Pramana – J. Phys. (2016) 87: 95
DOI 10.1007/s12043-016-1292-9

Comparison between the generalized tanh–coth and the (G'/G)-expansion methods for solving NPDEs and NODEs

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MS received 15 July 2015; revised 2 February 2016; accepted 7 March 2016; published online 4 November 2016

Abstract. In this paper, we find exact solutions of some nonlinear evolution equations by using generalized tanh–coth method. Three nonlinear models of physical significance, i.e. the Cahn–Hilliard equation, the Allen–Cahn equation and the steady-state equation with a cubic nonlinearity are considered and their exact solutions are obtained. From the general solutions, other well-known results are also derived. Also in this paper, we shall compare the generalized tanh–coth method and generalized (G'/G)-expansion method to solve partial differential equations (PDEs) and ordinary differential equations (ODEs). Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions have been found. These solutions might play important roles in engineering fields. The generalized tanh–coth method was used to construct periodic wave and solitary wave solutions of nonlinear evolution equations. This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that the generalized tanh–coth method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear problems.

Keywords. Generalized tanh–coth method; generalized (G'/G)-expansion method; Cahn–Hilliard equation; Allen–Cahn equation; steady-state equation.

PACS Nos 02.30.Jr; 02.70.−c; 02.70.Wz

1. Introduction

Considerable attention has been directed towards the study of nonlinear problems in all areas of physics and engineering. Nonlinear partial differential equations are widely used to describe complex phenomena in many fields of applied sciences, such as chemistry, physics and the engineering. Over the last few decades, seeking new travelling wave solutions of nonlinear partial differential equations became a mandatory task, to significantly comprehend and describe complex phenomena. Well-designed mathematical models accurately describing the studied phenomena, can only enhance the chances of achieving analytical solutions, thereby yielding a better physical understanding of the phenomena [1]. The construction of the explicit and exact solutions to nonlinear evolution equations (NEEs), by using different methods, is the goal for many researchers. Due to the complexity of nonlinear system, finding explicit and exact solutions for a real nonlinear physical model equation is often difficult. Fortunately, an enormous number of powerful methods for seeking explicit and exact solutions of NEEs have been proposed and developed. Among them are the inverse scattering transform [2], Hirota's bilinear method [3], homotopy analysis method [4,5], modified homotopy analysis method with Fourier transform [6], variational iteration method [7,8], homotopy perturbation method [9], homotopy perturbation transform method [10], sine–cosine method [11], tanh method [12], tanh–coth method [13], tanh–sech method [14], fractional sub-equation method [15,16], Bäcklund transformation [17], (G'/G)-expansion method [18], exp-function method [19–21], modified simple equation method [22] and so on. Here, we use two effective methods for constructing a range of exact solutions for the following nonlinear partial differential equations that in this article we developed solutions as well. The standard tanh method is a well-known analytical method which was first presented by Malfliet [23] and developed in [23,24]. In

[13], we applied the generalized tanh–coth method for solving some nonlinear partial differential equations. Also in [25], the new approach of generalized (G'/G)-expansion method to obtain exact travelling wave solutions of NLEEs is presented. In this paper, the generalized tanh–coth and generalized (G'/G)-expansion methods are presented to look for travelling wave solutions of nonlinear evolution equations. Heris and Lakestani [26] obtained exact solutions for the integrable sixth-order Drinfeld–Sokolov–Satsuma–Hirota system by the generalized tanh–coth and generalized (G'/G)-expansion methods. Chand and Malik [27] have applied the (G'/G)-expansion method for finding exact solutions of some nonlinear evolution equations. For further information about these methods, refer to [28–32]. To illustrate the basic idea of the generalized tanh–coth method, we consider the Cahn–Hilliard, Allen–Cahn and steady-state equations in the form

$$u_t - \gamma u_x - 6u(u_x)^2 - (3u^2-1)u_{xx} + u_{xxxx} = 0, \quad (1.1)$$

$$u_t - u_{xx} + u^3 - u = 0, \quad (1.2)$$

$$\alpha u''(x) - \beta u(x)(u(x)-m)(u(x)+m) = 0, \quad (1.3)$$

where α, β, γ and m are constants. The Cahn–Hilliard equation was introduced by Cahn and Hilliard in [33] to describe the complicated phase separation and coarsening phenomena in a solid. On the other hand, the Allen–Cahn equation was originally introduced by Allen and Cahn in [34] to describe the motion of antiphase boundaries in crystalline solids. The Allen–Cahn and Cahn–Hilliard equations have been widely used in many complicated moving interface problems in materials science and fluid dynamics through a phase-field approach [35–41].

This article is organized as follows: In §2 and 3, first we briefly give the steps of these methods and apply these methods to solve the nonlinear partial differential equations. In §4, comparison of these methods is presented. In §5, 6 and 7 we examine the Cahn–Hilliard equation, the Allen–Cahn equation and the steady-state equation respectively. Conclusion is given in §8.

2. Basic idea of the generalized tanh–coth method

Step 1. Suppose the given nonlinear partial differential equation for $u(x,t)$ is in the form

$$\mathcal{P}(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \quad (2.1)$$

which can be converted to an ODE

$$\mathcal{Q}(u, -cu', u', u', u', u'', \ldots) = 0, \quad (2.2)$$

where $\xi = x - ct$ is the wave variable. Also, c is a constant to be determined later.

Step 2. We introduce the Riccati equation as follows:

$$\Phi' = r + p\Phi + q\Phi^2, \quad \Phi = \Phi(\xi), \quad \xi = x - ct, \quad (2.3)$$

and this leads to the change of derivatives

$$\frac{d}{d\xi} = (r + p\Phi + q\Phi^2)\frac{d}{d\Phi}, \quad (2.4)$$

$$\frac{d^2}{d\xi^2} = (r + p\Phi + q\Phi^2) \times \left[(p + 2q\Phi)\frac{d}{d\Phi} + (r + p\Phi + q\Phi^2)\frac{d^2}{d\Phi^2}\right], \quad (2.5)$$

$$\frac{d^3}{d\xi^3} = (r + p\Phi + q\Phi^2) \times \left[(6q^2\Phi^2 + 6pq\Phi + 2rq + p^2)\frac{d}{d\Phi} + (6q^2\Phi^3 + 9pq\Phi^2 + 3(p^2+2rq)\Phi + 3rp)\frac{d^2}{d\Phi^2} + (r + p\Phi + q\Phi^2)^2\frac{d^3}{d\Phi^3}\right], \quad (2.6)$$

which admits the use of a finite series of functions of the form:

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} a_k\Phi^k + \sum_{k=1}^{m} b_k\Phi^{-k}, \quad (2.7)$$

where a_k $(k = 0, 2, \ldots, m)$, b_k $(k = 1, 2, \ldots, m)$, p, r and q are constants to be determined later. But, the positive integer m can be determined by considering a homogeneous balance between the highest-order derivatives and nonlinear terms appearing in eq. (2.2). If m is not an integer, then a transformation formula should be used to overcome this difficulty. For the aforementioned method, expansion (2.7) reduces to the standard tanh method [23] for $b_k = 0$, $1 \le k \le m$.

Step 3. Substitute eqs (2.3)–(2.6) into eq. (2.2) with the value of m obtained in Step 2. Collecting the coefficients of Φ^k $(k = 0, 1, 2, \ldots)$, then setting each coefficient to zero, we can get a set of overdetermined equations for a_0, a_i $(i = 1, 2, \ldots, m)$, b_i $(i = 1, 2, \ldots, m)$, p, q and r with the aid of the symbolic computation *Maple* 13.

Step 4. Solve the algebraic equations in Step 3, then substitute $a_0, a_1, b_1, \ldots, a_m, b_m, c$ in eq. (2.7).

We shall consider the following special solutions of the Riccati equation (2.3):

Family 1: For each p, r and $q \neq 0$, eq. (2.3) has the following solutions:

$$\Phi(\xi) = \frac{-p}{2q} + \frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right),$$

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right), \tag{2.8}$$

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2} + C\right),$$

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2} + C\right), \tag{2.9}$$

where $\Delta = p^2 - 4qr$, $\xi = x - ct$ and C is a constant.

Family 2: For $p \neq 0, r \neq 0$ and $q = 0$, eq. (2.3) has the following solution:

$$\Phi(\xi) = C\mathrm{e}^{p\xi} - \frac{r}{p}, \quad \xi = x - ct. \tag{2.10}$$

Family 3: For $r \neq 0$, $p = 0$ and $q \neq 0$, eq. (2.3) has the following solution:

$$\Phi(\xi) = \sqrt{\frac{r}{q}}\tan(\sqrt{rq}\xi + C). \tag{2.11}$$

But, we know

$$\tanh\left(\frac{\xi}{2}\right) = \coth(\xi) - \mathrm{csch}(\xi),$$

$$\coth\left(\frac{\xi}{2}\right) = \coth(\xi) + \mathrm{csch}(\xi). \tag{2.12}$$

Family 4: For $r = 0$, $p \neq 0$ and $q \neq 0$, eq. (2.3) has the following solution:

$$\Phi(\xi) = \frac{p}{-q + C\mathrm{e}^{-p\xi}}. \tag{2.13}$$

3. Description of the new generalized (G'/G)-expansion method

Step 1. Consider the general nonlinear partial differential equation

$$\mathcal{P}(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{3.1}$$

which can be converted to an ODE

$$\mathcal{Q}(u, -cu', u', u', u', u'', \ldots) = 0, \tag{3.2}$$

where $\xi = x - ct$ is the wave variable. Also, c is a constant to be determined later.

Step 2. Suppose the travelling wave solution of (3.2) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} d_i(p + M(\xi))^i + \sum_{i=1}^{N} e_i(p + M(\xi))^{-i}, \tag{3.3}$$

where either d_N or e_N may be zero, but both of them could not be zero at the same time. d_i ($i = 0, 1, 2, \ldots, N$), e_i ($i = 1, 2, \ldots, N$) and p are constants to be determined later and $M(\xi) = (G'/G)$, where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$k_1GG'' - k_2GG' - k_3(G')^2 - k_4G^2 = 0, \tag{3.4}$$

where the prime stands for derivative with respect to ξ; k_1, k_2, k_3 and k_4 are real parameters.

Step 3. Determine the positive integer N, taking the homogeneous balance between the highest-order nonlinear terms and the derivatives of the highest-order come out in (3.2).

Step 4. Substitute eq. (3.4) including eq. (3.3) into eq. (3.2) and the value of N obtained in Step 3, we obtain polynomials in $(p + M(\xi))^N$ ($N = 1, 2, \ldots$) and $(p + M(\xi))^{-N}$ ($N = 1, 2, \ldots$). Then, we collect each coefficient of the resulted polynomials to zero to yield a set of algebraic equations for d_i ($i = 0, 1, 2, \ldots, N$) and e_i ($i = 1, 2, \ldots, N$), p and c.

Step 5. Suppose that the value of the constants d_i ($i = 0, 1, 2, \ldots, N$) and e_i ($i = 1, 2, \ldots, N$), p and c can be found by solving the algebraic equations obtained in Step 4. As the general solution of eq. (3.4) is well known, by inserting the values of d_i, e_i, p and c into eq. (3.2), we obtain more general type and new exact travelling wave solutions of the nonlinear partial differential eq. (3.1). Using the general solution of eq. (3.4), we have the following solutions of eq. (3.3):

Family 1: When $k_2 \neq 0, \alpha = k_1 - k_3, \beta = k_2^2 + 4k_4$ and $\alpha > 0$.

$$M(\xi) = \frac{G'(\xi)}{G(\xi)} = \frac{k_2}{2\alpha} + \frac{\sqrt{\beta}}{2\alpha}\left(\frac{A\sinh\left(\frac{\sqrt{\beta}\xi}{2k_1}\right) + B\cosh\left(\frac{\sqrt{\beta}\xi}{2k_1}\right)}{A\cosh\left(\frac{\sqrt{\beta}\xi}{2k_1}\right) + B\sinh\left(\frac{\sqrt{\beta}\xi}{2k_1}\right)}\right). \tag{3.5}$$

Family 2: When $k_2 \neq 0, \alpha = k_1 - k_3, \beta = k_2^2 + 4k_4$ and $\alpha < 0$.

$$M(\xi) = \frac{G'(\xi)}{G(\xi)} = \frac{k_2}{2\alpha} + \frac{\sqrt{-\beta}}{2\alpha}\left(\frac{-A\sin\left(\frac{\sqrt{-\beta}\xi}{2k_1}\right) + B\cos\left(\frac{\sqrt{-\beta}\xi}{2k_1}\right)}{A\cos\left(\frac{\sqrt{-\beta}\xi}{2k_1}\right) + B\sin\left(\frac{\sqrt{-\beta}\xi}{2k_1}\right)}\right). \quad (3.6)$$

Family 3: When $k_2 \neq 0, \alpha = k_1 - k_3, \beta = k_2^2 + 4k_4$ and $\alpha = 0$.

$$M(\xi) = \frac{G'(\xi)}{G(\xi)} = \frac{k_2}{2\alpha} + \frac{C_2}{C_1 + C_2\xi}. \quad (3.7)$$

Family 4: When $k_2 = 0, \alpha = k_1 - k_3$ and $\gamma = \alpha k_4 > 0$.

$$M(\xi) = \frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{\gamma}}{\alpha}\left(\frac{A\sinh\left(\frac{\sqrt{\gamma}\xi}{k_1}\right) + B\cosh\left(\frac{\sqrt{\gamma}\xi}{k_1}\right)}{A\cosh\left(\frac{\sqrt{\gamma}\xi}{k_1}\right) + B\sinh\left(\frac{\sqrt{\gamma}\xi}{k_1}\right)}\right). \quad (3.8)$$

Family 5: When $k_2 = 0$, $\alpha = k_1 - k_3$ and $\gamma = \alpha k_4 < 0$.

$$M(\xi) = \frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{-\gamma}}{\alpha} \times \left(\frac{-A\sin\left(\frac{\sqrt{-\gamma}\xi}{k_1}\right) + B\cos\left(\frac{\sqrt{-\gamma}\xi}{k_1}\right)}{A\cos\left(\frac{\sqrt{-\gamma}\xi}{k_1}\right) + B\sin\left(\frac{\sqrt{-\gamma}\xi}{k_1}\right)}\right). \quad (3.9)$$

4. Comparison of the generalized tanh–coth and the (G'/G)-expansion methods

We consider (3.4) as follows:

$$k_1 GG'' - k_2 GG' - k_3(G')^2 - k_4 G^2 = 0$$

$$\Longrightarrow \frac{k_1 GG'' - k_2 GG' - k_3(G')^2 - k_4 G^2}{G^2} = 0 \quad (4.1)$$

or

$$k_1\left(\frac{G''}{G}\right) - k_2\left(\frac{G'}{G}\right) - k_3\left(\frac{G'}{G}\right)^2 - k_4 = 0. \quad (4.2)$$

Set $F = G'/G$ and then

$$F' = \frac{G''G - G'^2}{G^2} = \left(\frac{G''}{G}\right) - \left(\frac{G'}{G}\right)^2 = \left(\frac{G''}{G}\right) - F^2. \quad (4.3)$$

Thus, we conclude that relation (4.2) is transformed to

$$k_1 F' + (k_1 - k_3)F^2 - k_2 F - k_4 = 0 \quad (4.4)$$

or

$$F' = \frac{k_4}{k_1} + \frac{k_2}{k_1}F + \frac{k_3 - k_1}{k_1}F^2 = r + pF + qF^2, \quad (4.5)$$

where $r = k_4/k_1$, $p = k_2/k_1$ and $q = (k_3 - k_1)/k_1$. We found that relation (3.3) is the same as relation (2.3). Thus, we obtain that the exact solutions derived by the generalized (G'/G)-expansion are the same as the ones by the generalized tanh–coth methods. Hence we use only the generalized tanh–coth method.

5. The Cahn–Hilliard equation

We consider the Cahn–Hilliard equation in the form [38,41]

$$u_t - \gamma u_x - 6u(u_x)^2 - (3u^2 - 1)u_{xx} + u_{xxxx} = 0. \quad (5.1)$$

Using the wave variable $\xi = x - ct$, we get,

$$-(c + \gamma)u - 3u^2u' + u' + u''' = 0 \quad (5.2)$$

which is obtained by integrating and neglecting the constant of integration. In order to determine the value of m, we balance the linear term of the highest-order u''' with the highest-order nonlinear term u^2u' in eq. (5.2) and by using eq. (2.7) we obtain $m = 1$. We can suppose that the solutions of eq. (5.1) is of the form

$$u(\xi) = a_0 + a_1\Phi + \frac{b_1}{\Phi}. \quad (5.3)$$

Using Family 1–4 in §2, substituting (5.3) into eq. (5.2) and collecting all terms with the same order of $\Phi(\xi)$ together, we can obtain a set of algebraic equations for a_0, a_1, b_1, p, q, r and c as follows:

Coefficients of Φ^k:

$$
\begin{cases}
\Phi^0:\ 6b_1r^3 - 3b_1^3r = 0,\\
\Phi^1:\ 12b_1r^2p - 6a_0b_1^2r - 3b_1^3p = 0,\\
\Phi^2:\ -3a_1b_1^2r + 8r^2qb_1 + 7rp^2b_1 + rb_1 - 3b_1^3q\\
\quad -3a_0^2rb_1 - 6a_0b_1^2p = 0,\\
\Phi^3:\ 8rpqb_1 + p^3b_1 + Bb_1 - 6a_0b_1^2q - 3a_1b_1^2p\\
\quad -3a_0^2pb_1 + cb_1 + \gamma b_1 = 0,\\
\Phi^4:\ \gamma a_0 + ca_0 - 2r^2qa_1 + 3a_0^2ra_1 - rp^2a_1\\
\quad +2q^2rb_1 - ra_1 + qp^2b_1 - 3a_1b_1^2q + qb_1\\
\quad +3a_1^2rb_1 - 3a_0^2qb_1 = 0,\\
\Phi^5:\ 3a_0^2pa_1 - 8rpqa_1 + ca_1 + 6a_0a_1^2r - p^3a_1\\
\quad -pa_1 + \gamma a_1 + 3a_1^2pb_1 = 0,\\
\Phi^6:\ -7qp^2a_1 + 3a_1^2qb_1 + 3a_1^3r + 3a_0^2qa_1\\
\quad +6a_0a_1^2p - 8q^2ra_1 - qa_1 = 0,\\
\Phi^7:\ -12pq^2a_1 + 3a_1^3p + 6a_0a_1^2q = 0,\\
\Phi^8:\ 3a_1^3q - 6q^3a_1 = 0.
\end{cases}
\tag{5.4}
$$

Solving eqs (5.4), we have the following sets of coefficients for the solutions of (5.3)

Set I

$$a_0 = \pm\frac{p}{\sqrt{2}},\quad a_1 = 0,\quad b_1 = b_1,\quad c = -\gamma,$$

$$\Delta = p^2 \mp (p^2-2),\quad u(\xi) = \pm\frac{p}{\sqrt{2}} + b_1\Phi^{-1}. \tag{5.5}$$

By using Family 1 we have

$$u_1(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{(p^2-2)\sqrt{2}} \pm\frac{\sqrt{-2p^2 \pm 2(p^2-2)}}{p^2-2} \times\tan\left(\frac{\sqrt{-p^2\pm(p^2-2)}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.6}$$

$$u_2(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{(p^2-2)\sqrt{2}} \mp\frac{\sqrt{-2p^2 \pm 2(p^2-2)}}{p^2-2} \times\cot\left(\frac{\sqrt{-p^2\pm(p^2-2)}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.7}$$

$$u_3(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{(p^2-2)\sqrt{2}} \mp\frac{\sqrt{2p^2\mp2(p^2-2)}}{p^2-2} \times\tanh\left(\frac{\sqrt{p^2\mp(p^2-2)}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.8}$$

$$u_4(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{(p^2-2)\sqrt{2}} \mp\frac{\sqrt{2p^2\mp2(p^2-2)}}{p^2-2} \times\coth\left(\frac{\sqrt{p^2\mp(p^2-2)}(x+\gamma t)}{2}+C\right)\right]^{-1}. \tag{5.9}$$

Set II

$$a_0 = \pm\frac{p}{\sqrt{2}},\quad a_1 = a_1,\quad b_1 = 0,\quad c = -\gamma,$$

$$\Delta = p^2 \mp (p^2-2),\quad u(\xi) = \pm\frac{p}{\sqrt{2}} + a_1\Phi. \tag{5.10}$$

By using Family 1 we have

$$u_5(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \pm \frac{\sqrt{-p^2\pm(p^2-2)}}{\sqrt{2}} \times\tan\left(\frac{\sqrt{-p^2\pm(p^2-2)}(x+\gamma t)}{2}+C\right)\right], \tag{5.11}$$

$$u_6(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{-p^2\pm(p^2-2)}}{\sqrt{2}} \times\cot\left(\frac{\sqrt{-p^2\pm(p^2-2)}(x+\gamma t)}{2}+C\right)\right], \tag{5.12}$$

$$u_7(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2\mp(p^2-2)}}{\sqrt{2}} \times\tanh\left(\frac{\sqrt{p^2\mp(p^2-2)}(x+\gamma t)}{2}+C\right)\right], \tag{5.13}$$

$$u_8(x,t) = \pm\frac{p}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2\mp(p^2-2)}}{\sqrt{2}} \times\coth\left(\frac{\sqrt{p^2\mp(p^2-2)}(x+\gamma t)}{2}+C\right)\right]. \tag{5.14}$$

Set III

$$a_0 = \pm\frac{p}{\sqrt{2}},\quad a_1 = a_1,\quad b_1 = -\frac{p^2-2}{4a_1},$$

$$c = -\gamma,\quad \Delta = p^2 \mp (p^2-2),$$

$$u(\xi) = \pm\frac{p}{\sqrt{2}} + a_1\Phi - \frac{p^2-2}{4a_1}\Phi^{-1}, \tag{5.15}$$

By using Family 1 we have

$$u_9(x,t)=\pm\frac{p}{\sqrt{2}}+\left[\mp\frac{p}{\sqrt{2}}\pm\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\tan\left(\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]-\frac{p^2-2}{4}\left[\mp\frac{p}{\sqrt{2}}\pm\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\tan\left(\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.16}$$

$$u_{10}(x,t)=\pm\frac{p}{\sqrt{2}}+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\cot\left(\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]-\frac{p^2-2}{4}\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\cot\left(\frac{\sqrt{-p^2\mp\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.17}$$

$$u_{11}(x,t)=\pm\frac{p}{\sqrt{2}}+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\tanh\left(\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]-\frac{p^2-2}{4}\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\tanh\left(\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]^{-1}, \tag{5.18}$$

$$u_{12}(x,t)=\pm\frac{p}{\sqrt{2}}+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\coth\left(\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]-\frac{p^2-2}{4}\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}}{\sqrt{2}}\times\coth\left(\frac{\sqrt{p^2\pm\frac{(p^2-2)}{2}}(x+\gamma t)}{2}+C\right)\right]^{-1}. \tag{5.19}$$

Set IV

$$a_0=\pm1,\quad a_1=0,\quad b_1=b_1,\quad c=-\gamma,$$
$$u(\xi)=\pm1+b_1\Phi^{-1}. \tag{5.20}$$

By using Family 2 we have

$$u_{13}(x,t)=\pm1+b_1\left(C\exp[\pm\sqrt{2}(x+\gamma t)]-\frac{b_1}{2}\right)^{-1}. \tag{5.21}$$

Set V

$$a_0=0,\quad a_1=a_1,\quad b_1=\frac{1}{4a_1},\quad c=-\gamma,$$
$$u(\xi)=a_1\Phi+\frac{1}{4a_1}\Phi^{-1}. \tag{5.22}$$

By using Family 3 we have

$$u_{14}(x,t)=\frac{1+16a_1^4\left[i\tan\left(\frac{i}{2\sqrt{2}}(x+\gamma t)+C\right)\right]^2}{8a_1^2\left[i\tan\left(\frac{i}{2\sqrt{2}}(x+\gamma t)+C\right)\right]}=\frac{1+16a_1^4\tanh^2\left(\frac{1}{2\sqrt{2}}(x+\gamma t)+C\right)}{-8a_1^2\tanh\left(\frac{1}{2\sqrt{2}}(x+\gamma t)+C\right)}. \tag{5.23}$$

Set VI

$$a_0=\pm1,\quad a_1=a_1,\quad b_1=0,\quad c=-\gamma,$$
$$u(\xi)=\pm1+a_1\Phi. \tag{5.24}$$

By using Family 4 we have

$$u_{15}(x,t)=\pm1+a_1\times\left(\frac{\pm\sqrt{2}}{\mp\frac{1}{\sqrt{2}}a_1+C\exp[\mp\sqrt{2}(x+\gamma t)]}\right). \tag{5.25}$$

6. The Allen–Cahn equation

Now we consider the Allen–Cahn equation in the form [39,41]

$$u_t - u_{xx} + u^3 - u = 0. \tag{6.1}$$

Using the wave variable $\xi = x - ct$, we get

$$cu' + u'' - u^3 + u = 0. \tag{6.2}$$

In order to determine m, we balance the linear term of the highest-order u'' with the highest-order nonlinear term u^3 in eq. (6.2) and by using eq. (2.7) we obtain $m = 1$. We can suppose that the solutions of eq. (6.1) is of the form

$$u(\xi) = a_0 + a_1\Phi + \frac{b_1}{\Phi}. \tag{6.3}$$

Using Family 1–4 in §2 and substituting (6.3) into eq. (6.2) and collecting all terms with the same order of $\Phi(\xi)$ together, we can obtain a set of algebraic equations for a_0, a_1, b_1, p, q, r and c as follows:

Coefficients of Φ^k:

$$\begin{cases} \Phi^0: 2b_1r^2 - b_1^3 = 0, \\ \Phi^1: -crb_1 - 3a_0b_1^2 + 3rpb_1 = 0, \\ \Phi^2: -3a_1b_1^2 + p^2b_1 - 3a_0^2b_1 - cpb_1 \\ \quad +2qb_1r + b_1 = 0, \\ \Phi^3: cra_1 - a_0^3 + rpa_1 - 6a_0a_1b_1 + a_0 \\ \quad -cqb_1 + pqb_1 = 0, \\ \Phi^4: 2rqa_1 + p^2a_1 + a_1 + cpa_1 - 3a_1^2b_1 \\ \quad -3a_0^2a_1 = 0, \\ \Phi^5: 3pqa_1 - 3a_0a_1^2 + cqa_1 = 0, \\ \Phi^6: -a_1^3 + 2q^2a_1 = 0. \end{cases} \tag{6.4}$$

Solving eqs (6.4), we have the following sets of coefficients for the solutions of (6.3):

Set I

$$a_0 = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right), \quad a_1 = 0, \quad b_1 = b_1,$$

$$c = \pm\frac{3}{\sqrt{2}}, \quad \Delta = p^2 \mp \frac{(2p^2-1)}{2},$$

$$u(\xi) = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right) + b_1\Phi^{-1}. \tag{6.5}$$

By using Family 1 we have

$$u_1(x,t) = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right) + \left[\mp\frac{2\sqrt{2}p}{2p^2-1} \pm \frac{\sqrt{-8p^2 \pm 4(2p^2-1)}}{2p^2-1} \times \tan\left(\frac{\sqrt{-p^2 \pm \frac{(2p^2-1)}{2}}\left(x \mp \frac{3}{\sqrt{2}}t\right)}{2} + C\right)\right]^{-1}, \tag{6.6}$$

$$u_2(x,t) = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right) + \left[\mp\frac{2\sqrt{2}p}{2p^2-1} \mp \frac{\sqrt{-8p^2 \pm 4(2p^2-1)}}{2p^2-1} \times \cot\left(\frac{\sqrt{-p^2 \pm \frac{(2p^2-1)}{2}}\left(x \mp \frac{3}{\sqrt{2}}t\right)}{2} + C\right)\right]^{-1}, \tag{6.7}$$

$$u_3(x,t) = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right) + \left[\mp\frac{2\sqrt{2}p}{2p^2-1} \mp \frac{\sqrt{8p^2 \mp 4(2p^2-1)}}{2p^2-1} \times \tanh\left(\frac{\sqrt{p^2 \mp \frac{(2p^2-1)}{2}}\left(x \mp \frac{3}{\sqrt{2}}t\right)}{2} + C\right)\right]^{-1}, \tag{6.8}$$

$$u_4(x,t) = \pm\frac{1}{\sqrt{2}}\left(p \mp \frac{1}{\sqrt{2}}\right) + \left[\mp\frac{2\sqrt{2}p}{2p^2-1} \mp \frac{\sqrt{8p^2 \mp 4(2p^2-1)}}{2p^2-1} \times \coth\left(\frac{\sqrt{p^2 \mp \frac{(2p^2-1)}{2}}\left(x \mp \frac{3}{\sqrt{2}}t\right)}{2} + C\right)\right]^{-1}. \tag{6.9}$$

Set II

$$a_0 = \pm\frac{1}{\sqrt{2}}\left(p \pm \frac{1}{\sqrt{2}}\right), \quad a_1 = a_1, \quad b_1 = 0,$$

$$c = \pm\frac{3}{\sqrt{2}}, \quad \Delta = p^2 \mp \frac{(2p^2-1)}{2},$$

$$u(\xi) = \pm\frac{1}{\sqrt{2}}\left(p \pm \frac{1}{\sqrt{2}}\right) + a_1\Phi. \tag{6.10}$$

By using Family 1 we have

$$u_5(x,t)=\pm\frac{1}{\sqrt{2}}\left(p\pm\frac{1}{\sqrt{2}}\right)+\left[\mp\frac{p}{\sqrt{2}}\pm\frac{\sqrt{-p^2\pm\frac{(2p^2-1)}{2}}}{\sqrt{2}}\times\tan\left(\frac{\sqrt{-p^2\pm\frac{(2p^2-1)}{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)}{2}+C\right)\right],\tag{6.11}$$

$$u_6(x,t)=\pm\frac{1}{\sqrt{2}}\left(p\pm\frac{1}{\sqrt{2}}\right)+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{-p^2\pm\frac{(2p^2-1)}{2}}}{\sqrt{2}}\times\cot\left(\frac{\sqrt{-p^2\pm\frac{(2p^2-1)}{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)}{2}+C\right)\right],\tag{6.12}$$

$$u_7(x,t)=\pm\frac{1}{\sqrt{2}}\left(p\pm\frac{1}{\sqrt{2}}\right)+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\mp\frac{(2p^2-1)}{2}}}{\sqrt{2}}\times\tanh\left(\frac{\sqrt{p^2\mp\frac{(2p^2-1)}{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)}{2}+C\right)\right],\tag{6.13}$$

$$u_8(x,t)=\pm\frac{1}{\sqrt{2}}\left(p\pm\frac{1}{\sqrt{2}}\right)+\left[\mp\frac{p}{\sqrt{2}}\mp\frac{\sqrt{p^2\mp\frac{(2p^2-1)}{2}}}{\sqrt{2}}\times\coth\left(\frac{\sqrt{p^2\mp\frac{(2p^2-1)}{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)}{2}+C\right)\right].\tag{6.14}$$

Set III

$$a_0=\pm\frac{1}{2},\quad a_1=a_1,\quad b_1=0,\quad c=\pm\frac{3}{\sqrt{2}},$$
$$r=\mp\frac{1}{4\sqrt{2}a_1},\quad p=0,\quad q=\pm\frac{1}{\sqrt{2}}a_1,$$
$$u(\xi)=\pm\frac{1}{2}+a_1\Phi.\tag{6.15}$$

By using Family 3 we have

$$\begin{aligned}u_9(x,t)&=\pm\frac{1}{2}+\frac{i}{2}\tan\left[\frac{1}{2\sqrt{2}}i\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right]\\&=\pm\frac{1}{2}-\frac{1}{2}\tanh\left[\frac{1}{2\sqrt{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right].\end{aligned}\tag{6.16}$$

Set IV

$$a_0=-\frac{1}{2},\quad a_1=a_1,\quad b_1=\frac{1}{16a_1},\quad c=\mp\frac{3}{\sqrt{2}},$$
$$r=\mp\frac{1}{16\sqrt{2}a_1},\quad p=0,\quad q=\pm\frac{a_1}{\sqrt{2}},$$
$$u(\xi)=-\frac{1}{2}+a_1\Phi+\frac{1}{16a_1}\Phi^{-1}.\tag{6.17}$$

By using Family 3 we have

$$\begin{aligned}u_{10}(x,t)&=-\frac{1}{2}+\frac{i}{4}\tan\left[\frac{1}{4\sqrt{2}}i\left(x\pm\frac{3}{\sqrt{2}}t\right)+C\right]\\&\quad-\frac{i}{4}\cot\left[\frac{1}{4\sqrt{2}}i\left(x\pm\frac{3}{\sqrt{2}}t\right)+C\right]\\&=-\frac{1}{2}-\frac{1}{4}\tanh\left[\frac{1}{4\sqrt{2}}\left(x\pm\frac{3}{\sqrt{2}}t\right)+C\right]\\&\quad-\frac{1}{4}\coth\left[\frac{1}{4\sqrt{2}}\left(x\pm\frac{3}{\sqrt{2}}t\right)+C\right].\end{aligned}\tag{6.18}$$

Set V

$$a_0=\frac{1}{2},\quad a_1=a_1,\quad b_1=\frac{1}{16a_1},\quad c=\pm\frac{3}{\sqrt{2}},$$
$$r=\mp\frac{1}{16\sqrt{2}a_1},\quad p=0,\quad q=\pm\frac{a_1}{\sqrt{2}},$$
$$u(\xi)=-\frac{1}{2}+a_1\Phi+\frac{1}{16a_1}\Phi^{-1}.\tag{6.19}$$

By using Family 3 we have

$$\begin{aligned} u_{11}(x,t) &= \frac{1}{2}+\frac{i}{4}\tan\left[\frac{1}{4\sqrt{2}}i\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right] \\ &\quad -\frac{i}{4}\cot\left[\frac{1}{4\sqrt{2}}i\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right] \\ &= \frac{1}{2}-\frac{1}{4}\tanh\left[\frac{1}{4\sqrt{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right] \\ &\quad -\frac{1}{4}\coth\left[\frac{1}{4\sqrt{2}}\left(x\mp\frac{3}{\sqrt{2}}t\right)+C\right]. \end{aligned} \tag{6.20}$$

7. The steady-state equation

At last, we consider the steady-state equation as follows [40,41]:

$$\alpha u''(x)-\beta u(x)(u(x)-m)(u(x)+m)=0, \tag{7.1}$$

where α, β and m are constants. By making the transformation

$$v(x)=m^{-1}(u(\varepsilon x)+m), \tag{7.2}$$

where $\varepsilon=\sqrt{\alpha/\beta m^2}$, eq. (7.1) becomes

$$v''(x)+v(x)+v^3(x)=0. \tag{7.3}$$

By balancing term v'' with term v^3 in eq. (7.3) we obtain $m=1$. We can suppose that the solutions of eq. (7.1) is of the form

$$v(x)=a_0+a_1\Phi+\frac{b_1}{\Phi}. \tag{7.4}$$

Using Family 1–4 in §2 and substituting (7.4) into eq. (7.3) and collecting all terms with the same order of $\Phi(\xi)$ together, we can obtain a set of algebraic equations for a_0, a_1, b_1, p, q, r and c as follows:

Coefficients of Φ^k:

$$\begin{cases} \Phi^0: -b_1^3+2b_1r^2=0, \\ \Phi^1: 3b_1^2+3rpb_1-3a_0b_1^2=0, \\ \Phi^2: -3a_0^2b_1+6a_0b_1+p^2b_1+2qb_1r \\ \qquad -2b_1-3a_1b_1^2=0, \\ \Phi^3: rpa_1-6a_0a_1b_1-2a_0+pqb_1-a_0^3 \\ \qquad +3a_0^2+6a_1b_1=0, \\ \Phi^4: -3a_0^2a_1-2a_1+p^2a_1+2rqa_1 \\ \qquad -3a_1^2b_1+6a_0a_1=0, \\ \Phi^5: 3pqa_1-3a_0a_1^2+3a_1^2=0, \\ \Phi^6: -a_1^3+2q^2a_1=0. \end{cases} \tag{7.5}$$

Solving eqs (7.5), we have the following sets of coefficients for the solutions of (7.4) as given below:

Set I

$$\begin{aligned} &a_0=1\pm\frac{1}{\sqrt{2}}p,\quad a_1=0,\quad b_1=b_1, \\ &\Delta=p^2\mp(p^2-2),\quad v(x)=1\pm\frac{1}{\sqrt{2}}p+b_1\Phi^{-1}. \end{aligned} \tag{7.6}$$

By using Family 1 we have

$$v_1(x)=1\pm\frac{1}{\sqrt{2}}p+\left[\mp\frac{\sqrt{2}p}{p^2-2}\pm\frac{\sqrt{-2p^2\pm2(p^2-2)}}{p^2-2}\times\tan\left(\frac{\sqrt{-p^2\pm(p^2-2)}}{2}x+C\right)\right]^{-1},$$

$$u_1(x)=\pm\frac{mp}{\sqrt{2}}+m\left[\mp\frac{\sqrt{2}p}{p^2-2}\pm\frac{\sqrt{-2p^2\pm2(p^2-2)}}{p^2-2}\times\tan\left(\frac{\sqrt{-p^2\pm(p^2-2)}}{2}\times m\sqrt{\frac{\beta}{\alpha}}x+C\right)\right]^{-1}, \tag{7.7}$$

$$v_2(x)=1\pm\frac{1}{\sqrt{2}}p+\left[\mp\frac{\sqrt{2}p}{p^2-2}\mp\frac{\sqrt{-2p^2\pm2(p^2-2)}}{p^2-2}\times\cot\left(\frac{\sqrt{-p^2\pm(p^2-2)}}{2}x+C\right)\right]^{-1},$$

$$u_2(x)=\pm\frac{mp}{\sqrt{2}}+m\left[\mp\frac{\sqrt{2}p}{p^2-2}\mp\frac{\sqrt{-2p^2\pm2(p^2-2)}}{p^2-2}\times\cot\left(\frac{\sqrt{-p^2\pm(p^2-2)}}{2}m\sqrt{\frac{\beta}{\alpha}}x+C\right)\right]^{-1}, \tag{7.8}$$

$$v_3(x) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{\sqrt{2}p}{p^2-2} \mp\frac{\sqrt{2p^2 \mp 2(p^2-2)}}{p^2-2} \times \tanh\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}x + C\right)\right]^{-1},$$

$$u_3(x) = \pm\frac{mp}{\sqrt{2}} + m\left[\mp\frac{\sqrt{2}p}{p^2-2} \mp\frac{\sqrt{2p^2 \mp 2(p^2-2)}}{p^2-2} \times \tanh\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}m \times \sqrt{\frac{\beta}{\alpha}}x + C\right)\right]^{-1}, \quad (7.9)$$

$$v_4(x) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{\sqrt{2}p}{p^2-2} \mp\frac{\sqrt{2p^2 \mp 2(p^2-2)}}{p^2-2} \times \coth\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}x + C\right)\right]^{-1},$$

$$u_4(x) = \pm\frac{mp}{\sqrt{2}} + m\left[\mp\frac{\sqrt{2}p}{p^2-2} \mp\frac{\sqrt{2p^2 \mp 2(p^2-2)}}{p^2-2} \times \coth\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}m \times\sqrt{\frac{\beta}{\alpha}}x + C\right)\right]^{-1}. \quad (7.10)$$

Set II

$$a_0 = 1 \pm \frac{1}{\sqrt{2}}p, \quad a_1 = a_1, \quad b_1 = 0,$$
$$\Delta = p^2 \mp (p^2-2), \quad v(x) = 1 \pm \frac{1}{\sqrt{2}}p + a_1\Phi. \quad (7.11)$$

By using Family 1 we have

$$v_5(x,t) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{p}{\sqrt{2}} \pm\frac{\sqrt{-p^2 \pm (p^2-2)}}{\sqrt{2}} \times \tan\left(\frac{\sqrt{-p^2 \pm (p^2-2)}x}{2} + C\right)\right],$$

$$u_5(x,t) = \pm\frac{mp}{\sqrt{2}} + m\left[\mp\frac{p}{\sqrt{2}} \pm \frac{\sqrt{-p^2 \pm (p^2-2)}}{\sqrt{2}} \times \tan\left(\frac{\sqrt{-p^2 \pm (p^2-2)}}{2}m\sqrt{\frac{\beta}{\alpha}}x + C\right)\right], \quad (7.12)$$

$$v_6(x,t) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{-p^2 \pm (p^2-2)}}{\sqrt{2}} \times \cot\left(\frac{\sqrt{-p^2 \pm (p^2-2)}x}{2} + C\right)\right],$$

$$u_6(x,t) = \pm\frac{mp}{\sqrt{2}} + m\left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{-p^2 \pm (p^2-2)}}{\sqrt{2}} \times \cot\left(\frac{\sqrt{-p^2 \pm (p^2-2)}}{2}m \times \sqrt{\frac{\beta}{\alpha}}x + C\right)\right], \quad (7.13)$$

$$v_7(x,t) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2 \mp (p^2-2)}}{\sqrt{2}} \times \tanh\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}x + C\right)\right],$$

$$u_7(x,t) = \pm\frac{mp}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2 \mp (p^2-2)}}{\sqrt{2}} \times\tanh\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}m\sqrt{\frac{\beta}{\alpha}}x + C\right)\right], \quad (7.14)$$

$$v_8(x) = 1 \pm \frac{1}{\sqrt{2}}p + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2 \mp (p^2-2)}}{\sqrt{2}} \times \coth\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}x + C\right)\right],$$

$$u_8(x) = \pm\frac{mp}{\sqrt{2}} + \left[\mp\frac{p}{\sqrt{2}} \mp \frac{\sqrt{p^2 \mp (p^2-2)}}{\sqrt{2}} \times \coth\left(\frac{\sqrt{p^2 \mp (p^2-2)}}{2}m\sqrt{\frac{\beta}{\alpha}}x + C\right)\right]. \quad (7.15)$$

Set III

$$a_0 = 1, \quad a_1 = \pm\sqrt{2}q, \quad b_1 = \pm\frac{\sqrt{2}}{8q}, \quad q = q,$$

$$r = -\frac{1}{8q}, \quad p = 0, \quad u(x) = 1 \pm \sqrt{2}q\Phi \pm \frac{\sqrt{2}}{8q}\Phi^{-1}. \quad (7.16)$$

By using Family 3 we have

$$\begin{aligned} v_9(x) &= 1 \pm \frac{qi}{2}\tan\left[\frac{1}{2\sqrt{2}}ix + C\right] \mp\frac{i}{2q}\cot\left[\frac{1}{2\sqrt{2}}ix + C\right] \\ &= 1 \mp \frac{q}{2}\tanh\left[\frac{1}{2\sqrt{2}}x + C\right] \mp\frac{1}{2q}\coth\left[\frac{1}{2\sqrt{2}}x + C\right], \\ u_9(x) &= \pm\frac{mqi}{2}\tan\left[\frac{1}{2\sqrt{2}}im\sqrt{\frac{\beta}{\alpha}}x + C\right] \mp\frac{i}{2q}\cot\left[\frac{1}{2\sqrt{2}}im\sqrt{\frac{\beta}{\alpha}}x + C\right] \\ &= \mp\frac{mq}{2}\tanh\left[\frac{1}{2\sqrt{2}}m\sqrt{\frac{\beta}{\alpha}}x + C\right] \mp\frac{m}{2q}\coth\left[\frac{1}{2\sqrt{2}}m\sqrt{\frac{\beta}{\alpha}}x + C\right]. \end{aligned} \quad (7.17)$$

Set IV

$$a_0 = 1, \quad a_1 = \pm\sqrt{2}q, \quad b_1 = \pm\frac{\sqrt{2}}{8q}, \quad q = q,$$

$$r = \frac{1}{4q}, \quad p = 0, \quad u(x) = 1 \pm \sqrt{2}q\Phi \pm \frac{\sqrt{2}}{8q}\Phi^{-1}. \quad (7.18)$$

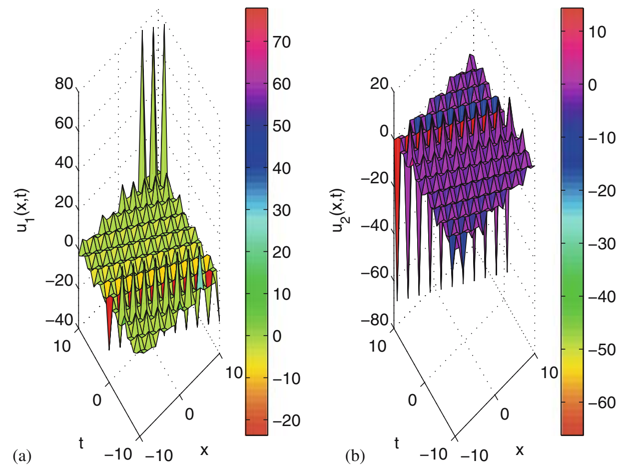

Figure 1. Graphs of Cahn–Hilliard equation (**a**) u_{1-} and (**b**) u_{2-} real values of (5.6) and (5.7) are demonstrated at $p = 1/2$ and $\gamma = 2$, when $-10 < x < 10$, $-10 < t < 10$.

By using Family 3 we have

$$\begin{aligned} v_{10}(x) &= 1 \pm \frac{q}{\sqrt{2}}\tan\left[\frac{1}{2}x + C\right] \mp\frac{\sqrt{2}}{4q}\cot\left[\frac{1}{2}x + C\right], \\ u_{10}(x) &= \pm\frac{mq}{\sqrt{2}}\tan\left[\frac{1}{2}m\sqrt{\frac{\beta}{\alpha}}x + C\right] \mp\frac{\sqrt{2}m}{4q}\cot\left[\frac{1}{2}m\sqrt{\frac{\beta}{\alpha}}x + C\right]. \end{aligned} \quad (7.19)$$

Please note that all the obtained results have been checked with *Maple* 13 by putting them back into the original equation and found to be correct.

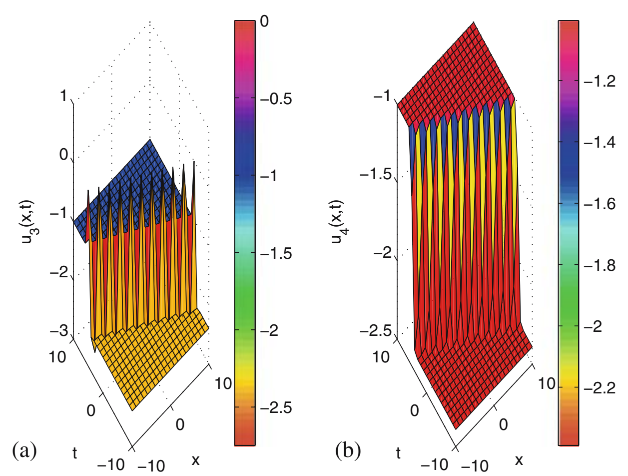

Figure 2. Graphs of Cahn–Hilliard equation (**a**) u_{3-} and (**b**) u_{4-} real values of (5.8) and (5.9) are demonstrated at $p = 2$ and $\gamma = 2$, when $-10 < x < 10$, $-10 < t < 10$.

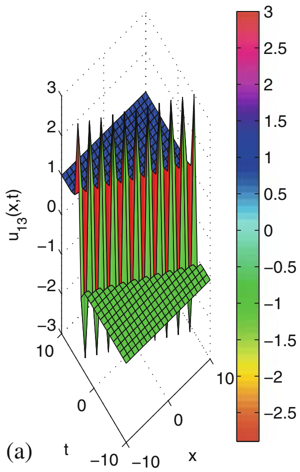

Figure 3. Graphs of Cahn–Hilliard equation (**a**) u_{13+} and (**b**) u_{15+} real values of (5.21) and (5.25) are demonstrated at $a_1 = 2$, $b_1 = 2$, $C = 2$ and $\gamma = 2$, when $-10 < x < 10$, $-10 < t < 10$.

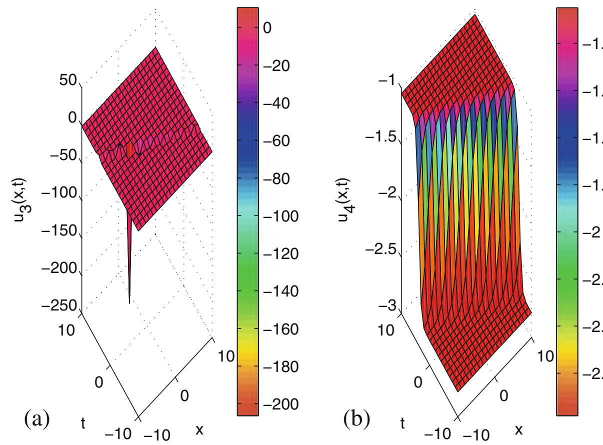

Figure 5. Graphs of Allen–Cahn equation (**a**) u_{3-} and (**b**) u_{4-} real values of (6.8) and (6.9) are demonstrated at $p = 1$, when $-10 < x < 10$ and $-10 < t < 10$.

Physical interpretations of the solutions

Remark 1. In figures 1–4, we plot three-dimensional graphics of real values of the Cahn–Hilliard equation and the Allen–Cahn equation respectively, which denote the dynamics of solutions with appropriate parametric selections. Figures 1 and 4 represent the exact periodic travelling wave solutions. The other figures are ignored for simplicity.

Remark 2. Figures 2 and 5 represent the exact soliton solutions of the Cahn–Hilliard equation and the Allen–Cahn equation respectively. Solitons are special kinds of solitary waves. Solitons have the remarkable property of keeping their identity upon interacting with other solitons. Soliton solutions have particle-like structures, for example, magnetic monopoles, and extended structures, like, domain walls and cosmic strings, that have implications in cosmology of the early Universe. The other figures are ignored for simplicity.

Remark 3. Figure 3 represents the singular kink-type travelling wave solution of the Cahn–Hilliard equation. For convenience the other figures are omitted.

Remark 4. Figures 6–8 show the shape of two-dimensional graphics of real values of the steady-state equation. The other figures are ignored for simplicity.

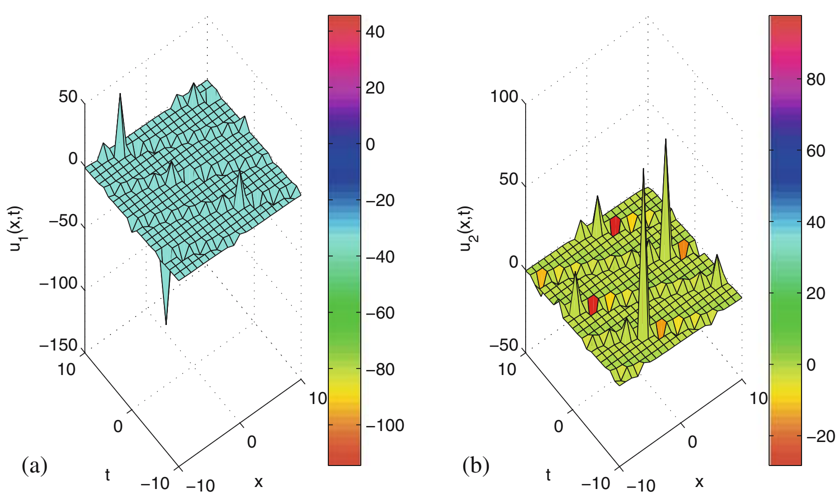

Figure 4. Graphs of Allen–Cahn equation (**a**) u_{1-} and (**b**) u_{2-} real values of (6.6) and (6.7) are demonstrated at $p = 1/3$, when $-10 < x < 10$ and $-10 < t < 10$.

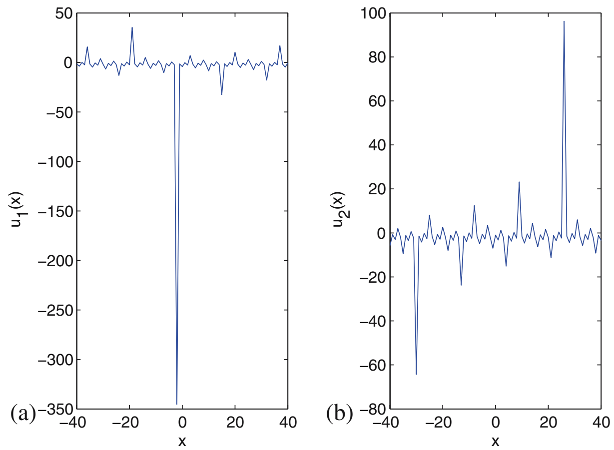

Figure 6. Graphs of steady-state equation (**a**) u_{1-} and (**b**) u_{2-} real values of (7.7) and (7.8) are demonstrated at $p = 2/3$, $m = 2$, $\alpha = 2$ and $\beta = 3$, when $-40 < x < 40$.

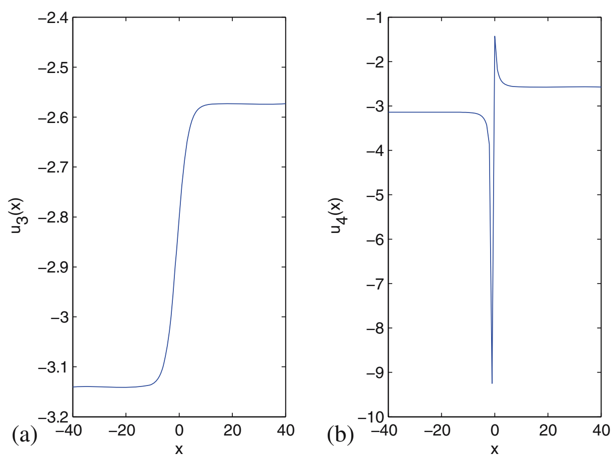

Figure 7. Graphs of steady-state equation (**a**) u_{3-} and (**b**) u_{4-} real values of (7.9) and (7.10) are demonstrated at $p = 1.01$, $m = 2$, $\alpha = 2$ and $\beta = 3$, when $-40 < x < 40$.

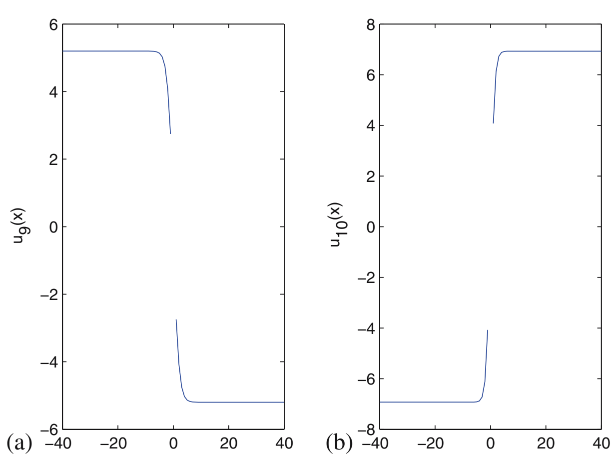

Figure 8. Graphs of steady-state equation (**a**) u_9 and (**b**) u_{10} real values of (7.17) and (7.19) are demonstrated at $q = 5$, $m = 2$, $\alpha = 2$ and $\beta = 4$, when $-40 < x < 40$.

8. Conclusion

In this article, we obtained exact solutions for the Cahn–Hilliard equation, Allen–Cahn equation and steady-state equation by using the generalized tanh–coth method. Generalized tanh–coth method is a useful method for finding travelling wave solutions of nonlinear evolution equations. In §4, we found that the generalized tanh–coth method is the same as the generalized (G'/G)-expansion method [25]. Thus, we obtained exact solutions derived by generalized (G'/G)-expansion which are the same as the ones by the generalized tanh–coth method. Hence we used only the generalized tanh–coth method. Also, new results are formally developed in this article, see [41]. It can be concluded that this method is a very powerful and efficient method to find exact solutions for wide classes of problems. The crucial advantage of the new approach against the generalized and improved (G'/G)-expansion method is that the method provides more general and abundant exact travelling wave solutions with real parameters. The exact solutions of PDEs disclose the internal mechanism of the complex physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

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