mechanical systems.

Appendix A

To illustrate, as an alternative two-experiment set (one for finding J_1 and the other for finding Y_1) where only l_0 is allowed to change among the set $\{g, l_0, v\}$, let us consider the same system in (6.2). But now we have: $\{g = g' \wedge l_0' \neq l_0 \wedge v' = v\}$ [34d], and let us consider different initial values as different alternatives, say: $x_0' \neq x_0 \Rightarrow \{\alpha_1' \neq \alpha_1 \neq 0, \beta_1' = \beta_1 = 0\}$. According to Lemma 4.2, we do not have to choose the experimental initial values necessarily as the nth zeros of Y_1 and J_1 as we did in (4.17f)–(4.18f) to obtain the desired Bessel functions: $J_1 \approx \tilde{J}_1$ and $Y_1 \approx \tilde{Y}_1$ entirely. Here, although we now choose different initial values in our alternative two-experiment set, in effect, we should have the same JWKB-approximated Bessel functions if Lemma 4.2 without requiring the nth root of Y_1 and J_1 is true. Choosing such an x_0 point on J_1 in part (i) and such an x_0' point on Y_1 in part (ii) according to Lemma 4.2 with (4.16a) and (4.16b), we have the following experimental details giving the same results as we obtained above:

(i) For finding $\tilde{J}_1(x)$ ($\approx J_1(x)$) via Lemma 4.2:

$$\begin{aligned}
&x_0 = 5.33144(2773525031...) \Rightarrow \rho_0 = c - 2\ln x_0 \\
&= c - 3.34724(37815294818...) \\
&\Rightarrow \{y_{\mathrm{EX}}(x_0) = J_1(x_0) = \alpha_1 = -0.346126, \\
&\quad y'_{\mathrm{EX}}(x_0) = J'_1(x_0) = \beta_1 = 0\} \\
&\Rightarrow \left\{\alpha_2 = \alpha_1 = -0.346126,\ \beta_2 = \frac{-\beta_1 x_0}{2} = 0\right\} \\
&\Rightarrow \text{Exp. parameters: } l_0 = ?,\ v = 1\ \mathrm{m/s},\ g = 10\ \mathrm{m/s^2} \\
&\Rightarrow l_0 = \frac{x_0^2 v^2}{4g} = \boxed{0.710607\ \mathrm{m}} \\
&\Rightarrow \left\{\theta(x_0) = \frac{\alpha_1}{x_0} = -0.0649217,\right. \\
&\quad \left.\theta_{x_0} = -x_0^{-2}\alpha_1 + x_0^{-1}\beta_1 = 0.0121771\right\} \\
&\Rightarrow \text{Exp. initial values: } \left\{\theta(t_0 = 0) = \frac{\alpha_1}{x_0} = -0.0649217\,\mathrm{rad},\right. \\
&\quad \left.\theta_{t_0=0} = \frac{2\theta_{x_0}}{x_0 v} = 0.00456804\ \mathrm{rad/s}\right\} \\
&\Rightarrow \text{Exp. result: } \theta(t) = [x(t)]^{-1} J_1(t) \approx \tilde{\theta}(t) \\
&\qquad = [x(t)]^{-1}\tilde{J}_1(t). \qquad \text{(A.1)}
\end{aligned}$$

(ii) For finding $\tilde{Y}_1(x)$ ($\approx Y_1(x)$) via Lemma 4.2:

$$\begin{aligned}
x'_0 &= 3.68302(28565851777...) \\
&\Rightarrow \rho_0 = c - 2\ln x_0 \\
&= c - 2.60746(7686826076...) \\
&\Rightarrow \{y_{\mathrm{EX}}(x'_0) = Y_1(x'_0) = \alpha'_1 = 0.41673 \neq \alpha_1, \\
&\quad y'_{\mathrm{EX}}(x'_0) = Y'_1(x'_0) = \beta'_1 = 0 = \beta_1\} \\
&\Rightarrow \left\{\alpha'_2 = \alpha'_1 = 0.41673,\ \beta'_2 = \frac{-\beta'_1 x'_0}{2} = 0\right\} \\
&\Rightarrow \text{Exp. parameters: } l'_0 = ? \neq l_0, \\
&\quad v' = v = 1\ \mathrm{m/s},\ g' = g = 10\ \mathrm{m/s^2} \\
&\Rightarrow l'_0 = \frac{x_0'^2 v^2}{4g} = \boxed{0.339116\ \mathrm{m}} \\
&\Rightarrow \left\{\theta(x'_0) = \frac{\alpha'_1}{x'_0} = 0.113149,\right. \\
&\quad \left.\theta_{x'_0} = -x_0'^{-2}\alpha'_1 + x_0'^{-1}\beta'_1 = -0.0307217\right\} \\
&\Rightarrow \text{Exp. initial values: } \left\{\theta(t'_0 = 0) = \frac{\alpha'_1}{x'_0}\right. \\
&= 0.113149\ \mathrm{rad}, \\
&\quad \left.\theta_{t'_0=0} = \frac{2\theta_{x'_0}}{x'_0 v} = -0.0166829\ \mathrm{rad/s}\right\} \\
&\Rightarrow \text{Exp. Result: } \theta(t) = [x(t)]^{-1} Y_1(t) \approx \tilde{\theta}(t) \\
&= [x(t)]^{-1}\tilde{Y}_1(t). \qquad \text{(A.2)}
\end{aligned}$$

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[22a] Here we have the following alternative equivalent expressions:
$\theta(x) = \theta[x, y(x)] = x^{-1}y(x) = \theta[x(l)] =: \theta(l)$
$= \theta\{x[l(t)]\} =: \theta(t)$.
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[31a] See eq. (3.9) for the representation of the corresponding terms without tildes (c_1 and c_2) in the exact solution in our notation.
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[34a] Note that the locations of the CAR and CIR are the same, i.e., CAR lies on the LHS and CIR lies on the RHS, for both SLP in [6] and LP-BDE system here. So, the conventional JWKB connection formulas give the same form of solutions for both systems in both regions.
[34b] However, we shall pay for it by missing the first zero of $Y_{v=0,n=0}(x)$ (see table 3).
[34c] Note that, from (3.12), x is dimensionless.
[34d] Remember that we preferred the following two experiments for the same LP system in (6.2): LP= $\{g, l_0, v\}$ and LP$' = \{g', l_0', v'\}$ as $\{g = g', l_0 = l_0', v \neq v'\}$ and we determined the necessary experimental v and v' values in eqs. (6.4) and (6.6), respectively.