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The properties of *C*-parameter and coupling constants

R SALEH-MOGHADDAM and M E ZOMORRODIAN*

Department of Physics, Faculty of Physics, Ferdowsi University of Mashhad, 91775-1436, Mashhad, Iran
*Corresponding author. E-mail: Zomorrod@um.ac.ir

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Abstract. We present the properties of the *C*-parameter as an event-shape variable. We calculate the coupling constants in the perturbative and also in the non-perturbative parts of the QCD theory, using the dispersive as well as the shape function models. By fitting the corresponding theoretical predictions to our data, we find $\alpha_s(M_{Z^0}) = 0.117 \pm 0.014$ and $\alpha_0(\mu_I) = 0.491 \pm 0.043$ for dispersive model and $\alpha_s(M_{Z^0}) = 0.124 \pm 0.015$ and $\lambda_1 = 1.234 \pm 0.052$ for the shape function model. Our results are consistent with the world average value of $\alpha_s(M_{Z^0}) = 0.118 \pm 0.002$. All these features are explained in the main text.

Keywords. Quantum chromodynamics; perturbative calculations; non-perturbative theory.

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1. Introduction

Event-shape variables in e^+e^- annihilation provide an ideal testing ground to study quantum chromodynamics (QCD) and these variables have been measured and studied extensively in the last three decades. In particular, event-shape variables are useful for studying the interplay between perturbative and non-perturbative dynamics [1]. One of the most common and successful ways of testing QCD is investigating the distribution of event shapes in $e^+e^- \to$ hadrons, which have been measured accurately over a range for centre-of-mass energies, and provide a useful way of evaluating the strong coupling constant α_s. The main obstruction for obtaining an accurate value of α_s from distributions is not due to a lack of precise data but to dominant errors in the theoretical calculation of the distributions. In particular, there are non-perturbative effects that cannot yet be calculated from first principles but cause power-suppressed corrections that can be significant at experimentally accessible energy scales [2].

In this article, we show cross-section as a function of the event-shape variable *C*. We also use both perturbative and non-perturbative theory for calculating coupling constants by using these two models. It is worth mentioning that we have already performed some analyses on a few event-shape observables in our previous publications [3,4].

The outline of the paper is as follows: In §2 we define and review the *C*-parameter and show the cross-section distribution for different energies. In §3, we present the calculations of the perturbative theory as well as the non-perturbative theory up to the next-to-next-to-leading order (NNLO) for both the dispersive and the shape function models. Finally, we extract the coupling constant from our analysis. The last section summarizes our conclusions.

2. The *C*-parameter

The *C*-parameter [5,6] for electron−positron annihilation events is derived from the eigenvalues λ_i of the linearized momentum tensor θ_{jk}.

$$\theta_{jk} = \frac{\sum_i p_j^i p_k^i / |p^i|}{\sum_i |p^i|}, \tag{1}$$

where p^i are the spatial components ($j, k = 1, 2, 3$) of the *i*th particle momentum in the centre-of-mass frame. The sum on *i* runs over all the final-state particles. The *C*-parameter is defined as

$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1). \tag{2}$$

The real symmetric matrix θ_{jk} has eigenvalues λ_i with $0 \le \lambda_3 \le \lambda_2 \le \lambda_1 \le 1$. It describes an ellipsoid with orthogonal axes named minor, semimajor and major corresponding to the three eigenvalues. The C-parameter varies in the range $0 \leq C \leq 1$. $C = 0$ corresponds to a perfect two-jet event (with massless jets), while $C = 1$ characterizes a spherical event. For the planar events which are described by three-body $e^+e^- \to q\bar{q}g$ events (Mercedes events), we have $0 \leq C \leq 3/4$ in perturbative region.

3. Different models with power corrections

QCD is based on NLO as well as NNLO theories [7]. One special aspect of QCD is the interplay between perturbative and non-perturbative corrections.

Event-shape variables y measure geometrical properties of hadronic final states at high-energy particle collisions. The nth moment for y is defined as

$$\langle y^n \rangle = \frac{1}{\sigma_{\text{had}}} \int y^n \frac{\mathrm{d}\sigma}{\mathrm{d}y} \, \mathrm{d}y, \tag{3}$$

where $y_{\max}$ is the kinematically allowed upper limit of the observable. In event-shape moments, one expects the hadronization corrections to be additive such that they can be divided into perturbative (pt) and also non-perturbative (np) contributions [8].

$$\langle y^n \rangle = \langle y^n \rangle_{\text{pt}} + \langle y^n \rangle_{\text{np}}. \tag{4}$$

In the following, we explain the theory and models in details.

3.1 *NLO and NNLO corrections*

The perturbative expansion of a differential distribution of the generic observable y can be written for any infrared-safe observable in the $e^+e^- \to$ hadrons process. If we just assume the NLO theory, its corresponding perturbative expansion is

$$\frac{1}{\sigma_{\text{tot}}} \frac{\mathrm{d}\sigma}{\mathrm{d}y} = \left(\frac{\alpha_s(\mu)}{2\pi}\right) \frac{\mathrm{d}\bar{A}}{\mathrm{d}y} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \frac{\mathrm{d}\bar{B}}{\mathrm{d}y}. \tag{5}$$

$\bar{A}$ gives the LO result and $\bar{B}$ the NLO correction. σ_{tot} denotes the total hadronic cross-section calculated up to the relevant order. The arbitrary renormalization scale is denoted by μ.

The perturbative expansion up to the third order is

$$\frac{1}{\sigma_{\text{tot}}} \frac{\mathrm{d}\sigma}{\mathrm{d}y} = \left(\frac{\alpha_s(\mu)}{2\pi}\right) \frac{\mathrm{d}\bar{A}}{\mathrm{d}y} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \frac{\mathrm{d}\bar{B}}{\mathrm{d}y} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^3 \frac{\mathrm{d}\bar{C}}{\mathrm{d}y}. \tag{6}$$

$\bar{C}$ gives the NNLO correction. The perturbative contribution to $\langle y^n \rangle$ is given up to NNLO in terms of the dimensionless coefficients $\bar{A}_{y,n}$, $\bar{B}_{y,n}$ and $\bar{C}_{y,n}$ as [1]

$$\begin{aligned}\langle y^n \rangle_{\text{pt}} = {} & \left(\frac{\alpha_s(\mu)}{2\pi}\right) \bar{A}_{y,n} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \\ & \times \left(\bar{B}_{y,n} + \bar{A}_{y,n}\beta_0 \log \frac{\mu^2}{E_{\text{cm}}}\right) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^3 \\ & \times \left(\bar{C}_{y,n} + 2\bar{B}_{y,n}\beta_0 \log \frac{\mu^2}{E_{\text{cm}}} + \bar{A}_{y,n} \right. \\ & \left. \times \left(\beta_0^2 \log^2 \frac{\mu^2}{E_{\text{cm}}} + \beta_1 \log \frac{\mu^2}{E_{\text{cm}}}\right)\right). \end{aligned} \tag{7}$$

$A_{y,n}$, $B_{y,n}$ and $C_{y,n}$ are directly related to $\bar{A}_{y,n}$, $\bar{B}_{y,n}$ and $\bar{C}_{y,n}$ by

$$\begin{aligned} \bar{A}_{y,n} &= A_{y,n}, \\ \bar{B}_{y,n} &= B_{y,n} - \frac{3}{2} C_F A_{y,n}, \\ \bar{C}_{y,n} &= C_{y,n} - \frac{3}{2} C_F B_{y,n} + \left(\frac{9}{4} C_F^2 - K_2\right) A_{y,n}. \end{aligned} \tag{8}$$

The constant K_2 is also given by [9,10]

$$\begin{aligned} K_2 = \frac{1}{4} \bigg[& -\frac{3}{2} C_F^2 + C_F C_A \left(\frac{123}{2} - 44\zeta_3\right) \\ & + C_F T_R N_F (-22 + 16\zeta_3) \bigg]. \end{aligned} \tag{9}$$

E_{cm} denotes the centre-of-mass energy squared, μ is the QCD renormalization scale and $\zeta_3 = 1.202056$ [11]. The NLO expression is obtained by suppressing all terms at order α_s^3. The two coefficients of the QCD β-function are

$$\begin{aligned} \beta_0 &= \frac{11 C_A - 4 T_R N_F}{6}, \\ \beta_1 &= \frac{17 C_A^2 - 10 C_A T_R N_F - 6 C_F T_R N_F}{6}, \end{aligned} \tag{10}$$

where $C_A = N = 3$, $C_F = (N^2 - 1)/2N = \frac{4}{3}$ and $T_R = \frac{1}{2}$ are the QCD colour factors [1,2]. Also N_F is the number of light quark flavours.

The perturbative coefficients ($A_{y,n}$, $B_{y,n}$ and $C_{y,n}$) are independent of the centre-of-mass energy. They are obtained by integrating parton-level distributions, which were calculated recently to NNLO accuracy [1]. These values are shown in table 1.

Table 1. Contributions to C-parameter at LO, NLO and NNLO from ref. [1].

n	$A_{C,n}$	$B_{C,n}$	$C_{C,n}$
1	8.6379	172.778 ± 0.007	3212.2 ± 88.7
2	2.4317	81.184 ± 0.005	2220.9 ± 12.0
3	1.0792	42.771 ± 0.003	1296.6 ± 6.7

Next we explain the dispersive as well as the shape function models as follows:

3.2 *The dispersive model*

Non-perturbative power corrections can be related to infrared renormalizations in the perturbative QCD expansion for the event-shape variables. The dispersive model for the strong coupling leads to a shift in the distributions [8]:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y}(y) = \frac{\mathrm{d}\sigma_{\mathrm{pt}}}{\mathrm{d}y}(y - a_y P), \tag{11}$$

where the numerical factor a_y depends on the event shape, for C-parameter it is $a_y = 3\pi$ [11], while P is believed to be universal and scales with the CMS energy like μ_I/Q [8]. μ_I is the renormalization scale in the non-perturbative part of the theory. Q is also the centre-of-mass energy. This scale factor for Λ_{QCD} is $\mu_I = 2$ GeV.

Then we obtain the non-perturbative parameter from this model as follows:

$$\begin{aligned}\langle y^n \rangle_{\mathrm{np}} &= a_y P \\ &= a_y \frac{4C_F}{\pi^2} M \frac{\mu_I}{E_{\mathrm{cm}}} \left[\alpha_0(\mu_I) - \alpha_{\mathrm{s}}(\mu) \right. \\ &\quad \left. - \left(\ln\frac{\mu}{\mu_I} + 1 + \frac{k}{4\pi\beta_0} \right) 2\beta_0 \alpha_{\mathrm{s}}^2(\mu) \right].\end{aligned} \tag{12}$$

In the $\overline{\mathrm{MS}}$ renormalization scheme the constant k has the value

$$k = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_F.$$

The Milan factor M is known in two loops as $M = 1.49 \pm 0.20$ [12,13], for the number of flavours $N_F = 3$ at the relevant low scales [14].

As a result for M, the so-called non-inclusive Milan factor, we have [15]

$$\begin{aligned}M &= 1 + \frac{3.299 C_A}{\beta_0} + 2 \times \frac{-0.862 C_A - 0.052 N_F}{\beta_0} \\ &= 1 + \frac{1.575 C_A - 0.104 N_F}{\beta_0} = 1.49 \pm 0.20.\end{aligned} \tag{13}$$

At this stage we use power corrections to calculate the perturbative and the non-perturbative theories for the moments of y up to third order, and we have [8]

$$\langle y^1 \rangle = \langle y^1 \rangle_{\mathrm{NLO}} + a_y P, \tag{14}$$

$$\langle y^2 \rangle = \langle y^2 \rangle_{\mathrm{NLO}} + 2\langle y^1 \rangle_{\mathrm{NLO}}(a_y P) + (a_y P)^2, \tag{15}$$

$$\begin{aligned}\langle y^3 \rangle &= \langle y^3 \rangle_{\mathrm{NLO}} + 3\langle y^2 \rangle_{\mathrm{NLO}}(a_y P) \\ &\quad + 3\langle y^1 \rangle_{\mathrm{NLO}}(a_y P)^2 + (a_y P)^3.\end{aligned} \tag{16}$$

In eq. (12), α_{s} is the strong coupling constant and α_0 defines the non-perturbative parameter accounting for the contributions to an infrared matching scale $\mu_I \cong 2$. We are using the AMY data taken from TRISTAN at KEK, as well as DELPHI and ALEPH at CERN. We have also used the PYTHIA data in this analysis.

Figure 1 shows the distributions obtained for the PYTHIA event generator as well as for the distributions obtained for the real data. As the dispersive model (solid line) includes both perturbative and non-perturbative parts of the theory, it is more consistent with our data, when compared with the distributions obtained from NLO and NNLO, where the non-perturbative part of the theory is absent.

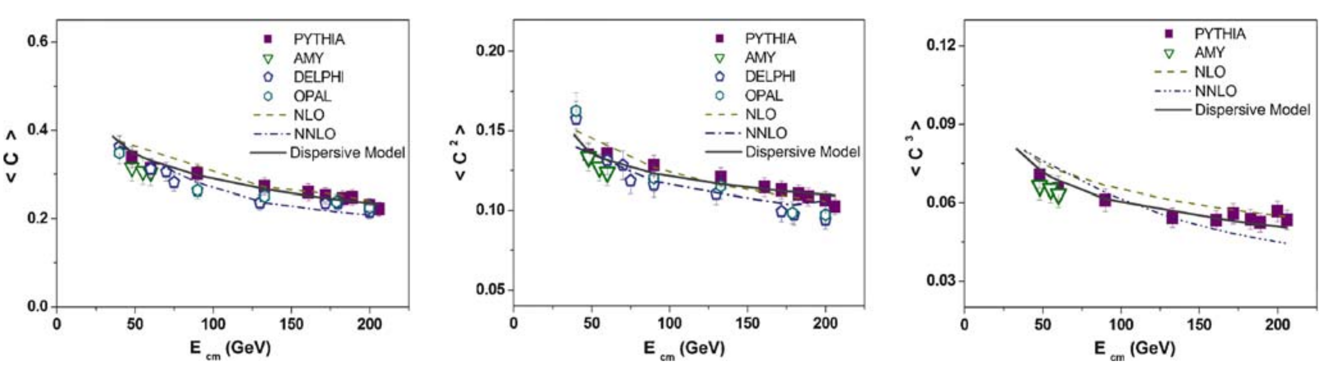

Figure 1. Fitting the C-parameter in the dispersive model with the data.

Table 2. Measurements of the coupling constants using the dispersive model.

n	1	2	3
$\alpha_s(M_{Z^0})$	$0.12171 \pm 0.0021 \pm 0.0123$	$0.1169 \pm 0.0013 \pm 0.0125$	$0.1120 \pm 0.0028 \pm 0.0127$
$\alpha_0(\mu_I)$	$0.5030 \pm 0.0045 \pm 0.0415$	$0.4702 \pm 0.0069 \pm 0.0305$	$0.4992 \pm 0.0067 \pm 0.0396$
$\chi^2/(\text{d.o.f.})$	1.069	1.057	1.019

Table 3. Measurements of the coupling constant for the C-parameter [16].

Exp.	ALEPH	DELPHI	L3
$\alpha_s(M_{Z^0})$	0.1228 ± 0.0027	0.1222 ± 0.0036	0.1164 ± 0.0047
$\alpha_0(\mu_I)$	0.461 ± 0.016	0.444 ± 0.022	0.457 ± 0.040

The values for $\alpha_s(M_{Z^0})$ and $\alpha_0(\mu_I)$ up to third power correction are indicated in table 2. The errors include the statistical and the hadronization parts of corrections. The statistical uncertainty is that of the fitting procedure used to determine the coupling constant. The hadronization uncertainty is estimated by changing the Monte Carlo generator PYTHIA to real data at similar energies. The difference between the two results gives us the hadronization uncertainties.

The mean value is the average of the three values quoted in table 2, which is: $\alpha_s(M_{Z^0}) = 0.11687 \pm 0.0021(\text{stat.}) \pm 0.0125(\text{had.})$ and $\alpha_0(\mu_I) = 0.4908 \pm 0.0060(\text{stat.}) \pm 0.0372(\text{had.})$.

Our results are consistent with the values obtained from other experiments cited in table 3 [16].

3.3 *The shape function model*

The shape function model [17] includes the perturbative as well as the non-perturbative parts of the theory. This model is a combination of both the NLO prediction and the power correction terms (eq. (4)). We are using eq. (7) to calculate the strong coupling constant in perturbative theory. We also use the following expansion for measuring the free parameter in the non-perturbative part of the theory:

$$\langle C^1 \rangle = \langle C^1 \rangle_{\text{NLO}} + \frac{\lambda_1}{E_{\text{cm}}}, \tag{17}$$

where the first part shows the perturbative and the second part shows the non-perturbative parts of the theory.

Analogously, for the power corrections (the second and the third moments), we have [18]

$$\langle C^2 \rangle = \langle C^2 \rangle_{\text{NLO}} + 2\frac{\lambda_1}{E_{\text{cm}}}\langle C^1 \rangle_{\text{NLO}} + \frac{\lambda_2}{E_{\text{cm}}^2}, \tag{18}$$

$$\langle C^3 \rangle = \langle C^3 \rangle_{\text{NLO}} + 3\frac{\lambda_1}{E_{\text{cm}}}\langle C^2 \rangle_{\text{NLO}} + 3\frac{\lambda_2}{E_{\text{cm}}^2}\langle C^1 \rangle_{\text{NLO}} + \frac{\lambda_3}{E_{\text{cm}}^3}. \tag{19}$$

The coefficient λ_1 is the first moment and λ_2 is the second moment of the shape function as universal scales [19]. By doing a similar fitting procedure with the corresponding distribution for the dispersive model (figure 2), our obtained results are tabulated in table 4.

The mean values obtained are: $\alpha_s(M_{Z^0}) = 0.1241 \pm 0.0024(\text{stat.}) \pm 0.0122(\text{had.})$ and $\lambda_1 = 1.2339 \pm 0.0145(\text{stat.}) \pm 0.0372(\text{had.})$.

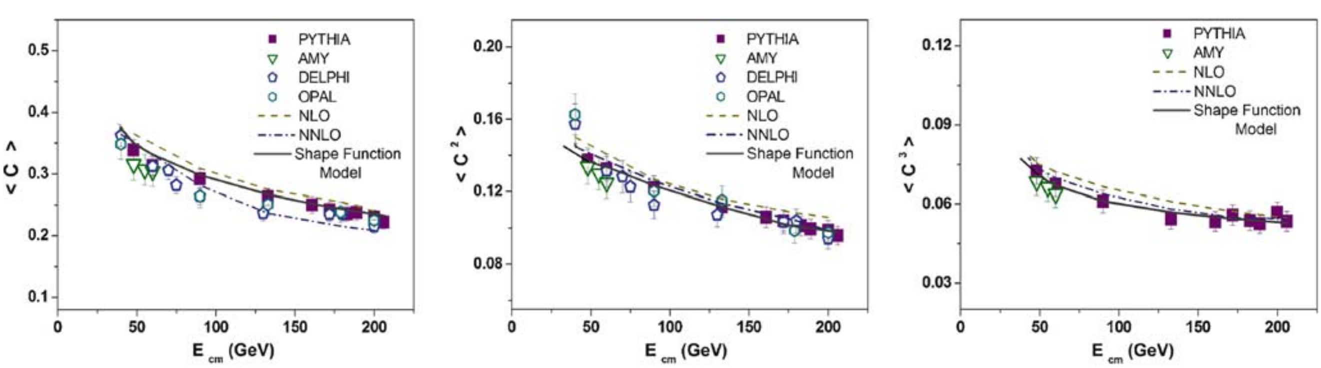

Figure 2. Fitting the C-parameter in the shape function model with the data.

Table 4. Measurements of the coupling constants using the shape function model.

n	1	2	3
$\alpha_s(M_{Z^0})$	$0.1258 \pm 0.001 \pm 0.0131$	$0.1209 \pm 0.0014 \pm 0.0123$	$0.1257 \pm 0.0050 \pm 0.0112$
λ_1	$1.2478 \pm 0.0142 \pm 0.0421$	$1.2314 \pm 0.0179 \pm 0.0357$	$1.2224 \pm 0.0114 \pm 0.0338$
$\chi^2/(\text{d.o.f.})$	1.046	1.010	1.040

These values are in good agreement with our results for other event-shape variables carried out in our previous works [3,4]. Our results are also consistent with the QCD predictions [14].

3.4 *The variance*

The simple prediction for the variance of the event-shape variable y on hadron level is [20]:

$$\text{Var}(y) = \langle y^2 \rangle - \langle y \rangle^2. \tag{20}$$

We obtain a purely perturbative expression for the variance in the dispersive model and also in the shape function model, up to strongly suppressed corrections $O(\alpha_s^4)$. NLO and NNLO do not include non-perturbative region. Thus, in the case of the C-parameter, we have

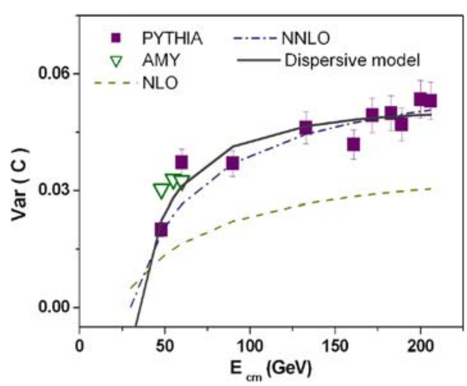

Figure 3. Fitting the variance of the dispersive model with the data.

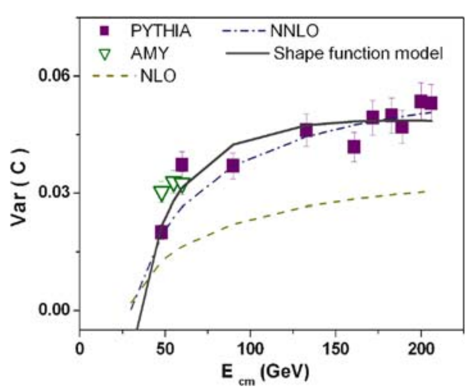

Figure 4. Fitting the variance of the shape function model with the data.

Table 5. Measurements of the coupling constants using the variance of dispersive model.

Observable	$\alpha_s(M_{Z^0})$	$\alpha_0(\mu_I)$	$\chi^2/(\text{d.o.f.})$
C-parameter	0.1161 ± 0.0047	0.5392 ± 0.0333	1.026

Table 6. Measurements of coupling constants using the variance of shape function model.

Observable	$\alpha_s(M_{Z^0})$	λ_1	$\chi^2/(\text{d.o.f.})$
C-parameter	0.1185 ± 0.0042	1.0304 ± 0.0837	1.052

$$\text{Var}(C) = \langle C^2 \rangle_{\text{NLO}} - \langle C \rangle^2_{\text{NLO}}. \tag{21}$$

On the other hand, if we also take into account the non-perturbative part of the model, we have

$$\text{Var}(C) = \langle C^2 \rangle_{\text{total}} - \langle C \rangle^2_{\text{total}}, \tag{22}$$

where the subscript on the right includes both regions. Figures 3 and 4 show our distribution. Our results are summarized in tables 5 and 6.

The values obtained in both tables indicate that our results are in good agreement with the QCD predictions [21]. They are also consistent with the values for other experiments [14].

4. Conclusion

The C-parameter is explained as an event-shape variable in this article. The coupling constant is calculated in perturbative as well as in non-perturbative regions. To achieve this, we use the dispersive and the shape function models. By fitting these models with the corresponding distributions, we find the mean values $\alpha_s(M_{Z^0}) = 0.117 \pm 0.014$ and $\alpha_0(\mu_I) = 0.491 \pm 0.043$ for the dispersive model and the mean values $\alpha_s(M_{Z^0}) = 0.124 \pm 0.015$ and $\lambda_1 = 1.234 \pm 0.052$ for the shape function model. Finally, we extract the coupling constant through the definition of the variance in both regions. The results obtained by this method are also consistent with QCD predictions.

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