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Lie symmetry analysis and soliton solutions of time-fractional $K(m, n)$ equation

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Abstract. In this note, method of Lie symmetries is applied to investigate symmetry properties of time-fractional $K(m, n)$ equation with the Riemann–Liouville derivatives. Reduction of time-fractional $K(m, n)$ equation is done by virtue of the Erdélyi–Kober fractional derivative which depends on a parameter α. Then soliton solutions are extracted by means of a transformation.

Keywords. Lie symmetries; time-fractional $K(m, n)$ equation; Erdélyi–Kober fractional derivative; Riemann–Liouville derivatives; soliton solutions.

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1. Introduction

Nonlinear wave phenomena which appear in a wide variety of scientific applications, are often related to wave and dispersive equations. Searching for solitary solutions of such equations in mathematical physics are of fundamental importance. It is well-known that a soliton is a self-reinforcing solitary wave caused by a delicate balance between nonlinear and dispersive effects in the medium.

Factional differential equations are increasingly used to model problems in physics, such as fluid mechanics, biology, viscoelasticity, engineering etc. [1–4]. In recent years, due to the numerous applications of fractional partial differential equations (FPDEs), solving such equations and some analytical schemes for fractional differential equations have become important.

This paper mainly focusses on the Lie group analysis of time-fractional $K(m, n)$ equation [5]:

$$\partial_t^\alpha u - a(u^n)_x + b(u^m)_{xxx} = 0. \tag{1}$$

The well-known $K(m, n)$ equation which is a generalization of the KdV equation, describes the evolution of weakly nonlinear and weakly dispersive wave used in various fields such as solid-state physics, plasma physics, fluid physics and quantum field theory. After that, Odibat [5] introduced eq. (1) and considered three special cases $K(2, 2)$, $K(3, 3)$ and $K(n, n)$ and Koçak *et al* [6] considered the nonlinear dispersive $K(m, n, 1)$ equations with fractional time derivatives by using the homotopy perturbation method. Wazwaz in [7] apart from obtaining the compactons, solitons and periodic solutions of standard nonlinear dispresive $K(n, n)$ equation, demonstrated that this equation is non-integrable. Mirzazadeh and Eslami [8] utilized the functional variable method to investigate this equation.

Here $\partial_t^\alpha u := D_t^\alpha u$ stands for Riemann–Liouville derivative of order α, which is defined by [1–4]

$$D_t^\alpha u(x, t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-\xi)^{n-\alpha-1} u(x, \xi) \mathrm{d}\xi, \\ \qquad n-1 < \alpha < n, \\ \dfrac{\partial^n u}{\partial t^n}, \qquad \alpha = n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\Gamma(z)$ is the Euler gamma function.

2. Lie symmetry analysis of fractional partial differential equations

In this section, brief details of the Lie symmetry analysis of FPDEs are presented. Consider a FPDE having the form [9–18]

$$\partial_t^\alpha u = F(x, t, u, u_x, u_{xx}, u_{xxx}), \quad 0 < \alpha < 1. \tag{3}$$

One-parameter Lie group of infinitesimal transformations of this equation is

$$\bar{t} = \bar{t}(x, t, u; \epsilon), \quad \bar{x} = \bar{x}(x, t, u; \epsilon),$$
$$\bar{u} = \bar{u}(x, t, u; \epsilon), \tag{4}$$

where ϵ is the group parameter and its associated Lie algebra is spanned by

$$V = \xi^1(x, t, u)\frac{\partial}{\partial t} + \xi^2(x, t, u)\frac{\partial}{\partial x} + \phi(x, t, u)\frac{\partial}{\partial u}, \tag{5}$$

where

$$\xi^1 = \left.\frac{\mathrm{d}\bar{t}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \quad \xi^2 = \left.\frac{\mathrm{d}\bar{x}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \quad \phi = \left.\frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}. \tag{6}$$

If the vector field (5) generates a symmetry of (3), then V must satisfy the Lie symmetry condition

$$\mathrm{Pr}^{(\alpha,3)} V(\Delta)|_{\Delta=0} = 0, \quad \Delta = \partial_t^\alpha u - F. \tag{7}$$

The prolongation operator $\mathrm{Pr}^{(\alpha,3)} V$ takes the form

$$\mathrm{Pr}^{(\alpha,3)} V = V + \phi_\alpha^0 \partial_{\partial_t^\alpha u} + \phi^x \partial_{u_x} + \phi^{xx} \partial_{u_{xx}} + \phi^{xxx} \partial_{u_{xxx}}, \tag{8}$$

where

$$\phi^x = D_x(\phi - \xi^1 u_t - \xi^2 u_x) + \xi^1 u_{tx} + \xi^2 u_{xx},$$
$$\phi^{xx} = D_x^2(\phi - \xi^1 u_t - \xi^2 u_x) + \xi^1 u_{txx} + \xi^2 u_{xxx},$$
$$\phi^{xxx} = D_x^3(\phi - \xi^1 u_t - \xi^2 u_x) + \xi^1 u_{txxx} + \xi^2 u_{xxxx},$$
$$\phi_\alpha^0 = D_t^\alpha(\phi) + \xi^2 D_t^\alpha(u_x) - D_t^\alpha(\xi^2 u_x) + D_t^\alpha(D_t(\xi^1)u) - D_t^{\alpha+1}(\xi^1 u) + \xi^1 D_t^{\alpha+1}(u).$$

The invariance condition

$$\xi^1(x, t, u)|_{t=0} = 0, \tag{9}$$

is necessary to transformations (4), because of the conservative property of fractional derivative operator (2).

The αth extended infinitesimal has the form

$$\phi_\alpha^0 = D_t^\alpha(\phi) + \xi^2 D_t^\alpha(u_x) - D_t^\alpha(\xi^2 u_x) + D_t^\alpha(D_t(\xi^1)u) - D_t^{\alpha+1}(\xi^1 u) + \xi^1 D_t^{\alpha+1}(u), \tag{10}$$

where the operator D_t^α expresses the total fractional derivative operator. The generalized Leibnitz rule in the fractional sense is given by

$$D_t^\alpha [u(t)v(t)] = \sum_{n=0}^{\infty} \binom{a}{n} D_t^{\alpha-n} u(t) D_t^n v(t), \quad \alpha > 0, \tag{11}$$

where

$$\binom{a}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{12}$$

Thus, from (11) we can rewrite (10) as follows:

$$\phi_\alpha^0 = D_t^\alpha(\phi) - \alpha D_t(\xi^1)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty} \binom{a}{n+1} D_t^{n+1}(\xi^1) D_t^{\alpha-n}(u). \tag{13}$$

Also from the chain rule

$$\frac{\mathrm{d}^m f(g(t))}{\mathrm{d}t^m} = \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!}[-g(t)]^r \frac{\mathrm{d}^m}{\mathrm{d}t^m}[g(t)^{k-r}] \times \frac{\mathrm{d}^k f(g)}{\mathrm{d}g^k} \tag{14}$$

and setting $f(t) = 1$, one can get

$$D_t^\alpha(\phi) = \frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{a}{n} \frac{\partial^n \phi_u}{\partial t^n} D_t^{\alpha-n}(u) + \mu, \tag{15}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \binom{a}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r \frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}. \tag{16}$$

Therefore

$$\phi_\alpha^0 = \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\xi^1))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{a}{n+1} D_t^{n+1}(\xi^1)\right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n}(u_x).$$

3. Lie symmetry analysis of time-fractional $K(m, n)$ equation

According to the Lie theory, applying the prolongation $\mathrm{Pr}^{(\alpha,3)} V$ to eq. (1), we can get the following invariance criterion:

$$
\begin{aligned}
&\phi_\alpha^0 - an(n-1)\phi u^{n-2}u_x - anu^{n-1}\phi^x \\
&\quad + bm(m-1)(m-2)(m-3)u^{m-4}\phi u_x^3 \\
&\quad + 3bm(m-1)(m-2)u^{m-3}u_x^2\phi^x \\
&\quad + 3bm(m-1)(m-2)u^{m-3}\phi u_x u_{xx} \\
&\quad + 3bm(m-1)u^{m-2}\phi^x u_{xx} \\
&\quad + 3bm(m-1)u^{m-2}u_x\phi^{xx} \\
&\quad + bm(m-1)u^{m-2}\phi u_{xxx} + bmu^{m-1}\phi^{xxx} = 0. \quad (17)
\end{aligned}
$$

Substituting (13) into (17), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, one can find the determining equations for the symmetry group of eq. (1). Solving these equations, we obtain the following forms of coefficient functions:

$$
\xi^1 = (m+2-3n)tc_2, \quad \xi^2 = c_1 + \alpha(m-n)xc_2, \quad \phi = 2\alpha uc_2, \quad (18)
$$

where c_1 and c_2 are arbitrary constants. Thus, the Lie algebra g of infinitesimal symmetry of eq. (1) is spanned by the two vector fields:

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= (m+2-3n)t\frac{\partial}{\partial t} + \alpha(m-n)x\frac{\partial}{\partial x} + 2\alpha u\frac{\partial}{\partial u}. \quad (19)
\end{aligned}
$$

For the symmetry of V_2, the corresponding characteristic equation is

$$
\frac{\mathrm{d}t}{(m+2-3n)t} = \frac{\mathrm{d}x}{\alpha(m-n)x} = \frac{\mathrm{d}u}{2\alpha u}, \quad (20)
$$

which after solving yields the similarity variables

$$
\zeta = xt^{\frac{\alpha(n-m)}{m+2-3n}}, \quad u(x,t) = t^{\frac{2\alpha}{m+2-3n}}\mathcal{F}(\zeta). \quad (21)
$$

Now, from the following theorem we reduce the FPDE (1) to a fractional ordinary differential equation (FODE).

Theorem 3.1. *The transformation* (21) *reduces* (1) *to the following nonlinear ordinary differential equation of fractional order*:

$$
\begin{aligned}
\left(\mathcal{P}^{1-\alpha+\frac{2\alpha}{m+2-3n},\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta) &= an\mathcal{F}^{n-1}\mathcal{F}' - bm\mathcal{F}^{m-1}\mathcal{F}''' \\
&\quad - bm(m-1)(m-2)\mathcal{F}^{m-3}(\mathcal{F}')^3 \\
&\quad - 3bm(m-1)\mathcal{F}^{m-2}\mathcal{F}'\mathcal{F}'',
\end{aligned}
$$

with the Erdélyi–Kober fractional differential operator $\mathcal{P}_\beta^{\tau,\alpha}$ *of order*:

$$
(\mathcal{P}_\beta^{\tau,\alpha}\mathcal{F}) := \prod_{j=0}^{k-1}\left(\tau + j - \frac{1}{\beta}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)(\mathcal{K}_\beta^{\tau+\alpha,k-\alpha}\mathcal{F})(\zeta),
$$

$$
k = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N} \end{cases}, \quad (22)
$$

where

$$
(\mathcal{K}_\beta^{\tau,\alpha}\mathcal{F})(\zeta) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}\mathcal{F}(\zeta u^{1/\beta})\mathrm{d}u, & \alpha > 0 \\ \mathcal{F}(\zeta), & \alpha = 0, \end{cases} \quad (23)
$$

is the Erdélyi–Kober fractional integral operator.

Proof. Let $k-1 < \alpha < k$, $k = 1, 2, 3, \ldots$. Based on the Reimann–Liouville fractional derivative, one can have

$$
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^k}{\partial t^k}\left[\frac{1}{\Gamma(k-\alpha)} \times \int_0^t (t-s)^{k-\alpha-1}s^{\frac{2\alpha}{m+2-3n}}\mathcal{F}\left(xs^{\frac{\alpha(n-m)}{m+2-3n}}\right)\mathrm{d}s\right]. \quad (24)
$$

Letting $v = t/s$, one can get $\mathrm{d}s = -(t/v^2)\mathrm{d}v$. Therefore (24) can be written as

$$
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^k}{\partial t^k}\left[t^{k-\alpha+\frac{2\alpha}{m+2-3n}}\left(\mathcal{K}^{1+\frac{2\alpha}{m+2-3n},k-\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta)\right]. \quad (25)
$$

Taking the relation, $\zeta = xt^{\frac{\alpha(n-m)}{m+2-3n}}$, into account, we can obtain

$$
t\frac{\partial}{\partial t}\phi(\zeta) = t\frac{\partial\zeta}{\partial t}\frac{\mathrm{d}\phi(\zeta)}{\mathrm{d}\zeta} = \frac{\alpha(n-m)}{m+2-3n}\zeta\frac{\mathrm{d}\phi(\zeta)}{\mathrm{d}\zeta}. \quad (26)
$$

Therefore, one can get

$$
\begin{aligned}
&\frac{\partial^k}{\partial t^k}\left[t^{k-\alpha+\frac{\alpha}{m+2-3n}}\left(\mathcal{K}^{1+\frac{2\alpha}{m+2-3n},k-\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta)\right] \\
&= \frac{\partial^{k-1}}{\partial t^{k-1}}\left[\frac{\partial}{\partial t}\left(t^{k-\alpha+\frac{2\alpha}{m+2-3n}}\left(\mathcal{K}^{1+\frac{2\alpha}{m+2-3n},k-\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta)\right)\right] \\
&= \frac{\partial^{k-1}}{\partial t^{k-1}}\left[t^{k-\alpha+\frac{2\alpha}{m+2-3n}-1}\left(k-\alpha+\frac{2\alpha}{m+2-3n} + \frac{\alpha(n-m)}{m+2-3n}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\left(\mathcal{K}^{1+\frac{2\alpha}{m+2-3n},k-\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta)\right] \\
&= \ldots \\
&= t^{-\alpha+\frac{2\alpha}{m+2-3n}}\prod_{j=0}^{k-1}\left(1-\alpha+\frac{2\alpha}{m+2-3n}+j + \frac{\alpha(n-m)}{m+2-3n}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\left(\mathcal{K}^{1+\frac{2\alpha}{m+2-3n},k-\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta) \\
&= t^{-\alpha+\frac{2\alpha}{m+2-3n}}\left(\mathcal{P}^{1-\alpha+\frac{2\alpha}{m+2-3n},\alpha}_{\frac{m+2-3n}{\alpha(m-n)}}\mathcal{F}\right)(\zeta). \quad (27)
\end{aligned}
$$

This completes the proof. □

4. Soliton solutions

In this section, we focus on constructing the soliton solution to eq. (1). First, we use the following transformation [20]:

$$u(x,t) = \frac{A}{\cosh^p \tau} \tag{28}$$

and

$$\tau = B\left(x - v\frac{t^\alpha}{\Gamma(1+\alpha)}\right). \tag{29}$$

Here, A and B are free parameters of the shock wave and v denotes the speed of the wave. The exact value of the unknown parameter p will be determined later. The free parameters A and B respectively denote the dilation factor and the steepening factor of the shock wave. Now we plug (28) and (29) into (1), and get:

$$u_t = pvAB\frac{\tanh\tau}{\cosh^p\tau}, \tag{30}$$

$$u_x = -pAB\frac{\tanh\tau}{\cosh^p\tau}, \tag{31}$$

$$(u^n)_x = -npA^nB\frac{\tanh\tau}{\cosh^{np}\tau}, \tag{32}$$

$$(u^m)_{xxx} = -m^3p^3A^mB^3\frac{\tanh\tau}{\cosh^{mp}\tau} + mp(mp+1)(mp+2)A^mB^3\frac{\tanh\tau}{\cosh^{mp+2}\tau}. \tag{33}$$

Substituting (30)–(33) into (1), we arrive at

$$pvAB\frac{\tanh\tau}{\cosh^p\tau} + anpA^nB\frac{\tanh\tau}{\cosh^{np}\tau} - bm^3p^3A^mB^3\frac{\tanh\tau}{\cosh^{mp}\tau} + bmp(mp+1)(mp+2)A^mB^3\frac{\tanh\tau}{\cosh^{mp+2}\tau} = 0. \tag{34}$$

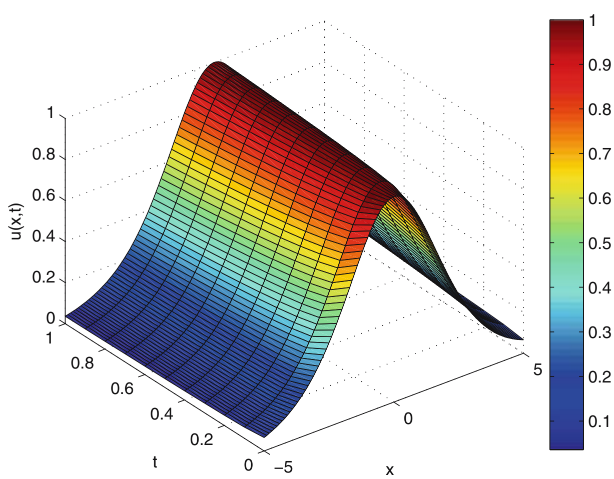

Figure 1. Solution of eq. (1) with $\alpha = 0.5$.

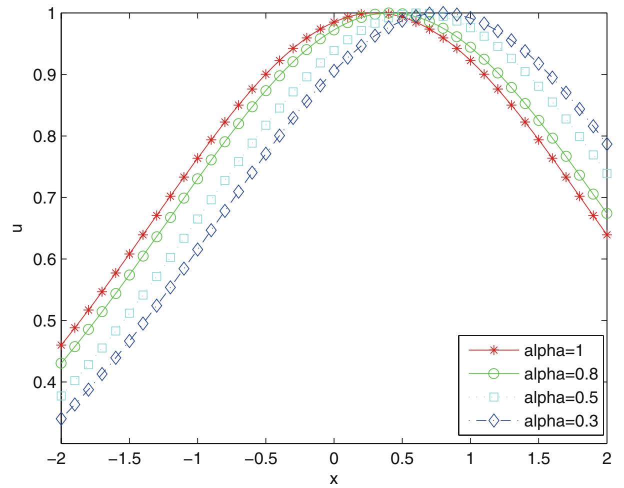

Figure 2. Solution (44) for various α for $n = 2, b = 1$, $a = -1$, $A = 1$, $B = \sqrt{6}/6$, $t = 0.3$.

By the balancing principle, equating the exponents $mp + 2$ and np yields

$$mp + 2 = np, \tag{35}$$

that is

$$p = \frac{2}{n-m}. \tag{36}$$

Also, equating the exponents mp and p, one can get

$$mp = p. \tag{37}$$

This way,

$$m = 1. \tag{38}$$

Therefore, for a $K(m,n)$ eq. (1) solitons exist only when $m = 1$. It should be noted that, to our knowledge, this is the first time that this result is presented. Thus, from (36), one can obtain

$$p = \frac{2}{n-1}. \tag{39}$$

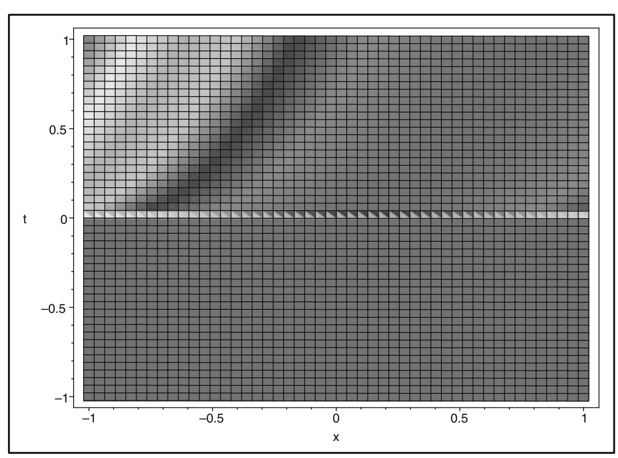

Figure 3. Density of solution (44) when $\alpha = 0.5$.

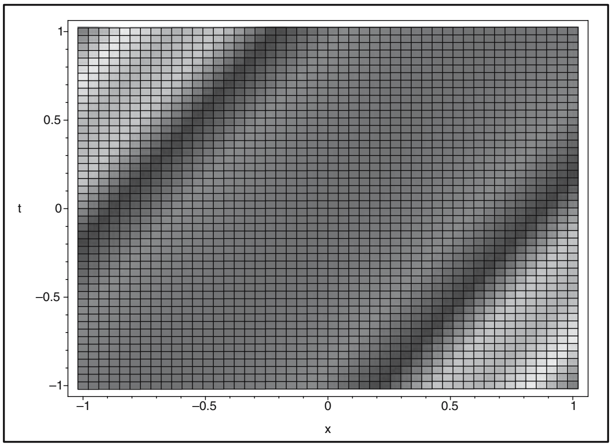

Figure 4. Density of solution (44) when $\alpha = 1$.

Now, from their coefficients it is possible to get

$$v = bp^2B^2 \tag{40}$$

and

$$B = \sqrt{-\frac{aA^{n-1}(n-1)^2}{2b(n+1)}}, \tag{41}$$

which means that the solitons will exist for

$$ab(n+1) < 0. \tag{42}$$

Thus, finally the $K(m, n)$ equation reduces to

$$u_t^{\alpha} - a(u^n)_x + bu_{xxx} = 0, \tag{43}$$

and the 1-soliton solution to $K(1, n)$ equation is as follows:

$$u(x, t) = \frac{A}{\cosh^{2/(n-1)} B(x - v\frac{t^{\alpha}}{\Gamma(1+\alpha)})}. \tag{44}$$

Here the free parameters are connected by (41), the constraint condition is determined by (42) while the speed of the wave is given by (40). Odibat in [5] considered the case $K(n, n)$ which is different from our solution (44).

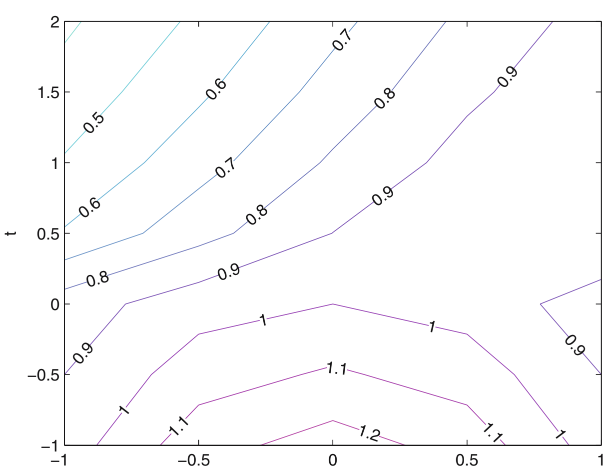

Figure 5. Contour of solution (44) when $\alpha = 0.5$.

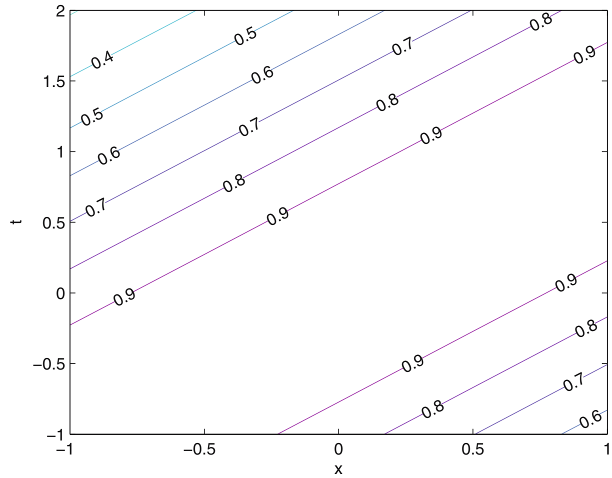

Figure 6. Contour of solution (44) when $\alpha = 1$.

5. Numerical simulations

This section shows some soliton solutions of $K(1, n)$. We consider the case when $n = 2, b = 1, a = -1$, $A = 1, B = \sqrt{6}/6$. Figure 1 shows the solution of eq. (1) while $\alpha = 0.5$ is fixed. Figure 2 shows solution (44) for different values of α when $n = 2$, b = 1, $a = -1, A = 1, B = \sqrt{6}/6, t = 0.3$. Figures 3 and 4 respectively show the density of solution (44) when $\alpha = 0.5$ and $\alpha = 1$. Figures 5 and 6 respectively display the contour of solution (44) when $\alpha = 0.5$ and $\alpha = 1$.

Remark 1. If $\alpha = 1$, eq. (1) can become the conventional integer-order $K(m, n)$ equation. Compared with the conventional ones, it can be seen that the key advantage of time-fractional $K(m, n)$ equation is its non-local property.

6. Conclusion

This research aims to investigate symmetry properties, similarity reduction form and explicit soliton solutions of the time-fractional $K(m, n)$ equation by means of the Lie symmetry groups. We have shown that eq. (1) can be transformed into a third-order nonlinear ODE of fractional order. Finally, a transformation is used to construct a soliton solution of eq. (1).

References

[1] D Baleanu, K Diethelm, E Scalas and J Trujillo, *Fractional calculus models and numerical methods*, Series on Complexity, Nonlinearity and Chaos (World Scientific, 2012)

[2] R Hilfer, *Applications of fractional calculus in physics* (Academic Press, Orlando, 1999)

[3] A Kilbas, H Srivastava and J Trujillo, *Theory and applications of fractional differential equations* (Elsevier, 2006)
[4] I Podlubny, *Fractional differential equations* (Academic Press, San Diego, CA, 1999)
[5] Z M Odibat, *Phys. Lett. A* **370**, 295 (2007)
[6] H Koçak, T Özis and A Yildirim, *Int. J. Numer. Method H* **20(2)**, 174 (2010)
[7] A M Wazwaz, *Appl. Math. Comput.* **163**, 1081 (2005)
[8] M Mirzazadeh and M Eslami, *Pramana – J. Phys.* **81**, 911 (2013)
[9] G W Wang, X Q Liu and Y Y Zhang, *Commun. Nonlinear Sci. Numer. Simul.* **18**, 2321 (2013)
[10] G W Wang and T Xu, *Bound Value Probl.* **2013**, 232 (2013)
[11] G W Wang and T Z Xu, *Nonlinear Dyn.* **76**, 571 (2014)
[12] Q Huang and R Zhdanov, *Physica A* **409**, 110 (2014)
[13] J Hu, Y Ye, S Shen and J Zhang, *Appl. Math. Comput.* **233**, 439 (2014)
[14] G W Wang, T Z Xu and T Feng, *PLoS ONE* **9(2)**, e88336 (2014)
[15] H Z Liu, *Stud. Appl. Math.* **131**, 317 (2013)
[16] M S Hashemi, *Phys. Stat. Mech. Appl.* **417**, 141 (2015)
[17] R Sahadevan and T Bakkyaraj, *J. Math. Anal. Appl.* **393**, 341 (2012)
[18] R K Gazizov, A A Kasatkin and S Y Lukashchuk, *Phys. Scr. T* **136**, 014016 (2009)
[19] V E Tarasov, *Commun. Nonlinear Sci. Numer. Simulat.* **18**, 2945 (2013)
[20] S Guo, L Mei, Y Fang and Z Qiu, *Phys. Lett. A* **376**, 158 (2012)