Pramana – J. Phys. (2017) 88: 16
DOI 10.1007/s12043-016-1315-6

Symbolic computation and abundant travelling wave solutions to KdV–mKdV equation

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MS received 5 July 2015; revised 21 April 2016; accepted 6 May 2016; published online 9 December 2016

Abstract. In this article, the novel (G'/G)-expansion method is successfully applied to construct the abundant travelling wave solutions to the KdV–mKdV equation with the aid of symbolic computation. This equation is one of the most popular equation in soliton physics and appear in many practical scenarios like thermal pulse, wave propagation of bound particle, etc. The method is reliable and useful, and gives more general exact travelling wave solutions than the existing methods. The solutions obtained are in the form of hyperbolic, trigonometric and rational functions including solitary, singular and periodic solutions which have many potential applications in physical science and engineering. Many of these solutions are new and some have already been constructed. Additionally, the constraint conditions, for the existence of the solutions are also listed.

Keywords. Novel (G'/G)-expansion method; KdV–mKdV equation; travelling wave solutions.

PACS Nos 02.30.Jr; 05.45.Yv

1. Introduction

Investigating the new and more exact travelling wave solutions to nonlinear evolution equations (NLEEs) has been the area under discussion in different branches of engineering, physics and mathematics. So, during the last several decades, many powerful and efficient methods have been developed by diverse groups of scientists to find exact travelling wave solutions of NLEE, i.e. the inverse scattering method [1,2], the Hirota's bilinear method [3], the Backlund transformation [4], the symmetry method [5], the sine–cosine function method [6], the tanh–coth function method [7], the exp-function method [8,9] and so on (see also [10–39]).

The aim of this paper is to construct new travelling wave solutions to KdV–mKdV equation with the aid of symbolic computation. The integration scheme, which is successfully used, is the novel (G'/G)-expansion method to stress its power in holding and obtaining the more general exact travelling wave solutions of KdV–mKdV equation. This equation is one of the fundamental equations of soliton physics and appears in many practical scenarios like thermal pulse, wave propagation of bound particle etc. The solutions obtained are in the form of hyperbolic, trigonometric, rational functions including solitary, singular and periodic solutions. Many of these solutions are new and some have already been constructed, because methods presented in [13,15,16,24] are only special cases of the applied novel (G'/G)-expansion method (for more details, see Remark 1).

The rest of the paper is organized as follows. In §2, we describe the novel (G'/G)-expansion method for finding travelling wave solutions of NLEEs. In §3, we illustrate the method in detail with the celebrated KdV–mKdV equation. In §4, some conclusions are given.

2. Description of the method

Suppose that we have a nonlinear partial differential equation in the following form:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{1}$$

where P is a polynomial in $u(x,t)$ and its partial derivatives wherein the highest-order partial derivatives and the nonlinear terms are involved. The main steps of the method are as follows:

Step 1. Combining the real variables x and t by a compound variable ξ, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x \pm Vt, \tag{2}$$

where V is the speed of the travelling wave. Equation (2) transforms eq. (1) into an ODE for $u = u(\xi)$:

$$Q(u, u'', u''', ...) = 0, \tag{3}$$

where Q is a function $u(\xi)$ and its derivatives wherein prime stands for derivative with respect to ξ.

Step 2. Assume that the solution of eq. (3) can be expressed in powers $\psi(\xi)$:

$$u(\xi) = \sum_{j=-N}^{N} \alpha_j (\psi(\xi))^j, \tag{4}$$

where

$$\psi(\xi) = (d + \phi(\xi))$$

and

$$\phi(\xi) = \frac{G'(\xi)}{G(\xi)}. \tag{5}$$

Here α_{-N} or α_N may be zero, but both of them could not be zero simultaneously. α_j ($j = 0, \pm 1, \pm 2, ..., \pm N$) and d are constants to be determined later and $G = G(\xi)$ satisfies the second-order nonlinear ODE

$$GG'' = \lambda GG' + \mu G^2 + \nu (G')^2, \tag{6}$$

where prime denotes the derivative with respect to ξ; λ; μ; and ν are real parameters. The Cole–Hopf transformation $\phi(\xi) = \ln(G(\xi))_\xi = (G'(\xi)/G(\xi))$ reduces eq. (6) into the Riccati equation:

$$\phi'(\xi) = \mu + \lambda \phi(\xi) + (\nu - 1)\phi^2(\xi). \tag{7}$$

Equation (7) has twenty five solutions.

Step 3. The value of the positive integer N can be determined by balancing the highest-order linear terms with the nonlinear terms of the highest order come out in eq. (3). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D\left[\frac{d^p u(\xi)}{d\xi^p}\right] = n + p,$$

$$D\left[u^p \left(\frac{d^q u(\xi)}{d(\xi)^q}\right)^s\right] = np + s(n+q).$$

Step 4. Substitute eqs (4) (6) into eq. (3). We obtain polynomials in $(d + (G'(\xi)/G(\xi)))^j$ and $(d + (G'(\xi)/G(\xi)))^{-j}$ ($j = 0, 1, 2, ..., N$). Collecting each coefficient of the resulted polynomials to zero yields an overdetermined set of algebraic equations for α_j, $j = 0, \pm 1, \pm 2, ..., \pm N$, d and v.

Step 5. Suppose the values of the constants are obtained by solving the algebraic equations obtained in Step 4. Substituting the values of the constants together with the solutions of eq. (6), we shall obtain new and comprehensive exact travelling wave solutions of the nonlinear evolution eq. (1).

Remark 1. It is noteworthy to observe that if we replace λ by $-\lambda$ and μ by $-\mu$ and put $\nu = 0$ in eq. (7), then the applied novel (G'/G)-expansion method coincides with the generalized and improved (G'/G)-expansion method. On the other hand, if we put $d = 0$ in eq. (6) and $\nu = 0$ in eq. (7) then the proposed method is identical to the improved (G'/G)-expansion method. Again, if we set $d = 0$, $\nu = 0$ in eq. (7) and α_j ($j = 1, 2, 3, ..., N$) are functions of x and t instead of constants then the proposed method is transformed into the generalized (G'/G)-expansion method developed by Zhang. Thus, the methods presented in [13,15,16,24] are only special cases of the applied novel (G'/G)-expansion method.

3. Mathematical analysis

In this section, we shall bring to bear the (G'/G)-expansion method to construct new and more general travelling wave solutions of KdV–mKdV equation. The KdV–mKdV equation is one of the most popular equation in soliton physics and appear in many practical scenarios like thermal pulse, wave propagation of bound particle and reads as

$$u_t + (\alpha + \beta u)uu_x + u_{xxx} = 0, \tag{8}$$

where α and β are constants. Using the travelling wave transformation $\xi = x - Vt$, eq. (8) is converted into the following ODE:

$$(\alpha u + \beta u^2 - V)u' + u''' = 0. \tag{9}$$

Integrating eq. (9), we obtain

$$K - Vu + \alpha\frac{u^2}{2} + \beta\frac{u^3}{3} + u'' = 0, \tag{10}$$

where K is an integration constant. Considering the homogeneous balance between the highest-order derivative u'' and nonlinear term of the highest order u^3 in eq. (10), we obtain for $N = 1$

$$u(\xi) = \alpha_{-1}(\psi(\xi))^{-1} + \alpha_0 + \alpha_1(\psi(\xi)). \tag{11}$$

Inserting eq. (11) into eq. (10), the left-hand side is transformed into polynomials in $(d + (G'(\xi)/G(\xi)))$ and $(d+(G'(\xi)/G(\xi)))^{-1}$. Equating the coefficients of like power of these polynomials to zero, we obtain a set of algebraic equations (because of space constraints, we leave out the display of these equations) for α_0, $\alpha_{-1}, \alpha_1, d, K$ and V. Solving the overdetermined set of algebraic equations by using the symbolic computation software, such as, *Maple*, we obtain

Set 1

$$K = 0, \quad \alpha_1 = \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}, \quad \alpha_0 = \alpha_0,$$
$$\alpha_{-1} = 0, \quad d = d, \tag{12}$$

where d, K, β and ν are all arbitrary constants.

Set 2

$$K = 0, \quad \alpha_0 = 0, \quad V = V, \quad \alpha_1 = \sqrt{\frac{-6(\nu-1)^2}{\beta}}$$

$$\alpha_{-1} = \frac{\sqrt{(-6/\beta)}(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2)}{-2(\nu-1)}, \tag{13}$$

where d, λ, β, α, and ν are arbitrary constants. Substituting eqs (12) and (13) into solution eq. (11), we obtain

$$u_1(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left(d + \left(\frac{G'}{G}\right)\right) \tag{14}$$

$$u_2(x,t) = \pm\sqrt{\frac{-6(\nu-1)^2}{\beta}}\left(d + \left(\frac{G'}{G}\right)\right) \pm \frac{\sqrt{\frac{-6}{\beta}}(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2)}{-2(\nu-1)} \times \left(d + \left(\frac{G'}{G}\right)\right)^{-1}, \tag{15}$$

where V, d, λ, μ, α, and ν are arbitrary constants.

Substituting the solutions $G(\xi)$ of eq. (6) into eq. (14) and simplifying, we achieve the following solutions:

When $\rho = \lambda^2 - 4\mu\nu + 4\mu > 0$ and $\lambda(\nu-1) \neq 0$ or $\mu(\nu-1) \neq 0$,

$$u_{1_1}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d - \frac{1}{(2\nu-1)}\left(\lambda + \sqrt{\rho}\tanh\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}, \tag{16}$$

$$u_{1_2}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{\rho}\coth\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}, \tag{17}$$

$$u_{1_3}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d - \frac{1}{2(\nu-1)}\{\lambda + \sqrt{\rho}(\tanh(\sqrt{\rho}\xi) \pm i\,\mathrm{sech}(\sqrt{\rho}\xi))\}\right], \tag{18}$$

$$u_{1_4}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d - \frac{1}{2(\nu-1)}\{\lambda + \sqrt{\rho}(\coth(\sqrt{\rho}\xi) \pm \mathrm{csch}(\sqrt{\rho}\xi))\}\right], \tag{19}$$

$$u_{1_5}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d - \frac{1}{4(\nu-1)}\left\{2\lambda + \sqrt{\rho}\left(\tanh\left(\frac{1}{4}\sqrt{\rho}\xi\right) + \coth\left(\frac{1}{4}\sqrt{\rho}\xi\right)\right)\right\}\right], \tag{20}$$

$$u_{1_6}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{2(\nu-1)} \times \left\{-\lambda + \frac{\pm\sqrt{\rho(A^2+B^2)} - A\sqrt{\rho}\cosh(\sqrt{\rho}\xi)}{A\sinh(\sqrt{\rho}\xi)+B}\right\}\right], \tag{21}$$

$$u_{1_7}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{2(\nu-1)} \times \left\{-\lambda + \frac{\pm\sqrt{\rho(A^2+B^2)}+A\sqrt{\rho}\cosh(\sqrt{\rho}\xi)}{A\sinh(\sqrt{\rho}\xi)+B}\right\}\right], \quad (22)$$

where both A and B are real constants;

$$u_{1_8}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d + \frac{2\mu\cosh(\frac{1}{2}\sqrt{\rho}\xi)}{\sqrt{\rho}\sinh\left(\frac{1}{2}\sqrt{\rho}\xi\right)-\lambda\cosh\left(\frac{1}{2}\sqrt{\rho}\xi\right)}\right\}, \quad (23)$$

$$u_{1_9}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d + \frac{2\mu\sinh(\frac{1}{2}\sqrt{\rho}\xi)}{\sqrt{\rho}\cosh\left(\frac{1}{2}\sqrt{\rho}\xi\right)-\lambda\sinh\left(\frac{1}{2}\sqrt{\rho}\xi\right)}\right\}, \quad (24)$$

$$u_{1_{10}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d + \frac{2\mu\cosh(\sqrt{\rho}\xi)}{\sqrt{\rho}\sinh(\sqrt{\rho}\xi)-\lambda\cosh(\sqrt{\rho}\xi)\pm i\sqrt{\rho}}\right\}, \quad (25)$$

$$u_{1_{11}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d + \frac{2\mu\sinh(\sqrt{\rho}\xi)}{\sqrt{\rho}\cosh(\sqrt{\rho}\xi)-\lambda\sinh(\sqrt{\rho}\xi)\pm\sqrt{\rho}}\right\}. \quad (26)$$

When $\rho = \lambda^2 - 4\mu\nu + 4\mu < 0$ and $\lambda(\nu-1) \neq 0$ or $\mu(\nu-1) \neq 0$,

$$u_{1_{12}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d + \frac{1}{2(\nu-1)}\left(-\lambda + \sqrt{-\rho}\tan\left(\frac{1}{2}\sqrt{-\rho}\xi\right)\right)\right\}, \quad (27)$$

$$u_{1_{13}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{-\rho}\cot\left(\frac{1}{2}\sqrt{-\rho}\xi\right)\right)\right\}, \quad (28)$$

$$u_{1_{14}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{2(\nu-1)} \times \{-\lambda + \sqrt{-\rho}(\tan(\sqrt{-\rho}\xi) \pm \sec(\sqrt{-\rho}\xi))\}\right], \quad (29)$$

$$u_{1_{15}}(x,t) = v\left[d - \frac{1}{2(\nu-1)} \times \{\lambda + \sqrt{-\rho}(\cot(\sqrt{-\rho}\xi) \pm \csc(\sqrt{-\rho}\xi))\}\right], \quad (30)$$

$$u_{1_{16}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{4(\nu-1)}\left\{-2\lambda + \sqrt{-\rho}\left(\tan\left(\frac{1}{4}\sqrt{-\rho}\xi\right) - \cot\left(\frac{1}{4}\sqrt{-\rho}\xi\right)\right)\right\}\right], \quad (31)$$

$$u_{1_{17}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{-\rho(A^2-B^2)} - A\sqrt{-\rho}\cos(\sqrt{-\rho}\xi)}{A\sin(\sqrt{-\rho}\xi)+B}\right\}\right], \quad (32)$$

$$u_{1_{18}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{-\rho(A^2-B^2)} + A\sqrt{-\rho}\cos(\sqrt{-\rho}\xi)}{A\sin(\sqrt{-\rho}\xi)+B}\right\}\right], \quad (33)$$

where both A and B are arbitrary constants such that $A^2 - B^2 > 0$

$$u_{1_{19}}(x,t) = \alpha_0 \pm \frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta} \times \left\{d - \frac{2\mu\cos(\sqrt{-\rho}\frac{\xi}{2})}{\sqrt{-\rho}\sin(\sqrt{-\rho}\frac{\xi}{2}) + \lambda\cos(\sqrt{-\rho}\frac{\xi}{2})}\right\}, \quad (34)$$

$$u_{1_{20}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d+\frac{2\mu\sin(\sqrt{-\rho}\frac{\xi}{2})}{\sqrt{-\rho}\cos(\sqrt{-\rho}\frac{\xi}{2})-\lambda\sin(\sqrt{-\rho}\frac{\xi}{2})}\right\}, \quad (35)$$

$$u_{1_{21}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d-\frac{2\mu\cos(\sqrt{-\rho}\xi)}{\sqrt{-\rho}\sin(\sqrt{-\rho}\xi)+\lambda\cos(\sqrt{-\rho}\xi)\pm\sqrt{-\rho}}\right\}, \quad (36)$$

$$u_{1_{22}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d+\frac{2\mu\sin(\sqrt{-\rho}\frac{\xi}{2})}{\sqrt{-\rho}\cos(\sqrt{-\rho}\frac{\xi}{2})-\lambda\sin(\sqrt{-\rho}\frac{\xi}{2})\pm\sqrt{-\rho}}\right\}. \quad (37)$$

When $\mu=0$ and $\lambda(\nu-1)\neq 0$, we have

$$u_{1_{23}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d-\frac{\lambda k}{(\nu-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}, \quad (38)$$

$$u_{1_{24}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d-\frac{\lambda\{\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}{(\nu-1)\{k+\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}\right\}, \quad (39)$$

where k is an arbitrary constant.

When $(\nu-1\neq 0)$ and $\lambda=\mu=0$, the solution of eq. (8) is

$$u_{1_{25}}(x,t)=\alpha_0\pm\frac{-4\lambda(\nu-1)+12d(\nu-1)^2}{\alpha+2\alpha_0\beta}\times\left\{d-\frac{1}{(\nu-1)\xi+c_1}\right\}, \quad (40)$$

where c_1 is an arbitrary constant.

By substituting solutions $G(\xi)$ of eq. (6) into eq. (15) and simplifying the resulting equation, we obtain the following solutions:

When $\rho=\lambda^2-4\mu\nu+4\mu>0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$),

$$u_{2_1}(x,t)=\sqrt{\frac{-6(\nu-1)^2}{\beta}}\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\rho}\tanh\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}$$
$$\pm\frac{\sqrt{\frac{-6}{\beta}}\left(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2\right)}{-2(\nu-1)}\times\left\{\left(\lambda+\sqrt{\rho}\tanh\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}^{-1}, \quad (41)$$

where $\xi=x-Vt$, d, λ, μ, and ν are arbitrary constants:

$$u_{2_2}(x,t)=\sqrt{\frac{-6(\nu-1)^2}{\beta}}\left\{d-\frac{1}{2(\nu-1)}\times\left(\lambda+\sqrt{\rho}\coth\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}$$
$$\pm\frac{\sqrt{\frac{-6}{\beta}}(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2)}{-2(\nu-1)}\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\rho}\coth\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right)\right\}^{-1}, \quad (42)$$

$$u_{2_3}(x,t)=\sqrt{\frac{-6(\nu-1)^2}{\beta}}\left\{d-\frac{1}{2(\nu-1)}\times(\lambda+\sqrt{\rho}(\tanh(\sqrt{\rho}\xi)\pm i\,\mathrm{sech}(\sqrt{\rho}\xi))\right\}$$
$$\pm\frac{\sqrt{\frac{-6}{\beta}}(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2)}{-2(\nu-1)}\times\left\{d-\frac{1}{2(\nu-1)}\times(\lambda+\sqrt{\rho}(\tanh(\sqrt{\rho}\xi)\pm i\,\mathrm{sech}(\sqrt{\rho}\xi)))\right\}^{-1}. \quad (43)$$

The other families of exact solutions of eq. (8) are omitted for convenience. When $\rho=\lambda^2-4\mu\nu+4\mu<0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$), we have

$$u_{2_{12}}(x,t)=\sqrt{\frac{-6(\nu-1)^2}{\beta}}\times\left\{d+\frac{1}{2(\nu-1)}\left(-\lambda+\sqrt{-\rho}\left(\tan\left(\sqrt{-\rho}\frac{\xi}{2}\right)\right)\right)\right\}$$
$$\pm\frac{\sqrt{\frac{-6}{\beta}}(\lambda^2-\nu+2\mu-4\lambda d(\nu-1)-6d^2(\nu-1)^2)}{-2(\nu-1)}\times\left\{d+\frac{1}{2(\nu-1)}\left(-\lambda+\sqrt{-\rho}\left(\tan\left(\sqrt{-\rho}\frac{\xi}{2}\right)\right)\right)\right\}^{-1}, \quad (44)$$

$$u_{2_{13}}(x,t)=\sqrt{\frac{-6(\nu-1)^2}{\beta}}\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{-\rho}\left(\cot\left(\sqrt{-\rho}\frac{\xi}{2}\right)\right)\right)\right\}$$

$$\pm \frac{\sqrt{\frac{-6}{\beta}}(\lambda^2 - \nu + 2\mu - 4\lambda d(\nu-1) - 6d^2(\nu-1)^2)}{-2(\nu-1)} \times \left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{-\rho}\left(\cot\left(\sqrt{-\rho}\frac{\xi}{2}\right)\right)\right)\right\}^{-1}, \quad (45)$$

$$u_{2_{14}}(x,t) = \sqrt{\frac{-6(\nu-1)^2}{\beta}}\left\{d - \frac{1}{2(\nu-1)} \times (-\lambda + \sqrt{-\rho}(\tan(\sqrt{-\rho}) \pm \sec(\sqrt{-\rho}\xi)))\right\} \pm \frac{\sqrt{\frac{-6}{\beta}}(\lambda^2 - \nu + 2\mu - 4\lambda d(\nu-1) - 6d^2(\nu-1)^2)}{-2(\nu-1)} \times \left\{d - \frac{1}{2(\nu-1)} \times (-\lambda + \sqrt{-\rho}(\tan(\sqrt{-\rho}) \pm \sec(\sqrt{-\rho}\xi)))\right\}^{-1}. \quad (46)$$

When $(\nu - 1) = 0$ and $\lambda = \mu = 0$, the solution of eq. (18) is

$$u_{2_{15}}(x,t)\sqrt{\frac{-6(\nu-1)^2}{\beta}}\left\{d - \frac{1}{(\nu-1)\xi + c_1}\right\} \pm \frac{\sqrt{\frac{-6}{\beta}}(\lambda^2 - \nu + 2\mu - 4\lambda d(\nu-1) - 6d^2(\nu-1)^2)}{-2(\nu-1)} \times \left\{d - \frac{1}{(\nu-1)\xi + c_1}\right\}^{-1}, \quad (47)$$

where c_1 is an arbitrary constant.

4. Conclusions

In this paper, the novel (G'/G)-expansion method has been used in order to find the abundant exact travelling wave solutions of the KdV–mKdV equation which is one of the fundamental equation of fluid mechanics. It can be observed that the method used is a powerful and more general tool for finding the exact solutions. These solutions are of the form of hyperbolic, trigonometric, and rational functions including solitary, singular, periodic, and plane solutions. It may be noted that the constraint conditions, for the existence of the solutions, are also listed.

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