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Periodic Hamiltonian hierarchies and non-uniqueness of superpotentials

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Abstract. In this article, a family of periodic quantum Hamiltonians, that is subject to a closure condition is considered. In the context of the factorization method, we address the question of non-uniqueness of the governing superpotentials and study an alternative factorization to generate new hierarchies of potentials.

Keywords. Supersymmetry; periodic Hamiltonian; isospectral deformation.

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1. Introduction

Through the past few decades, research in supersymmetric (SUSY) quantum mechanics (QM) has actively flourished opening up new directions of enquiry [1–6]. As it is well established by now, SUSYQM provides an elegant description of the mathematical structure and symmetry properties of the Schrödinger equation. The essence of SUSY lies in the possibility of factorizing [7,8] the Schrödinger equation in terms of a governing superpotential. In effect this amounts to solving a nonlinear differential equation that has a Riccati form.

The subject of periodic Hamiltonians has been examined for a long time [9–12]. Very recently, certain aspects of periodic Hamiltonian hierarchies have been reinvestigated [13,14] under cyclic permutations of the Hamiltonian indices for some particular classes of closed system. While these works have studied groups of permuted Hamiltonians by interpreting them in the framework of SUSYQM, the question of the possibility of non-uniqueness of superpotentials [15–17] that gives rise to an isospectral deformation has not been so far explored. We show, in this paper, that different choices of appropriate superpotentials do indeed give rise to new hierarchies that are distinct in their own right and different from the earlier ones.

The plan of this paper is as follows: In §2, a brief resume of SUSYQM is given; in §3, we consider in some detail the periodic closure condition imposed upon the superpotential and discuss some concrete cases of periodic SUSY Hamiltonians; in §4, we address the question of non-uniqueness of superpotentials and focus on an alternative factorization scenario to generate new hierarchies of potentials; finally in §5, a summary is given.

2. Brief resume of SUSYQM

The Schrödinger equation corresponding to a class of potentials $V_\lambda(x)$ reads as

$$H_\lambda \psi_{\lambda,n}(x) = \epsilon_{\lambda,n}(x)\psi_{\lambda,n}(x),\quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where the suffix λ identifies each member of the hierarchy of Hamiltonians given by

$$H_\lambda = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda(x), \quad \lambda = 1, 2, \ldots \tag{2.2}$$

in units $\hbar = m = 1$.

In (2.1) the eigenfunctions $\psi_{\lambda,n}$ are assumed to be normalized according to

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \psi^*_{\lambda,n}(x)\psi_{\lambda,m}(x) = \delta_{n,m}, \quad n, m = 0, 1, 2, \ldots. \tag{2.3}$$

In the language of SUSYQM [1–4,18], the ground-state wave function defines an underlying superpotential of system (2.1) in the manner

$$W_\lambda(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln|\psi_{\lambda,0}(x)| \Rightarrow$$
$$\psi_{\lambda,0}(x) = \mathcal{N}_{\lambda,0}\mathrm{e}^{-\int \mathrm{d}\xi\, W_\lambda(\xi)}, \tag{2.4}$$

where $\mathcal{N}_{\lambda,0}$ is a normalization factor.

In terms of $W_\lambda(x)$, we can choose intertwined Hamiltonians labelled by H_λ and $H_{\lambda+1}$ that belong to the same hierarchy such that

$$H_\lambda = A_\lambda^\dagger A_\lambda + \epsilon_{\lambda,0},$$
$$V_\lambda(x) = \frac{1}{2}[W_\lambda^2(x) + W_\lambda'(x)] + \epsilon_{\lambda,0} \tag{2.5}$$

and

$$H_{\lambda+1} = A_\lambda A_\lambda^\dagger + \epsilon_{\lambda,0} = A_{\lambda+1}^\dagger A_{\lambda+1} + \epsilon_{\lambda+1,0}$$
$$= -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}[W_\lambda^2(x) - W_\lambda'(x)] + \epsilon_{\lambda,0}, \tag{2.6}$$

where A_λ and $A_\lambda^\dagger$ are mutually adjoint differential operators defined by

$$A_\lambda = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_\lambda\right),$$
$$A_\lambda^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_\lambda\right). \tag{2.7}$$

It then easily transpires from (2.6) that corresponding to $H_{\lambda+1}$, the governing potential $V_{\lambda+1}$ has the form

$$V_{\lambda+1} = \frac{1}{2}[W_{\lambda+1}^2(x) + W_{\lambda+1}'(x)] + \epsilon_{\lambda+1,0} \tag{2.8}$$

tied with the constraint

$$W_{\lambda+1}^2(x) + W_{\lambda+1}'(x) = W_\lambda^2(x) - W_\lambda'(x) + 2(\epsilon_{\lambda,0} - \epsilon_{\lambda+1,0}). \tag{2.9}$$

It should be remarked that (2.6) and (2.7) are equivalent to the fulfillment of the intertwining relations given by

$$H_{\lambda+1}A_\lambda = A_\lambda H_\lambda, \tag{2.10}$$

$$H_\lambda A_\lambda^\dagger = A_\lambda^\dagger H_{\lambda+1}, \tag{2.11}$$

where $\lambda = 1, 2, \ldots$.

Since we are dealing with an unbroken case of SUSY, the ground state of H_λ is annihilated by A_λ, i.e. $A_\lambda\psi_{\lambda,0}(x) = 0$ while the remaining wavefunctions are furnished by the relations

$$\psi_{\lambda+1,n}(x) = \frac{1}{\sqrt{\epsilon_{\lambda,n+1} - \epsilon_{\lambda,0}}}A_\lambda\psi_{\lambda,n+1}(x) \tag{2.12}$$

$$\psi_{\lambda,n+1}(x) = \frac{1}{\sqrt{\epsilon_{\lambda,n+1} - \epsilon_{\lambda,0}}}A_\lambda^\dagger\psi_{\lambda+1,n}(x) \tag{2.13}$$

with the isospectrality condition $\epsilon_{\lambda+1,n} = \epsilon_{\lambda,n+1}$ being valid for $n = 0, 1, 2, \ldots$.

3. Periodic closure conditions

To solve an equation of the type (2.9), a possibility to obtain viable solutions is to impose some kind of closure condition

$$W_{\lambda+N}(x) = W_\lambda(x), \quad \Delta_{\lambda+N} = \Delta_\lambda, \tag{3.1}$$

where N is the periodicity of a hierarchy that involve $(N+1)$ Hamiltonians and we have set $\Delta_\lambda = \epsilon_{\lambda+1,0} - \epsilon_{\lambda,0}$. Note that the Δ_λ's sum to

$$\Omega \equiv \Delta_\lambda + \Delta_{\lambda+1} + \cdots + \Delta_{\lambda+N-1}$$
$$= \epsilon_{\lambda+N,0} - \epsilon_{\lambda,0} \neq 0. \tag{3.2}$$

A study of the cyclic condition (3.1) has revealed the following situations: the case $N = 1$ is consistent with the harmonic oscillator problem [9,11]; the case $N = 2$ yields the general conformal quantum mechanical result [19]

$$W_{1,2}(x) = \frac{1}{2}\left(\pm\frac{\Delta_1 - \Delta_2}{\Delta_1 + \Delta_2}\frac{1}{x} + \frac{\Delta_1 + \Delta_2}{2}x\right) \tag{3.3}$$

while the case $N = 3$ involves hierarchy of four Hamiltonians leading to potentials of transcendental form depending on the solutions of the Painlevé-IV nonlinear differential equation [20]:

$$W_1(x) = \frac{1}{2}\Omega x + g(x), \tag{3.4}$$

$$W_{2,3}(x) = -\frac{1}{2}g(x) \mp \frac{1}{2g(x)}(g'(x) + \Delta_2), \tag{3.5}$$

where the function $g(x)$ satisfies a nonlinear differential equation of the form

$$g''(x) = \frac{g'^2(x)}{2g(x)} + \frac{3}{2}g^3(x) + 2\Omega x g^2(x) + \left(\frac{1}{2}\Omega x^2 + \Delta_3 - \Delta_1\right)g(x) - \frac{\Delta_2^2}{2g(x)}. \tag{3.6}$$

The main purpose of this work is to investigate the viability of the non-uniqueness of the factorization schemes [21,22] within a SUSY framework in the context of a hierarchy of Hamiltonians enjoying a periodic closure condition. To embark upon such a possibility, we reassess the above results first. Then we also look for other solutions of isospectrality as induced by a different class of superpotentials that are tenable for the second-order case while we also write down new solutions of superpotentials for the third-order case that are guided by the error function or Airy function. In the following, we first review some specific cases of periodic SUSY Hamiltonians.

4. Periodic SUSY Hamiltonians

We focus on the simplest case of the hierarchial set-up with two Hamiltonians H_1 and H_2 subject to the closure condition

$$H_2 = H_1 - \Lambda, \tag{4.1}$$

where the factorized forms of the Hamiltonians are given by

$$H_1 = A_1^\dagger A_1 + \epsilon_{1,0}, \quad H_2 = A_1 A_1^\dagger + \epsilon_{1,0}. \tag{4.2}$$

In (4.2)

$$A_1 = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_1\right), \quad A_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_1\right),$$

W_1 is the superpotential and

$$H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x).$$

In terms of $W_1(x)$, the Hamiltonians H_1 and H_2 read as

$$H_1 = \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1'(x) + W_1^2(x)\right] + \epsilon_{1,0} \tag{4.3}$$

$$\begin{aligned} H_2 &= \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_1'(x) + W_1^2(x)\right] + \epsilon_{1,0} \\ &= H_1 - \Lambda = \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1'(x) + W_1^2(x)\right] + \epsilon_{1,0} - \Lambda. \end{aligned} \tag{4.4}$$

Comparison with (4.1) gives

$$W_1'(x) = \Lambda \Rightarrow W_1(x) = \Lambda(x - x_0). \tag{4.5}$$

Correspondingly

$$V_1(x) = \frac{1}{2}\Lambda^2(x - x_0)^2 + \left(\frac{1}{2}\Lambda + \epsilon_{1,0}\right),$$

which stands for the harmonic oscillator potential. The normalizable ground-state wave function then reads as

$$\psi_{1,0} \propto \exp\left(-\frac{\Lambda}{2}(x - x_0)^2\right), \quad \Lambda > 0. \tag{4.6}$$

We now turn to the hierarchy of three Hamiltonians

$$H_\lambda = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda(x), \quad \lambda = 1, 2, 3$$

with the closure condition

$$H_3 = H_1 - \Lambda. \tag{4.7}$$

These satisfy the intertwining relations as given by (2.10) and (2.11), i.e.

$$H_{\lambda+1} A_\lambda = A_\lambda H_\lambda, \quad H_\lambda A_\lambda^\dagger = A_\lambda^\dagger H_{\lambda+1} \tag{4.8}$$

for $\lambda = 1, 2$.

The factorized forms of the Hamiltonians read as

$$H_\lambda = A_\lambda^\dagger A_\lambda + \epsilon_{\lambda,0}, \quad H_{\lambda+1} = A_\lambda A_\lambda^\dagger + \epsilon_{\lambda,0}; \quad \lambda = 1, 2 \tag{4.9}$$

where

$$A_1 = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_1\right), \quad A_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_1\right),$$

$$A_2 = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_2\right), \quad A_2^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_2\right)$$

with $W_\lambda(x)$, $\lambda = 1, 2$ are the respective superpotentials.

Explicitly, H_1 has the form

$$\begin{aligned} H_1 &= A_1^\dagger A_1 + \epsilon_{1,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_1'(x) - W_1^2(x)\right] + \epsilon_{1,0} \\ &= A_2 A_2^\dagger + \epsilon_{2,0} + \Lambda = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_2'(x) - W_2^2(x)\right] \\ &\quad + \epsilon_{2,0} + \Lambda, \end{aligned} \tag{4.10}$$

where we have evidently

$$\begin{aligned} W_1'(x) + W_2'(x) &= -W_1^2(x) + W_2^2(x) \\ &\quad + 2(\Lambda + \Delta_1) \end{aligned} \tag{4.11}$$

and H_2 reads as

$$\begin{aligned} H_2 &= A_1 A_1^\dagger + \epsilon_{1,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1'(x) - W_1^2(x)\right] + \epsilon_{1,0} \\ &= A_2^\dagger A_2 + \epsilon_{2,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_2'(x) - W_2^2(x)\right] + \epsilon_{2,0}, \end{aligned} \tag{4.12}$$

where likewise the following constraint holds:

$$W_1'(x) + W_2'(x) = W_1^2(x) - W_2^2(x) - 2\Delta_1. \tag{4.13}$$

From (4.11) and (4.13) it readily follows

$$W_1'(x) + W_2'(x) = \Lambda \Rightarrow W_1(x) + W_2(x) = \Lambda(x - x_0) \tag{4.14}$$

which on solving gives

$$W_{1,2}(x) = \frac{1}{2}\Lambda(x - x_0) \pm \frac{2\Delta_1 + \Lambda}{2\Lambda}\frac{1}{x - x_0}. \tag{4.15}$$

Correspondingly

$$V_1(x) = \frac{\Delta_1 + \Lambda + 2\epsilon_1}{2} + \frac{\Lambda^2}{8}(x - x_0)^2 + \frac{\Delta_1^2 - (\Lambda^2/4)}{2\Lambda^2}\frac{1}{(x - x_0)^2} \tag{4.16}$$

and

$$V_2(x) = \frac{\Delta_1 - (\Lambda/2)}{4} + \frac{\Lambda^2}{8}(x - x_0)^2 - \frac{\Delta_1^2 - (\Lambda^2/4)}{2\Lambda^2}\frac{1}{(x - x_0)^2}. \tag{4.17}$$

We thus see that for $\Delta_1^2 > (\Lambda^2/4)$, while the potential $V_1(x)$ is an example of the isotonic oscillator, the centrifugal barrier term appears in $V_2(x)$ with a wrong sign.

For $\Lambda > 0$, from $A_\lambda^\dagger \psi_{\lambda,0} = 0$ we obtain the ground-state wave functions

$$\psi_{1,0} \propto (x - x_0)^{-\frac{1}{2} - \frac{\Delta_1}{\Lambda}} \exp\left(-\frac{\Lambda}{4}(x - x_0)^2\right) \tag{4.18}$$

$$\psi_{2,0} \propto (x - x_0)^{\frac{1}{2} + \frac{\Delta_1}{\Lambda}} \exp\left(-\frac{\Lambda}{4}(x - x_0)^2\right) \tag{4.19}$$

which are normalizable for $-1 < (\Delta_1/\Lambda) < 0$.

However, for $\Lambda < 0$, we have to use $A_\lambda \psi_{\lambda,0} = 0$ and the corresponding results are

$$\psi_{1,0} \propto (x - x_0)^{\frac{1}{2} + \frac{\Delta_1}{\Lambda}} \exp\left(\frac{\Lambda}{4}(x - x_0)^2\right) \tag{4.20}$$

$$\psi_{2,0} \propto (x - x_0)^{-\frac{1}{2} - \frac{\Delta_1}{\Lambda}} \exp\left(\frac{\Lambda}{4}(x - x_0)^2\right) \tag{4.21}$$

and note that here too normalization requires the condition $-1 < (\Delta_1/\Lambda) < 0$.

Now we focus on the hierarchy of four Hamiltonians

$$H_\lambda = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda(x), \quad \lambda = 1, 2, 3, 4$$

with the closure condition

$$H_4 = H_1 - \Lambda. \tag{4.22}$$

These satisfy the intertwining relations given by (2.10) and (2.11), i.e.

$$H_{\lambda+1}A_\lambda = A_\lambda H_\lambda, \quad H_\lambda A_\lambda^\dagger = A_\lambda^\dagger H_{\lambda+1}, \tag{4.23}$$

for $\lambda = 1, 2, 3$.

The factorized forms of the Hamiltonians are given by

$$H_\lambda = A_\lambda^\dagger A_\lambda + \epsilon_{\lambda,0}, \quad H_{\lambda+1} = A_\lambda A_\lambda^\dagger + \epsilon_{\lambda,0} \quad \lambda = 1, 2, 3, \tag{4.24}$$

where

$$A_\lambda = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_\lambda\right), \quad A_\lambda^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_\lambda\right)$$

and

$$W_\lambda(x), \quad \lambda = 1, 2, 3$$

are the respective superpotentials.

In terms of $W_\lambda(x)$'s

$$\begin{aligned} H_1 &= A_1^\dagger A_1 + \epsilon_{1,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_1' - W_1^2\right] + \epsilon_{1,0} \\ &= H_4 + \Lambda = A_3 A_3^\dagger + \epsilon_{3,0} \\ &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_3' - W_3^2\right] + (\Lambda + \epsilon_{3,0}), \end{aligned} \tag{4.25}$$

$$\begin{aligned} H_2 &= A_1 A_1^\dagger + \epsilon_{1,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1' - W_1^2\right] + \epsilon_{1,0} \\ &= A_2^\dagger A_2 + \epsilon_{2,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_2' - W_2^2\right] + \epsilon_{2,0}, \end{aligned} \tag{4.26}$$

$$\begin{aligned} H_3 &= A_2 A_2^\dagger + \epsilon_{2,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_2' - W_2^2\right] + \epsilon_{2,0} \\ &= A_3^\dagger A_3 + \epsilon_{3,0} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_3' - W_3^2\right] + \epsilon_{3,0}. \end{aligned} \tag{4.27}$$

Equating these two different factorizations associated with each H_λ we get the following system of constraint relations:

$$W_1' + W_3' = W_3^2 - W_1^2 + 2(\Delta_2 + \Delta_1 + \Lambda), \tag{4.28}$$

$$W_1' + W_2' = W_1^2 - W_2^2 - 2\Delta_1, \tag{4.29}$$

$$W_2' + W_3' = W_2^2 - W_3^2 - 2\Delta_2. \tag{4.30}$$

Adding these three constraints (4.28)–(4.30) we obtain $W_1' + W_2' + W_3' = \Lambda$, which after integration becomes

$$W_1 + W_2 + W_3 = \Lambda(x - x_0). \tag{4.31}$$

In a general approach [13] to solve these relations, one may obtain

$$W_{1,2} = -\frac{g}{2} \pm \frac{g'}{2g} \pm \frac{\epsilon_{1,0} - \epsilon_{2,0}}{g},$$
$$W_3 = g - \Lambda(x - x_0) \tag{4.32}$$

where $g(x)$ satisfies Painlevé-IV equation

$$gg'' = \frac{g'^2}{2} + \frac{3g^4}{2} + 4\Lambda(x - x_0)g^3 + 2[\Lambda^2(x - x_0)^2 - a]g^2 + b \tag{4.33}$$

and

$$a = -(\Lambda - \epsilon_{1,0} - \epsilon_{2,0} + 2\epsilon_{3,0}), \quad b = -2\Delta_1^2. \tag{4.34}$$

At this stage, we seek for an admissible formal solution of this system in the following form:

$$W_1 = \nu_1(x - x_0) + \frac{\mu_1}{x - x_0}, \tag{4.35}$$
$$W_2 = \nu_2(x - x_0) + \frac{\mu_2}{x - x_0}, \tag{4.36}$$
$$W_3 = \nu_3(x - x_0) - \frac{\mu_1 + \mu_2}{x - x_0}, \tag{4.37}$$

where $\nu_j, \mu_j \in \mathbb{R}$, $j = 1, 2, 3$.

From the constraint (4.31) we readily can get

$$\nu_1 + \nu_2 + \nu_3 = \Lambda. \tag{4.38}$$

Now substituting these expressions in (4.28), (4.29) and (4.30), we get

$$\nu_1^2 = \nu_2^2, \quad \mu_1^2 - \mu_2^2 = -(\mu_1 + \mu_2),$$
$$(\nu_1 + \nu_2) - 2(\nu_1\mu_1 - \nu_2\mu_2) = -2\Delta_1, \tag{4.39}$$
$$\nu_2^2 = \nu_3^2, \quad \mu_2^2 - (\mu_1 + \mu_2)^2 = \mu_1,$$
$$(\nu_2 + \nu_3) - 2(\nu_2\mu_2 + \nu_3\mu_1 + \nu_3\mu_2) = -2\Delta_2, \tag{4.40}$$
$$\nu_3^2 = \nu_1^2, \quad (\mu_1 + \mu_2)^2 - \mu_1^2 = \mu_2,$$
$$(\nu_1 + \nu_3) + 2(\nu_1\mu_1 + \nu_3\mu_1 + \nu_3\mu_2) = 2(\Delta_2 + \Delta_1 + \Lambda). \tag{4.41}$$

Solving for ν_i, μ_i we obtain

$$\mu_1 = \mu_2 = 0, \quad \nu_1 = \frac{\Lambda}{a}, \quad \nu_2 = \frac{\Lambda}{b}, \quad \nu_3 = \frac{\Lambda}{c},$$
$$a > 0, \quad b > 0, \quad c > 0;$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1, \quad \Delta_1 = -\frac{\Lambda}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$
$$\Delta_2 = -\frac{\Lambda}{2}\left(\frac{1}{b} + \frac{1}{c}\right) \tag{4.42}$$

and the corresponding superpotentials are

$$W_1 = \frac{\Lambda}{a}(x - x_0), \quad W_2 = \frac{\Lambda}{b}(x - x_0),$$
$$W_3 = \frac{\Lambda}{c}(x - x_0). \tag{4.43}$$

For $\Lambda > 0$, these superpotentials govern the corresponding normalized ground-state wave functions

$$\psi_{1,0} \propto \exp\left(-\frac{\Lambda}{2a}(x - x_0)^2\right), \quad \psi_{2,0} \propto \exp\left(-\frac{\Lambda}{2b}(x - x_0)^2\right),$$
$$\psi_{3,0} \propto \exp\left(-\frac{\Lambda}{2c}(x - x_0)^2\right). \tag{4.44}$$

Corresponding potentials are then given by

$$V_1(x) = \frac{1}{2}\left[\frac{\Lambda^2}{a^2}(x - x_0)^2 + \frac{\Lambda}{a}\right] + \epsilon_{1,0},$$
$$V_2(x) = \frac{1}{2}\left[\frac{\Lambda^2}{b^2}(x - x_0)^2 + \frac{\Lambda}{b}\right] + \epsilon_{2,0},$$
$$V_3(x) = \frac{1}{2}\left[\frac{\Lambda^2}{c^2}(x - x_0)^2 + \frac{\Lambda}{c}\right] + \epsilon_{3,0}. \tag{4.45}$$

The set (4.45) describes an isospectral family of shifted harmonic oscillators. Note that from the relations (4.39)–(4.41) we may find a non-trivial solution for μ_1 and μ_2, namely $\mu_1 = -\mu_2 = 1$. This solution for parameters $\mu_{1,2}$ is inadmissible as corresponding superpotentials are failed to produce normalized ground states.

5. Non-uniqueness of superpotentials

5.1 *The case of two Hamiltonians in the hierarchy*

If the superpotential is non-unique, then an alternative factorization of H_1 can be prescribed as follows [15]:

$$\tilde{H}_1 = B_1^\dagger B_1 + \tilde{\epsilon}_{1,0}, \tag{5.1}$$

where B_1 and $B_1^\dagger$ are mutually adjoint operators given by

$$B_1 = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_1 - \phi_1\right),$$
$$B_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_1 - \phi_1\right) \tag{5.2}$$

for some non-trivial function $\phi_1(x)$. Thus, a new hierarchy $\tilde{H}_\lambda, \lambda = 1, 2$ is set up with $\tilde{H}_1 = H_1$ and

$$\tilde{H}_2 = B_1 B_1^\dagger + \tilde{\epsilon}_{1,0} \tag{5.3}$$

with the closure condition

$$\tilde{H}_2 = \tilde{H}_1 - \tilde{\Lambda}. \tag{5.4}$$

In terms of $\phi_1(x)$ we can represent $\tilde{H}_1$ and $\tilde{H}_2$ as

$$\tilde{H}_1 = \frac{1}{2}\left[-\frac{d^2}{dx^2}+(W_1'+\phi_1')+(W_1+\phi_1)^2\right]+\tilde{\epsilon}_{1,0}, \quad (5.5)$$

$$\tilde{H}_2 = \frac{1}{2}\left[-\frac{d^2}{dx^2}-(W_1'+\phi_1')+(W_1+\phi_1)^2\right]+\tilde{\epsilon}_{1,0}$$
$$= \tilde{H}_1-\tilde{\Lambda}=\frac{1}{2}\left[-\frac{d^2}{dx^2}+(W_1'+\phi_1')+(W_1+\phi_1)^2\right] + \tilde{\epsilon}_{1,0}-\tilde{\Lambda}. \quad (5.6)$$

As such

$$W_1'+\phi_1' = \tilde{\Lambda} \Rightarrow \phi_1' = \tilde{\Lambda}-\Lambda, \quad (5.7)$$

where we use the result (4.5).

The choice of $\tilde{H}_1 = H_1$ gives us

$$\phi_1^2+2W_1\phi_1 = K,$$
$$K = (\Lambda-\tilde{\Lambda})+(\epsilon_{1,0}-\tilde{\epsilon}_{1,0})$$
$$\Rightarrow \phi_1 = -\Lambda(x-x_0)\pm\sqrt{K+\Lambda^2(x-x_0)^2} \quad (5.8)$$

and so

$$\phi_1' = -\Lambda \pm \frac{\Lambda^2(x-x_0)}{\sqrt{K+\Lambda^2(x-x_0)^2}} \quad (5.9)$$

which on comparison with (5.7) yields

$$\pm\frac{\Lambda^2(x-x_0)}{\sqrt{K+\Lambda^2(x-x_0)^2}} = \tilde{\Lambda}. \quad (5.10)$$

Equation (5.10) suggests $K=0$, $\tilde{\Lambda} = \pm\Lambda$ and implies a viable solution of $\phi_1(x)$, namely

$$\phi_1(x) = -2\Lambda(x-x_0), \quad \Lambda > 0. \quad (5.11)$$

Such a form is consistent with the harmonic oscillator potential as can be found out from (5.5) and (5.6). The corresponding normalized ground-state wave function $\psi_{1,0} \propto \exp\big(-(\Lambda/2)(x-x_0)^2\big)$ is obtained from $B_1\psi_{1,0}=0$.

5.2 *The case of three Hamiltonians in the hierarchy*

In the next level of complexity, the non-uniqueness of superpotential prescribes a new hierarchy of Hamiltonians $\tilde{H}_\lambda, \lambda = 1, 2, 3$ with its first member $\tilde{H}_1 = H_1$ with the closure condition

$$\tilde{H}_3 = \tilde{H}_1 - \tilde{\Lambda}. \quad (5.12)$$

These satisfy the intertwining relations as given by (2.10) and (2.11), i.e.

$$\tilde{H}_{\lambda+1}B_\lambda = B_\lambda\tilde{H}_\lambda, \quad \tilde{H}_\lambda B_\lambda^\dagger = B_\lambda^\dagger\tilde{H}_{\lambda+1},$$
$$\lambda = 1, 2, 3. \quad (5.13)$$

Explicitly, $\tilde{H}_\lambda$, $\lambda = 1, 2, 3$ are given by

$$\tilde{H}_\lambda = B_\lambda^\dagger B_\lambda + \tilde{\epsilon}_{\lambda,0}, \quad \tilde{H}_{\lambda+1} = B_\lambda B_\lambda^\dagger + \tilde{\epsilon}_{\lambda,0},$$
$$\lambda = 1, 2, 3, \quad (5.14)$$

where

$$B_\lambda = \frac{1}{\sqrt{2}}\left(\frac{d}{dx}-W_\lambda-\phi_\lambda\right)$$

and

$$B_\lambda^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx}-W_\lambda-\phi_\lambda\right)$$

are two mutually adjoint operators involving $W_\lambda(x)$, $\lambda = 1, 2, 3$ along with some non-trivial functions $\phi_\lambda(x)$.

In terms of $\phi_1(x)$ we can represent $\tilde{H}_1$ as

$$\tilde{H}_1 = -\frac{1}{2}\left[\frac{d^2}{dx^2}-(W_1'+\phi_1')-(W_1+\phi_1)^2\right]+\tilde{\epsilon}_{1,0} \quad (5.15)$$

which on comparison with H_1 (4.9) gives

$$\phi_1'+\phi_1^2+2W_1\phi_1 = 2(\epsilon_{1,0}-\tilde{\epsilon}_{1,0}). \quad (5.16)$$

Case I. If $\epsilon_{1,0} \neq \tilde{\epsilon}_{1,0}$ then for $\phi_1 = \xi'/\xi$, eq. (5.16) can be cast into a second-order differential equation

$$\frac{d^2\xi}{dy^2} + \frac{2\Delta_1+\Lambda}{\Lambda}\frac{1}{y}\frac{d\xi}{dy} + \Lambda y\frac{d\xi}{dy} + 2(\tilde{\epsilon}_{1,0}-\epsilon_{1,0}) = 0, \quad (5.17)$$

where we have set $y = x - x_0$.

If we choose $\Lambda = -2$ and $2\Delta_1+\Lambda = 0 \Rightarrow \Delta_1 = 1$ then for $\tilde{\epsilon}_{1,0}-\epsilon_{1,0} = n$ where n is a non-negative integer, (5.17) becomes the Hermite differential equation

$$\frac{d^2\xi}{dy^2} - 2y\frac{d\xi}{dy} + 2n\xi = 0. \quad (5.18)$$

Its solution is given by $\xi = H_n(y)$ where $H_n(y)$ represents the Hermite polynomial.

Hence we can write

$$\phi_1 = \frac{(d/dy)H_n(y)}{H_n(y)}$$
$$= 2(x-x_0) - \frac{H_{n+1}(x-x_0)}{H_n(x-x_0)}. \quad (5.19)$$

For our choice of Λ and Δ_1 we see that $W_1 = W_2 = -(x-x_0)$ and from two representations for $\tilde{H}_2$ in terms of ϕ_1 and ϕ_2 we can write

$$\tilde{H}_2 = -\frac{1}{2}\left[\frac{d^2}{dx^2}+(W_1'+\phi_1')-(W_1+\phi_1)^2\right]+\tilde{\epsilon}_{1,0}$$
$$= -\frac{1}{2}\left[\frac{d^2}{dx^2}-(W_2'+\phi_2')-(W_2+\phi_2)^2\right]+\tilde{\epsilon}_{2,0}. \quad (5.20)$$

As a result we are led to a constraint

$$\phi_2' + \phi_2^2 - 2(x-x_0)\phi_2 = -2\phi_1' + 2(\epsilon_{1,0} - \tilde{\epsilon}_{2,0} + 1). \quad (5.21)$$

Substituting (5.21) into the closure relation (5.12), the closed form expression for ϕ_2 reads as

$$\phi_2 = -\phi_1 + (\tilde{\Lambda}+2)(x-x_0) + K = \tilde{\Lambda}(x-x_0) + \frac{H_{n+1}(x-x_0)}{H_n(x-x_0)} + K, \quad (5.22)$$

where K is a constant. We have thus solved for the two non-trivial functions ϕ_1 and ϕ_2 and arrived at their closed form expressions given by (5.19) and (5.22). It is needless to mention that these functions play the role of isospectral deformation.

From $B_\lambda\psi_{\lambda,0} = 0$ we find the ground states

$$\psi_{1,0} \propto \exp\left(-\frac{1}{2}(x-x_0)^2\right) H_n(x-x_0), \quad (5.23)$$

$$\psi_{2,0} \propto \exp\left(\frac{\Lambda}{2}\{(x-x_0)+K\}^2\right) H_n(x-x_0), \quad \tilde{\Lambda} = -1, \quad (5.24)$$

where n has to be an even non-negative integer.

Case II. If $\epsilon_{1,0} = \tilde{\epsilon}_{1,0}$ then for the specific combination of the parameters $2\Delta_1 + \Lambda = 0$ we have from (4.15) $W_1 = W_2 = \frac{1}{2}\Lambda(x-x_0)$ and so the solution to eq. (5.16) will be

$$\phi_1(x) = \frac{e^{-(\Lambda/2)(x-x_0)^2}}{K_1 + \sqrt{(\pi/2\Lambda)}\mathrm{erf}(\sqrt{(\Lambda/2)}(x-x_0))}, \quad (5.25)$$

where K_1 is a constant and 'erf' is the error function.

Two different factorizations of $\tilde{H}_2$ in terms of ϕ_1 and ϕ_2 are given by

$$\tilde{H}_2 = -\frac{1}{2}\left[\frac{d^2}{dx^2} + (W_1' + \phi_1') - (W_1+\phi_1)^2\right] + \tilde{\epsilon}_{1,0} = -\frac{1}{2}\left[\frac{d^2}{dx^2} - (W_2' + \phi_2') - (W_2+\phi_2)^2\right] + \tilde{\epsilon}_{2,0} \quad (5.26)$$

implying

$$\phi_2' + \phi_2^2 + 2W_2\phi_2 = -\Lambda + 2(\tilde{\epsilon}_{1,0} - \tilde{\epsilon}_{2,0}) - 2\phi_1' - 2W_2. \quad (5.27)$$

On the other hand, the two different factorizations of $\tilde{H}_3$ in terms of ϕ_2 and ϕ_3 read as

$$\tilde{H}_3 = -\frac{1}{2}\left[\frac{d^2}{dx^2} + (W_2' + \phi_2') - (W_2+\phi_2)^2\right] + \tilde{\epsilon}_{2,0} = \tilde{H}_1 - \tilde{\Lambda} = -\frac{1}{2}\left[\frac{d^2}{dx^2} - W_1' - W_1^2\right] + \epsilon_{1,0} - \tilde{\Lambda}. \quad (5.28)$$

On comparison we find

$$-\phi_2' + \phi_2^2 + 2W_2\phi_2 = (W_1+W_2)' + (W_1^2 - W_2^2) - 2(\tilde{\Lambda} - \epsilon_{1,0} + \tilde{\epsilon}_{2,0}). \quad (5.29)$$

Using (5.27) and expressions $W_{1,2}$ as given in (4.15) we can directly solve (5.29) to obtain

$$\phi_2(x) = -\phi_1 - \frac{\Lambda}{4}(x-x_0)^2 - (\Lambda - \tilde{\Lambda})(x-x_0). \quad (5.30)$$

Corresponding to the above solutions of $\phi_{1,2}(x)$, the accompanying potential to (5.26) takes the form

$$\tilde{V}_2(x) = \frac{\Lambda^2}{8}(x-x_0)^2 - e^{-(\Lambda/2)(x-x_0)^2} + \tilde{\epsilon}_{1,0} - \frac{\Lambda}{4} \quad (5.31)$$

which for a small Λ approximates to a harmonic oscillator potential.

For $\Lambda > 0$, the ground-state wave functions are obtained from $B_\lambda^\dagger\psi\lambda, 0 = 0$ and read as

$$\psi_{1,0} \propto \exp\left(-\frac{\Lambda}{4}(x-x_0)^2\right) \times \left[K_1 + \sqrt{\frac{\pi}{2\Lambda}}\mathrm{erf}\left\{\sqrt{\frac{\Lambda}{2}}(x-x_0)\right\}\right]^{-\sqrt{\Lambda/2}}, \quad (5.32)$$

$$\psi_{2,0} \propto \exp\left(-\frac{\Lambda}{8}(x-x_0)^2\right) \times \left[K_1 + \sqrt{\frac{\pi}{2\Lambda}}\mathrm{erf}\left\{\sqrt{\frac{\Lambda}{2}}(x-x_0)\right\}\right]^{\sqrt{\Lambda/2}}, \quad (5.33)$$

where $\Lambda = \tilde{\Lambda}$, $K_1 > \sqrt{\pi/2\Lambda}$.

5.3 *The case of four Hamiltonians in the hierarchy*

In this level of complexity, the non-uniqueness of the superpotential prescribes a new hierarchy of Hamiltonians $\tilde{H}_\lambda$, $\lambda = 1, 2, 3, 4$ with its first member $\tilde{H}_1 = H_1$ along with the closure condition

$$\tilde{H}_4 = \tilde{H}_1 - \tilde{\Lambda}. \quad (5.34)$$

These satisfy the intertwining relations

$$\tilde{H}_{\lambda+1}B_\lambda = B_\lambda\tilde{H}_\lambda, \quad \tilde{H}_\lambda B_\lambda^\dagger = B_\lambda^\dagger\tilde{H}_{\lambda+1}, \quad \lambda = 1, 2, 3, 4 \quad (5.35)$$

and alternative factorizations of the Hamiltonians are as follows:

$$\tilde{H}_\lambda = B_\lambda^\dagger B_\lambda + \tilde{\epsilon}_{\lambda,0}, \quad \tilde{H}_{\lambda+1} = B_\lambda B_\lambda^\dagger + \tilde{\epsilon}_{\lambda,0}, \quad \lambda = 1, 2, 3, 4, \quad (5.36)$$

where

$$B_\lambda = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} - W_\lambda - \phi_\lambda\right)$$

and

$$B^{\dagger}_{\lambda} = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - W_{\lambda} - \phi_{\lambda}\right)$$

are two mutually adjoint operators involving $W_{\lambda}(x)$, $\lambda = 1, 2, 3, 4$. The latter plays the role of superpotentials in the hierarchy of H_{λ}, $\lambda = 1, 2, 3, 4$.

Explicitly, in terms of $\phi_1(x)$,

$$\tilde{H}_1 = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (W_1' + \phi_1') - (W_1 + \phi_1)^2\right] + \tilde{\epsilon}_{1,0}. \tag{5.37}$$

This on comparison with H_1 gives

$$\phi_1' + \phi_1^2 + 2W_1\phi_1 = 2\tilde{\delta}_1, \tag{5.38}$$

where we have set $\tilde{\delta}_{\lambda} = \epsilon_{\lambda,0} - \tilde{\epsilon}_{\lambda,0}$, $\lambda = 1, 2, 3, 4$.

For the simplest possibility, i.e. $\epsilon_{1,0} = \tilde{\epsilon}_{1,0} \Rightarrow \tilde{\delta}_1 = 0$ and then ϕ_1 takes the following form for the solution obtained in (4.43) as

$$\phi_1(x) = \frac{\exp(-(\Lambda/a)(x - x_0)^2)}{K_2 + \sqrt{(a\pi/4\Lambda)}\mathrm{erf}\left(\sqrt{(\Lambda/a)}(x - x_0)\right)}, \tag{5.39}$$

where K_2 is the integration constant.

Comparing two different factorizations for $\tilde{H}_2$ read as

$$\begin{aligned}\tilde{H}_2 &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (W_1' + \phi_1') - (W_1 + \phi_1)^2\right] + \tilde{\epsilon}_{1,0} \\ &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (W_2' + \phi_2') - (W_2 + \phi_2)^2\right] + \tilde{\epsilon}_{2,0}\end{aligned} \tag{5.40}$$

resulting in the constraint

$$\phi_2' + 2W_2\phi_2 + \phi_2^2 = -2\phi_1' + 2\tilde{\delta}_2. \tag{5.41}$$

In a similar way comparing two different factorizations for $\tilde{H}_3$ we get

$$\begin{aligned}\tilde{H}_3 &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (W_2' + \phi_2') - (W_2 + \phi_2)^2\right] + \tilde{\epsilon}_{2,0} \\ &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (W_3' + \phi_3') - (W_3 + \phi_3)^2\right] + \tilde{\epsilon}_{3,0}.\end{aligned} \tag{5.42}$$

As such we have

$$\phi_3' + 2W_3\phi_3 + \phi_3^2 = -2(\phi_1 + \phi_2)' + 2\tilde{\delta}_3. \tag{5.43}$$

The same treatment for $\tilde{H}_4$ reads as

$$\begin{aligned}\tilde{H}_4 &= -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (W_3' + \phi_3') - (W_3 + \phi_3)^2\right] + \tilde{\epsilon}_{3,0} \\ &= \tilde{H}_1 - \tilde{\Lambda} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - W_1' - W_1^2\right] + \epsilon_{1,0} - \tilde{\Lambda}\end{aligned} \tag{5.44}$$

and produces

$$(W_1 + W_3)' + W_1^2 - W_3^2 = -2(\phi_1 + \phi_2 + \phi_3)' + 2(\Delta_2 + \Delta_1 + \tilde{\Lambda}). \tag{5.45}$$

A simple calculation yields

$$(\phi_1 + \phi_2 + \phi_3)' = [\Delta_2 + \Delta_1 + \tilde{\Lambda}] - \Lambda\left(\frac{1}{a} + \frac{1}{c}\right) + \Lambda^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)(x - x_0)^2. \tag{5.46}$$

On setting $a = c$, we get the simplest possibility

$$(\phi_1 + \phi_2 + \phi_3)' = K_2,$$
$$K_2 = \left[\Delta_2 + \Delta_1 + \tilde{\Lambda} - \frac{2\Lambda}{a}\right]. \tag{5.47}$$

Using this result (5.43) gives us

$$\phi_3 = \frac{\exp\left((\Lambda/a)(x - x_0)^2\right)}{K_3 - \sqrt{(a\pi/4\Lambda)}\mathrm{erfi}\left(\sqrt{(\Lambda/a)}(x - x_0)\right)},$$
$$K_2 = \tilde{\delta}_3, \tag{5.48}$$

where K_3 is the integration constant and 'erfi' is the imaginary error function having the approximation

$$\mathrm{erfi}(x) \propto -i \pm \frac{1}{\sqrt{\pi}}\frac{1}{x}\exp(x^2), \quad |x| \to \infty,$$
$$i = \sqrt{-1}.$$

As a result

$$\phi_2 = K_2(x - x_0) - (\phi_1 + \phi_3). \tag{5.49}$$

We now list the corresponding ground-state wave functions

$$\psi_{1,0} \propto \exp\left(-\frac{\Lambda}{2a}(x - x_0)^2\right) \times \left[K_2 + \sqrt{\frac{a\pi}{4\Lambda}}\mathrm{erf}\left\{\sqrt{\frac{\Lambda}{a}}(x - x_0)\right\}\right]^{-\sqrt{\Lambda/a}}, \tag{5.50}$$

$$\psi_{3,0} \propto \exp\left(-\frac{\Lambda}{2a}(x - x_0)^2\right) \times \left[K_3 - \sqrt{\frac{a\pi}{4\Lambda}}\mathrm{erfi}\left\{\sqrt{\frac{\Lambda}{a}}(x - x_0)\right\}\right]^{\sqrt{\Lambda/a}}, \tag{5.51}$$

$$\psi_{2,0} \propto \exp\left(-\frac{1}{2}\left(\frac{\Lambda}{b} + K_2\right)(x - x_0)^2\right) \times \left[\frac{K_3 - \sqrt{(a\pi/4\Lambda)}\mathrm{erfi}\{\sqrt{(\Lambda/a)}(x - x_0)\}}{K_2 + \sqrt{(a\pi/4\Lambda)}\mathrm{erf}\{\sqrt{(\Lambda/a)}(x - x_0)\}}\right]^{\sqrt{\Lambda/a}}, \tag{5.52}$$

where

$$K_2 > \sqrt{\frac{a\pi}{4\Lambda}}, \quad \sqrt{\frac{\Lambda}{a}} < \frac{1}{2} \quad \text{and} \quad \tilde{\Lambda} \geq 1 + 3\frac{\Lambda}{a}.$$

The corresponding potentials emerge as

$$\tilde{V}_1 = V_1, \tag{5.53}$$

$$\tilde{V}_2 = V_2 - \frac{1}{2}[\phi_1' - \phi_1^2 - 2W_1\phi_1], \tag{5.54}$$

$$\tilde{V}_3 = V_3 - \frac{1}{2}[\phi_2' - \phi_2^2 - 2W_2\phi_2] - \tilde{\delta}_2 \tag{5.55}$$

and turn out to be complicated expressions.

6. Summary

The traditional method of factorization of the Schrödinger Hamiltonian is induced by a superpotential pertaining to the physical system under consideration. The non-uniqueness of the superpotential results in different isospectral partners of the parent potential in the Hamiltonian. A modified superpotential caused by the non-uniqueness in its basic form gives rise to what is called isospectral deformation. In this article, we have studied such a feature for a family of periodic Hamiltonians subject to a closure condition. The deformation of the superpotential is estimated for a class of Hamiltonians for a few consecutive members in the hierarchy starting from the simplest set up of two Hamiltonians. Some of our new solutions include the results that the non-unique form is controlled by the error function as in the case of three Hamiltonians in the hierarchy and by an error function as well as an imaginary error function as in the case of four Hamiltonian in the hierarchy. A particular matter that deserves mention is that the orthogonal polynomials obtained in the case of hierarchy of three Hamiltonians (period 2) are guided by Hermite polynomials. This emerges as a result of exploiting the non-uniqueness character of the superpotentials W_1 and W_2. As is well known, Hermite polynomials are also related to generalized Laguerre polynomials as $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2)$ and $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{-1/2}(x^2)$. Results for the eigenfunctions in terms of generalized Laguerre polynomials have been reported in [14], (see also [23]). In future we have a plan to find similar connections of orthogonal polynomials with these different hierarchies of periodic Hamiltonians.

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