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The Klein–Gordon–Zakharov equations with the positive fractional power terms and their exact solutions

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Abstract. In this paper, the famous Klein–Gordon–Zakharov (KGZ) equations are first generalized, and the new special types of KGZ equations with the positive fractional power terms (gKGZE) are presented. In order to derive exact solutions of the new special gKGZE, subsidiary higher-order ordinary differential equations (sub-ODEs) with the positive fractional power terms are introduced, and with the aid of the sub-ODE, exact solutions of four special types of the gKGZE are derived, which are the bell-type solitary wave solution, the algebraic solitary wave solution, the kink-type solitary wave solution and the sinusoidal travelling wave solution, provided that the coefficients of gKGZE satisfy certain constraint conditions.

Keywords. Klein–Gordon–Zakharov equation with the positive fractional power terms; sub-ordinary differential equations method; exact solution; constraint condition.

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1. Introduction

In this paper, we consider the famous Klein–Gordon–Zakharov (KGZ) equations:

$$\begin{cases} u_{tt} - k_1^2 u_{xx} + u = nu, \\ n_{tt} - k_2^2 n_{xx} = (|u|^2)_{xx}. \end{cases} \tag{1}$$

In eqs (1), the variable $u(x,t)$ is a complex function and $n(x,t)$ is a real function. Equations (1) appear in the area of plasma physics, and are used to describe the interaction of Langmuir waves and ion-acoustic waves in plasmas [1–3]. Therefore, their investigation is of physical significance.

Many researchers paid their attention to nonlinear KGZ system due to their potential application in plasma physics. Some exact solutions of eqs (1) are obtained using different methods [4–11]. In refs [4,5], using the F-expansion method, the periodic wave solutions expressed by Jacobi elliptic functions for eqs (1) are derived. In ref. [6], using the extended hyperbolic functions method, the multiple exact explicit solutions of eqs (1) are obtained. Using the solitary wave ansatz method, 1-soliton solution of the KGZ equation with power-law nonlinearity is given, and numerical simulations that support the analysis are included [7]. Bifurcation analysis and the travelling wave solutions of the KGZ equations are studied in [8]. The topological soliton solution of the KGZ equation in (1+1) dimensions with power-law nonlinearity is derived and bifurcation analysis is studied in ref. [9]. In refs [10,11], Jacobi elliptic function expansion method is used to derive the periodic solutions for the KGZ equations. Gan *et al* [12,13] studied the instability of standing waves for KGZ equations. Linear stability analysis for periodic travelling waves of KGZ equations are performed in refs [14,15]. In refs [16–19], finite difference schemes are proposed for the initial-boundary problem of the KGZ equations.

The rest of the paper is organized as follows: In §2, KGZ equations are generalized, and the three special types of KGZ equations with the positive fractional power terms are presented; in §3, the sub-ODEs with the positive fractional power terms are introduced, and the exact solutions are given; in §4, the exact solutions of three new special types of the KGZ equations (5), (6) and (7) are derived in detail with the aid of the sub-ODE with the positive fractional power terms, respectively; in §5, some conclusions are made briefly.

2. The Klein–Gordon–Zakharov equations with the positive fractional power terms

In ref. [20], the KGZ equations with power-law nonlinearity are considered as

$$\begin{cases} u_{tt} - k_1^2 u_{xx} + au + bnu = 0, \\ n_{tt} - k_2^2 n_{xx} = c(|u|^{2m})_{xx}, \end{cases} \tag{2}$$

and the soliton solutions are given. It is obvious that eqs (2) become eqs (1) when $a = 1$, $b = -1$, $c = 1$, $m = 1$. In ref. [21], (2+1)-dimensional KGZ equation with power-law nonlinearity are studied as

$$\begin{cases} q_{tt} - \lambda^2(q_{xx} + q_{yy}) + q + rq + \alpha|q|^{2m} q = 0, \\ r_{tt} - \lambda^2(r_{xx} + r_{yy}) = (|q|^{2m})_{xx} + (|q|^{2m})_{yy}, \end{cases} \tag{3}$$

and soliton solutions are presented.

Based on refs [20,21], we generalize the KGZ equation as

$$\begin{cases} q_{tt} - \lambda^2 \Delta q + aq + brf(|q|^2)q + \alpha g(|q|^2)q = 0, \\ r_{tt} - \lambda^2 \Delta r = \beta \Delta[h(|q|^2)]. \end{cases} \tag{4}$$

where f, g and h are functions of q. It is easy to see that eqs (4) become eqs (3) when $a = b = \alpha = \beta = 1$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Here, when we set $f(u) = g(u) = h(u) = u^{q/p}$ in eqs (4), the first new special type of KGZ equation with the positive fractional power terms is presented as

$$\begin{cases} u_{tt} - \lambda^2 u_{xx} + au + br|u|^{q/p}u + \alpha|u|^{q/p}u = 0, \\ r_{tt} - \lambda^2 r_{xx} = \beta(|u|^{q/p})_{xx}. \end{cases} \tag{5}$$

When $f(u) = 1$, $g(u) = u^{q/p} + u^{2q/p}$, $h(u) = u^{q/p}$ in eqs (4), the second new special type of KGZ equation with positive fractional power terms is presented as

$$\begin{cases} u_{tt} - \lambda^2 u_{xx} + au + bru + \alpha\,|u|^{q/p}u + \alpha\,|u|^{2q/p}u = 0, \\ r_{tt} - \lambda^2 r_{xx} = \beta(|u|^{q/p})_{xx}. \end{cases} \tag{6}$$

When $f(u) = 1$, $g(u) = u^{q/p} + u^{2q/p}$, $h(u) = u^{2q/p}$ in eqs (4), the third new special type of KGZ equation with the positive fractional power terms is presented as

$$\begin{cases} u_{tt} - \lambda^2 u_{xx} + au + bru + \alpha\,|u|^{q/p}u + \alpha\,|u|^{2q/p}u = 0, \\ r_{tt} - \lambda^2 r_{xx} = \beta(|u|^{2q/p})_{xx}. \end{cases} \tag{7}$$

In this paper, we are going to derive exact solutions of these new three special types of gKGZEs (5)–(7).

3. The sub-ODE with positive fractional power terms

Inspired by the subsidiary higher-order ordinary differential equations [22–29], we consider the nonlinear ODE with the positive fractional power terms as

$$F'^2(\xi) = AF^2(\xi) + BF^{(q/p)+2}(\xi) + CF^{(2q/p)+2}(\xi), \tag{8}$$

where $F(\xi)$ is a function of ξ, A, B and C are constants and p and q are positive integers.

Then eq. (8) admits exact solutions as follows:

(1) When $A > 0$, $B = 2\,\sigma\,A$, $C = (\sigma^2 - 1)\,A$, $-1 < \sigma < 1$,

$$F(\xi) = \left[\frac{1}{\cosh(\sqrt{A}\xi q/p) - \sigma}\right]^{p/q}. \tag{9}$$

(2) When $A = 0$, $B = 4p^2/q^2$, $C = -(4p^2/q^2)\sigma$, $\sigma > 0$,

$$F(\xi) = \left[\frac{1}{\xi^2 + \sigma}\right]^{p/q}. \tag{10}$$

(3) When $A > 0$, $B = -2\sqrt{AC}$, $C > 0$,

$$F(\xi) = \left\{\sqrt{\frac{A}{C}}\left[\frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{q}{2p}\sqrt{A}\xi\right)\right]\right\}^{p/q}. \tag{11}$$

(4) When $A = -1$, $B = 2\sigma$, $C = 1 - \sigma^2$,

$$F(\xi) = \left[\frac{1}{\sigma \pm \sin(q\xi/p)}\right]^{p/q}. \tag{12}$$

Note 1: It should be noted that ODE (8) can admit other solutions, for example, the negative solutions for odd integer q, and so on. But for the sake of simplicity, we neglect the cases here.

4. Exact solutions of some special types of eqs (4)

4.1 *The first special type KGZ equations (5)*

Here we suppose the exact solutions of eqs (5) are in the form

$$u(x,t) = v(\xi)\exp(i\eta), \quad r(x,t) = r(\xi),$$
$$\xi = kx - \omega t, \quad \eta = lx - \theta t, \tag{13}$$

where k, l, ω, θ are constants to be determined later.

Substituting (13) into (5) yields nonlinear equations as follows:

$$\theta\omega - lk\lambda^2 = 0, \tag{14}$$

$$(\omega^2 - k^2\lambda^2)v'' + (a + l^2\lambda^2 - \theta^2)v + brv^{(q/p)+1} + \alpha v^{(q/p)+1} = 0, \tag{15}$$

$$(\omega^2-\lambda^2k^2)r''-\beta k^2(v^{q/p})''=0. \tag{16}$$

Integrating (16) twice and setting constants to zero yield

$$r=\frac{\beta k^2}{\omega^2-\lambda^2k^2}v^{q/p}. \tag{17}$$

Substituting (17) into (15) yields

$$(\omega^2-k^2\lambda^2)v''+(a+l^2\lambda^2-\theta^2)v+\alpha v^{(q/p)+1} +\frac{b\beta k^2}{\omega^2-\lambda^2k^2}v^{(2q/p)+1}=0. \tag{18}$$

Suppose the solutions of (18) are in the form of

$$v=DF(\xi), \tag{19}$$

where F satisfies eq. (8) and A, B, C and D are constants.

Substituting (19) into (18) and considering eq. (8) simultaneously, the left-hand side of eq. (18) becomes a polynomial in $F(\xi)$. When $\omega^2-k^2\lambda^2\neq 0$, setting the coefficients of the polynomial in eq. (18) to zero yields

$$\begin{cases} A+\dfrac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2}=0,\\ (\omega^2-k^2\lambda^2)\left(\dfrac{q}{2p}+1\right)B+\alpha D^{q/p}=0,\\ (\omega^2-k^2\lambda^2)\left(\dfrac{q}{p}+1\right)C+\dfrac{b\beta k^2}{\omega^2-\lambda^2k^2}D^{2q/p}=0. \end{cases} \tag{20}$$

Solving the algebraic equations (14), (20) yields

$$\begin{cases} A=-\dfrac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2},\\ B=-\dfrac{2p\alpha D^{q/p}}{(\omega^2-k^2\lambda^2)(q+2p)},\\ C=-\dfrac{pb\beta k^2}{(\omega^2-k^2\lambda^2)^2(q+p)}D^{2q/p}, \end{cases} \tag{21}$$

where $D>0$, $\theta\omega-lk\lambda^2=0$, $\omega^2-k^2\lambda^2\neq 0$.

With the help of the sub-ODE (8), the exact solutions of eqs (5) are obtained as:

Case 4.1.1.

$$\begin{cases} u(x,t)=\sqrt[q]{\left[\dfrac{(q+2p)(a+l^2\lambda^2-\theta^2)}{p\alpha}\sigma\right]^p} \times\left[\dfrac{1}{\cosh(\sqrt{A}\xi q/p)-\sigma}\right]^{p/q}\exp(i\eta),\\ r(x,t)=\dfrac{\sigma\beta k^2(q+2p)(a+l^2\lambda^2-\theta^2)}{(\omega^2-\lambda^2k^2)p\alpha} \times\dfrac{1}{\cosh(\sqrt{A}\xi q/p)-\sigma}, \end{cases} \tag{22}$$

where

$$\xi=kx-\omega t,\quad \eta=lx-\theta t,\quad \omega=\lambda^2\frac{lk}{\theta},$$

$$\sigma^2=\frac{p\lambda^2\alpha^2(\lambda^2l^2-\theta^2)(q+p)}{p\lambda^2\alpha^2(\lambda^2l^2-\theta^2)(q+p)-b\beta\theta^2(q+2p)^2(a+l^2\lambda^2-\theta^2)}, \tag{23}$$

and the constants satisfy the additional conditions

$$\frac{a+l^2\lambda^2-\theta^2}{\lambda^2l^2-\theta^2}<0,$$

$$\frac{(a+l^2\lambda^2-\theta^2)\sigma}{\alpha}>0,$$

$$b\beta>0,\quad \lambda^2k^2\left(\frac{\lambda^2l^2}{\theta^2}-1\right)\neq 0. \tag{24}$$

Case 4.1.2.

$$\begin{cases} u(x,t)=\sqrt[q]{\left[-\dfrac{2p(\omega^2-k^2\lambda^2)(q+2p)}{\alpha q^2}\right]^p} \times\left[\dfrac{1}{\xi^2+\sigma}\right]^{p/q}\exp(i\eta),\\ r(x,t)=-\dfrac{2p\beta k^2(\omega^2-k^2\lambda^2)(q+2p)}{\alpha q^2(\omega^2-\lambda^2k^2)}\dfrac{1}{\xi^2+\sigma}, \end{cases} \tag{25}$$

where

$$\xi = kx - \omega t, \quad \eta = lx - \theta t, \quad \omega = \lambda^2 \frac{lk}{\theta},$$

$$\theta^2 = a + l^2\lambda^2, \quad \sigma = \frac{pb\beta k^2 (q+2p)^2}{\alpha^2 (q+p) q^2}, \tag{26}$$

and the constants satisfy the additional conditions

$$\frac{\lambda^2 l^2 - \theta^2}{\alpha} < 0, \quad b\beta > 0$$

and

$$\lambda^2 k^2 \left(\frac{\lambda^2 l^2}{\theta^2} - 1 \right) \neq 0. \tag{27}$$

Case 4.1.3.

$$\begin{cases} u(x,t) = D \left\{ \sqrt{\frac{A}{C}} \left[\frac{1}{2} \pm \frac{1}{2} \tanh\left(\frac{q}{2p} \sqrt{A} \xi \right) \right] \right\}^{p/q} \\ \qquad \times \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2 - \lambda^2 k^2} \\ \qquad \times D^{q/p} \sqrt{\frac{A}{C}} \left[\frac{1}{2} \pm \frac{1}{2} \tanh\left(\frac{q}{2p} \sqrt{A} \xi \right) \right], \end{cases} \tag{28}$$

where $\xi = kx - \omega t, \eta = lx - \theta t$,

$$A = -\frac{a + l^2\lambda^2 - \theta^2}{\omega^2 - k^2\lambda^2},$$

$$C = -\frac{pb\beta k^2}{(\omega^2 - k^2\lambda^2)^2 (q+p)} D^{2q/p},$$

$$\frac{p\alpha^2 (\omega^2 - k^2\lambda^2)(q+p)}{b\beta k^2 (q+2p)^2} = a + l^2\lambda^2 - \frac{l^2 k^2 \lambda^4}{\omega^2},$$

$$\theta = \frac{lk\lambda^2}{\omega}, \tag{29}$$

and the constants satisfy the additional conditions

$$\frac{\omega^2 (a + l^2\lambda^2) - l^2 k^2 \lambda^4}{\omega^2 - k^2\lambda^2} < 0,$$

$$b\beta < 0, \quad \alpha(\omega^2 - k^2\lambda^2) > 0,$$

$$D > 0, \quad \omega^2 - k^2\lambda^2 \neq 0. \tag{30}$$

Case 4.1.4.

$$u(x,t) = \sqrt[2q]{\left[\frac{(\omega^2 - k^2\lambda^2)^2 (q+2p)^2 (q+p)}{p^2\alpha^2 (q+p) - pb\beta k^2 (q+2p)^2} \right]^p} \times \left[\frac{1}{\sigma \pm \sin(q\xi/p)} \right]^{p/q} \exp(i\eta),$$

$$r(x,t) = \frac{\beta k^2}{\omega^2 - \lambda^2 k^2} \times \sqrt[q]{\left[\frac{(\omega^2 - k^2\lambda^2)^2 (q+2p)^2 (q+p)}{p^2\alpha^2 (q+p) - pb\beta k^2 (q+2p)^2} \right]^p} \times \frac{1}{\sigma \pm \sin(q\xi/p)}, \tag{31}$$

where

$$\xi = kx - \omega t, \quad \eta = lx - \theta t,$$

$$\theta^2 = k^2\lambda^2 \left(1 - \frac{a}{\omega^2 - k^2\lambda^2} \right), \quad l = \frac{\theta\omega}{k\lambda^2},$$

$$D = \sqrt[2q]{\left[\frac{(\omega^2 - k^2\lambda^2)^2 (q+2p)^2 (q+p)}{p^2\alpha^2 (q+p) - pb\beta k^2 (q+2p)^2} \right]^p}, \tag{32}$$

and the constants satisfy the additional conditions

$$\omega^2 - k^2\lambda^2 \neq 0, \quad \alpha(\omega^2 - k^2\lambda^2) < 0,$$

$$p\alpha^2 (q+p) - b\beta k^2 (q+2p)^2 > 0,$$

$$\sigma = -\frac{p\alpha D^{q/p}}{(\omega^2 - k^2\lambda^2)(q+2p)} > 1,$$

$$\frac{a}{\omega^2 - k^2\lambda^2} < 1. \tag{33}$$

4.2 *The second special type KGZ equations* (6)

Similar to §4.1, suppose the exact solutions of eqs (6) are in the form

$$u(x,t) = v(\xi) \exp(i\eta), \quad r(x,t) = r(\xi),$$

$$\xi = kx - \omega t, \quad \eta = lx - \theta t, \tag{34}$$

where k, l, ω, θ are constants to be determined later. Substituting (34) into (6) yields nonlinear equations as follows:

$$\theta\omega - lk\lambda^2 = 0, \tag{35}$$

$$r = \frac{\beta k^2}{\omega^2 - \lambda^2 k^2} v^{q/p}, \tag{36}$$

$$(\omega^2 - k^2\lambda^2)v'' + (a + l^2\lambda^2 - \theta^2)v + \left(\frac{\beta k^2 b}{\omega^2 - \lambda^2 k^2} + \alpha\right) v^{(q/p)+1} + \alpha v^{(2q/p)+1} = 0. \tag{37}$$

Suppose the solutions of (37) are in the form of

$$v = DF(\xi), \tag{38}$$

where F satisfies eq. (8) and A, B, C and D are constants.

Substituting (38) into (37) and considering eq. (8) simultaneously, the left-hand side of eq. (37) becomes a polynomial in $F(\xi)$, when $\omega^2 - k^2\lambda^2 \neq 0$. Considering (35) and setting the coefficients of the polynomial in eq. (37) to zero yields

$$\begin{cases} \theta\omega - lk\lambda^2 = 0, \\ (\omega^2 - k^2\lambda^2)A + (a + l^2\lambda^2 - \theta^2) = 0, \\ \left(\frac{q}{2p} + 1\right)(\omega^2 - k^2\lambda^2)B + \left(\frac{\beta k^2 b}{\omega^2 - \lambda^2 k^2} + \alpha\right) D^{q/p} = 0, \\ \left(\frac{q}{p} + 1\right)(\omega^2 - k^2\lambda^2)C + \alpha D^{2q/p} = 0. \end{cases} \tag{39}$$

Solving the algebraic equations (39) yields

$$\begin{cases} A = -\frac{a + l^2\lambda^2 - \theta^2}{\omega^2 - k^2\lambda^2}, \\ B = -\frac{\left(\frac{\beta k^2 b}{\omega^2 - \lambda^2 k^2} + \alpha\right)}{\left(\frac{q}{2p} + 1\right)(\omega^2 - k^2\lambda^2)} D^{q/p}, \\ C = -\frac{\alpha D^{2q/p}}{\left(\frac{q}{p} + 1\right)(\omega^2 - k^2\lambda^2)}, \\ \theta\omega - lk\lambda^2 = 0, \quad \omega^2 - k^2\lambda^2 \neq 0, \quad D > 0. \end{cases} \tag{40}$$

With the help of the sub-ODE (8), the exact solutions of eqs (6) are obtained as:

Case 4.2.1.

$$\begin{cases} u(x,t) = \sqrt[q]{\left\{\frac{(a + l^2\lambda^2 - \theta^2)(q + 2p)(\omega^2 - k^2\lambda^2)\sigma}{p[\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)]}\right\}^p} \times \left[\frac{1}{\cosh(\sqrt{A}\xi q/p) - \sigma}\right]^{p/q} \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2 - \lambda^2 k^2} \times \frac{(a + l^2\lambda^2 - \theta^2)(q + 2p)(\omega^2 - k^2\lambda^2)\sigma}{p[\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)]} \times \frac{1}{\cosh(\sqrt{A}\xi q/p) - \sigma}, \end{cases} \tag{41}$$

where

$$\xi = kx - \omega t, \quad \eta = lx - \theta t,$$

$$\sigma^2 = \frac{p(q + p)\,[\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)]^2}{p(q + p)[\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)]^2 - \alpha(q + 2p)^2(\omega^2 - k^2\lambda^2)^2},$$

$$\theta = \frac{lk\lambda^2}{\omega}, \tag{42}$$

and the constants satisfy the additional conditions

$$\omega^2 - k^2\lambda^2 \neq 0, \quad \frac{a + l^2\lambda^2 - \theta^2}{\omega^2 - k^2\lambda^2} < 0, \quad \alpha < 0, \quad \frac{\sigma}{\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)} < 0. \tag{43}$$

Case 4.2.2.

$$\begin{cases} u(x,t) = \left[\frac{1}{\xi^2+\sigma}\right]^{p/q} \\ \quad \times \sqrt[q]{\left[-\frac{2p(q+2p)(\omega^2-k^2\lambda^2)^2}{q^2(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))}\right]^p} \exp(i\eta), \\ r(x,t) = -\frac{2p\beta k^2(q+2p)(\omega^2-k^2\lambda^2)^2}{q^2(\omega^2-\lambda^2k^2)(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))} \\ \quad \times \frac{1}{\xi^2+\sigma}, \end{cases} \tag{44}$$

where

$$\xi = kx-\omega t, \quad \eta = lx-\theta t,$$

$$\sigma = \frac{\alpha p\,(q+2p)^2\,(\omega^2-k^2\lambda^2)^3}{q^2(q+p)(\omega^2-k^2\lambda^2)(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))^2},$$

$$l = \pm\sqrt{\frac{a\omega^2}{(k^2\lambda^2-\omega^2)\lambda^2}},$$

$$\theta = \pm\frac{k\lambda^2}{\omega}\sqrt{\frac{a\omega^2}{(k^2\lambda^2-\omega^2)\lambda^2}}, \tag{45}$$

and the constants satisfy the additional conditions

$$\omega^2-k^2\lambda^2 \neq 0,$$

$$\alpha(\omega^2-\lambda^2k^2) > 0, \quad \beta b k^2+\alpha(\omega^2-\lambda^2k^2) < 0,$$

$$a(k^2\lambda^2-\omega^2) > 0. \tag{46}$$

Case 4.2.3.

$$\begin{cases} u(x,t) = D\left\{\sqrt{\frac{A}{C}}\left[\frac{1}{2}\pm\frac{1}{2}\tanh\left(\frac{q}{2p}\sqrt{A}\xi\right)\right]\right\}^{p/q} \\ \quad \times \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2-\lambda^2k^2} \\ \quad \times D\sqrt{\frac{A}{C}}\left[\frac{1}{2}\pm\frac{1}{2}\tanh\left(\frac{q}{2p}\sqrt{A}\xi\right)\right], \end{cases} \tag{47}$$

where

$$\xi = kx-\omega t, \quad \eta = lx-\theta t,$$

$$A = -\frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2},$$

$$C = -\frac{\alpha D^{2q/p}}{\left(\frac{q}{p}+1\right)(\omega^2-k^2\lambda^2)},$$

$$\theta = \frac{lk\lambda^2}{\omega},$$

$$l = \pm\sqrt{\frac{(q+p)p\omega^2}{\alpha(q+2p)^2(\omega^2-k^2\lambda^2)\lambda^2}\left(\frac{\beta k^2 b+\alpha(\omega^2-\lambda^2k^2)}{\omega^2-\lambda^2k^2}\right)^2 - \frac{a\omega^2}{(\omega^2-k^2\lambda^2)\lambda^2}}, \tag{48}$$

and the constants satisfy the additional conditions

$$\frac{(q+p)p(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))^2 - a\alpha(q+2p)^2(\omega^2-k^2\lambda^2)^2}{\alpha(\omega^2-k^2\lambda^2)} \geq 0,$$

$$\omega^2-k^2\lambda^2 \neq 0, \quad \frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2} < 0, \quad \frac{\alpha}{\omega^2-k^2\lambda^2} < 0, \quad D > 0. \tag{49}$$

Case 4.2.4.

$$\begin{cases} u(x,t) = \left[\frac{1}{\sigma\pm\sin(q\xi/p)}\right]^{p/q}\sqrt[q]{\left[-\frac{\sigma(q+2p)(\omega^2-k^2\lambda^2)^2}{2p^2(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))}\right]^p}\exp(i\eta), \\ r(x,t) = -\frac{\sigma\beta k^2(q+2p)(\omega^2-k^2\lambda^2)^2}{2p^2(\omega^2-\lambda^2k^2)(\beta k^2 b+\alpha(\omega^2-\lambda^2k^2))}\frac{1}{\sigma\pm\sin(q\xi/p)}, \end{cases} \tag{50}$$

where

$$\xi = kx - \omega t, \quad \eta = lx - \theta t,$$

$$\sigma^2 = \frac{4p^3(q+p)(\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2))^2}{4p^3(q+p)(\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2))^2 - \alpha(q+2p)^2(\omega^2 - k^2\lambda^2)},$$

$$l = \pm\sqrt{\frac{\omega^2(\omega^2 - k^2\lambda^2 - a)}{\lambda^2(\omega^2 - k^2\lambda^2)}}, \quad \theta = \pm\frac{k\lambda^2}{\omega}\sqrt{\frac{\omega^2(\omega^2 - k^2\lambda^2 - a)}{\lambda^2(\omega^2 - k^2\lambda^2)}}, \tag{51}$$

and the constants satisfy the additional conditions

$$\omega^2 - k^2\lambda^2 \neq 0, \quad \frac{a}{\omega^2 - k^2\lambda^2} < 1,$$

$$\sigma[\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2)] < 0. \tag{52}$$

4.3 *The third special type KGZ equations (7)*

Similar to §4.1, suppose the exact solutions of eqs (7) are in the form

$$u(x,t) = v(\xi)\exp(i\eta), \quad r(x,t) = r(\xi),$$

$$\xi = kx - \omega t, \quad \eta = lx - \theta t, \tag{53}$$

where k, l, ω, θ are constants to be determined later. Substituting (53) into (7) yields nonlinear equations as follows:

$$\theta\omega - lk\lambda^2 = 0, \tag{54}$$

$$r = \frac{\beta k^2}{\omega^2 - \lambda^2 k^2} v^{2q/p}, \tag{55}$$

$$(\omega^2 - k^2\lambda^2)v'' + (a + l^2\lambda^2 - \theta^2)v + \alpha v^{(q/p)+1} + \left(\frac{\beta k^2 b}{\omega^2 - \lambda^2 k^2} + \alpha\right) v^{(2q/p)+1} = 0. \tag{56}$$

Suppose the solutions of (56) are in the form of

$$v = DF(\xi), \tag{57}$$

where F satisfies eq. (5) and A, B, C and D are constants.

Substituting (57) into (56) and considering eq. (8) simultaneously, the left-hand side of eq. (56) becomes a polynomial in $F(\xi)$, when $\omega^2 - k^2\lambda^2 \neq 0$. Setting the coefficients of the polynomial in eq. (56) to zero yields

$$\begin{cases} (\omega^2 - k^2\lambda^2)A + (a + l^2\lambda^2 - \theta^2) = 0, \\ \left(\frac{q}{2p} + 1\right)(\omega^2 - k^2\lambda^2)B + \alpha D^{q/p} = 0, \\ \left(\frac{q}{p} + 1\right)(\omega^2 - k^2\lambda^2)C \\ \quad + \left(\frac{\beta k^2 b}{\omega^2 - \lambda^2 k^2} + \alpha\right) D^{2q/p} = 0. \end{cases} \tag{58}$$

Solving the algebraic eqs (58) yields

$$\begin{cases} A = -\frac{a + l^2\lambda^2 - \theta^2}{\omega^2 - k^2\lambda^2}, \\ B = -\frac{2p\alpha D^{q/p}}{(q+2p)(\omega^2 - k^2\lambda^2)}, \\ C = -\frac{p(\beta k^2 b + \alpha(\omega^2 - \lambda^2 k^2))D^{2q/p}}{(q+p)(\omega^2 - k^2\lambda^2)^2}, \\ D > 0, \quad \omega^2 - k^2\lambda^2 \neq 0. \end{cases} \tag{59}$$

With the help of the sub-ODE (8), the exact solutions of eqs (7) are obtained as follows:

Case 4.3.1.

$$\begin{cases} u(x,t) = \sqrt[q]{\left[\frac{(q+2p)(a+l^2\lambda^2-\theta^2)}{p\alpha}\sigma\right]^p} \\ \qquad \times \left[\frac{1}{\cosh(\sqrt{A}\xi q/p)-\sigma}\right]^{p/q} \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2-\lambda^2k^2} \\ \qquad \times \left(\frac{\sigma(q+2p)(a+l^2\lambda^2-\theta^2)}{p\alpha(\cosh(\sqrt{A}\xi q/p)-\sigma)}\right)^2, \end{cases} \quad (60)$$

where

$$\xi = kx-\omega t, \quad \eta = lx-\theta t,$$
$$A = -\frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2},$$
$$\sigma^2 = \frac{1}{1-\frac{(q+2p)^2(a+l^2\lambda^2-\theta^2)(\beta k^2b+\alpha(\omega^2-\lambda^2k^2))}{p\alpha^2(q+p)(\omega^2-k^2\lambda^2)}}, \quad (61)$$

and the constants satisfy the additional conditions

$$\omega^2-k^2\lambda^2 \neq 0,$$
$$\frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2} < 0, \quad \frac{(a+l^2\lambda^2-\theta^2)\sigma}{\alpha} > 0,$$
$$\beta k^2b+\alpha(\omega^2-\lambda^2k^2) > 0. \quad (62)$$

Case 4.3.2.

$$\begin{cases} u(x,t) = \sqrt[q]{\left[-\frac{2p(q+2p)(\omega^2-k^2\lambda^2)}{\alpha q^2}\right]^p} \\ \qquad \times \left[\frac{1}{\xi^2+\sigma}\right]^{p/q} \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2-\lambda^2k^2} \\ \qquad \times \left(-\frac{2p(q+2p)(\omega^2-k^2\lambda^2)}{\alpha q^2}\frac{1}{\xi^2+\sigma}\right)^2, \end{cases} \quad (63)$$

where

$$\xi = kx-\omega t, \quad \eta = lx-\theta t,$$
$$\sigma = \frac{p(q+2p)^2(\beta k^2b+\alpha(\omega^2-\lambda^2k^2))}{\alpha^2q^2(q+p)}, \quad (64)$$

and the constants satisfy the additional conditions

$$a+l^2\lambda^2-\theta^2 = 0, \quad \beta k^2b+\alpha(\omega^2-\lambda^2k^2) > 0,$$
$$\omega^2-k^2\lambda^2 \neq 0, \quad \frac{\omega^2-k^2\lambda^2}{\alpha} < 0. \quad (65)$$

Case 4.3.3.

$$\begin{cases} u(x,t) = D\left\{\sqrt{\frac{A}{C}}\left[\frac{1}{2}\pm\frac{1}{2}\tanh\left(\frac{q}{2p}\sqrt{A}\xi\right)\right]\right\}^{p/q} \\ \qquad \times \exp(i\eta), \\ r(x,t) = \frac{\beta k^2D^{2q/p}}{\omega^2-\lambda^2k^2} \\ \qquad \times \left\{\sqrt{\frac{A}{C}}\left[\frac{1}{2}\pm\frac{1}{2}\tanh\left(\frac{q}{2p}\sqrt{A}\xi\right)\right]\right\}^2, \end{cases}$$

where

$$\xi = kx-\omega t, \quad \eta = lx-\theta t, \quad \omega^2-k^2\lambda^2 \neq 0,$$
$$A = -\frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2},$$
$$C = -\frac{p(\beta k^2b+\alpha(\omega^2-\lambda^2k^2))D^{2q/p}}{(q+p)(\omega^2-k^2\lambda^2)^2},$$
$$\theta^2 = a+l^2\lambda^2-\frac{p(\omega^2-k^2\lambda^2)(q+p)\alpha^2}{(q+2p)^2(\beta k^2b+\alpha(\omega^2-\lambda^2k^2))}, \quad (66)$$

and the constants satisfy the additional conditions

$$D > 0, \quad \frac{a+l^2\lambda^2-\theta^2}{\omega^2-k^2\lambda^2} < 0, \quad \frac{\alpha}{(\omega^2-k^2\lambda^2)} > 0,$$
$$\beta k^2b+\alpha(\omega^2-\lambda^2k^2) < 0. \quad (67)$$

Case 4.3.4.

$$\begin{cases} u(x,t) = \left[-\frac{\sigma(q+2p)(\omega^2-k^2\lambda^2)}{p\alpha(\sigma\pm\sin(q\xi/p))}\right]^{p/q} \\ \qquad \times \exp(i\eta), \\ r(x,t) = \frac{\beta k^2}{\omega^2-\lambda^2k^2} \\ \qquad \times \left(-\frac{\sigma(q+2p)(\omega^2-k^2\lambda^2)}{p\alpha(\sigma\pm\sin(q\xi/p))}\right)^2, \end{cases} \quad (68)$$

where

$$\xi = kx - \omega t, \quad \eta = lx - \theta t,$$
$$\omega^2 = a + l^2\lambda^2 - \theta^2 + k^2\lambda^2,$$
$$\sigma^2 = \frac{1}{1 - \frac{(q+2p)^2(\beta k^2 b + \alpha a + \alpha l^2\lambda^2 - \alpha\theta^2)}{p\alpha^2(q+p)}}, \tag{69}$$

and the constants satisfy

$$a + l^2\lambda^2 - \theta^2 > 0, \quad \frac{\sigma}{\alpha} < 0,$$
$$\frac{(q+2p)^2(\beta k^2 b + a\alpha + l^2\lambda^2\alpha - \alpha\theta^2)}{p\alpha^2(q+p)} < 1,$$
$$a \neq \theta^2 - l^2\lambda^2. \tag{70}$$

Note 2: There are other special types of KGZ equation if $f(u), g(u)$ and $h(u)$ are set to other special functions, and the exact solutions of these special type KGZ equations can be derived with the help of the sub-ODE (8). Here we do not discuss these problems in detail.

Note 3: The exact solutions of multidimensional KGZ equations with the positive fractional power terms can be obtained with the help of the sub-ODE (8). Here we do not discuss these problems in detail.

5. Conclusions and discussions

In this paper, the KGZ equations are first generalized, and special types of KGZ equations with the positive fractional power terms are introduced.

Secondly, subsidiary higher-order ordinary differential equations with the positive fractional power terms are presented.

Thirdly, exact solutions of four special types of KGZ equations with the positive fractional power terms are derived with the aid of the sub-ODE, which are the bell-type solitary wave solution, the algebraic solitary wave solution, the kink-type solitary wave solution and the sinusoidal travelling wave solution, provided the coefficients of gKGZE satisfy certain constraint conditions.

It is obvious that the method introduced in this paper may be applied to explore exact solutions for other nonlinear evolution equations with positive fractional power terms.

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