Pramana – J. Phys. (2016) 87: 90
DOI 10.1007/s12043-016-1295-6

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Folding model analysis of the nucleus–nucleus scattering based on Jacobi coordinates

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MS received 12 September 2015; revised 1 February 2016; accepted 7 March 2016; published online 3 November 2016

Abstract. This paper presents the results of scattering of $^{16}O + ^{209}Bi$ interaction near the Coulomb barrier. The interaction potential between two nuclei is calculated using the double folding model with the effective nucleon–nucleon (*NN*) interaction. The calculations of the exchange part of the interaction were assumed to be of finite-range and the density dependence of the *NN* interaction is accounted for. Also the results are compared with the zero-range approximation. All these calculations are done using the wave functions of the two colliding nuclei in place of their density distributions. The wave functions are obtained by the *D*-dimensional wave equation using the hyperspherical calculations on the basis of Jacobi coordinates. The numerical results for the interaction potential and the differential scattering are in good agreement with the previous works.

Keywords. Double folding model; M3Y interaction; differential equation; Yukawa potential; hyperspherical space; hypergeometric function; partial wave method; differential scattering cross-section.

PACS Nos 03.65.Nk; 13.75.Cs; 13.85.Dz; 25.60.Bx; 25.60.Dz

1. Introduction

Investigation of the scattering theory is important in any standard quantum text because many important discoveries in the nuclear and the atomic physics have been made by bombarding a nucleus or an atom by particles and measuring the number of particles scattered in various directions. The scattering can change the phase or the amplitude of the outgoing wave [1–6].

The interaction between two nuclei is one of the most important aspects in nuclear physics. An important step in investigating these interactions is the calculation of ion–ion interaction potential that can help us to estimate the cross-sections of the elastic, inelastic and fusion reactions [7,8]. A complicated many-body problem is considered, because in these interactions all nucleons of the target nucleus interact with each other and with the nucleons of the projectile nucleus. This complicated interaction can be approximated by the optical model. In this model, a two-body potential between the projectile and the target nucleus is investigated [9–11]. The real part of the optical potential is named as the folding model potential. The double folding model calculates the interaction potential between the two interacting nuclei using their densities. Density dependency in this model is a privilege when considering the effects of nuclei deformation. However, the aim of the present study is to use the wave functions of two nuclei instead of their densities and evaluate the efficiency of this method using the results of the scattering problem.

In §2, the folding method is described briefly. Next, the wave functions of the two nuclei are calculated using the differential wave equation. Solution of the scattering problem is available in §4 and finally the conclusions are presented in §5.

2. The double folding model (DFM)

The total interaction potential between the two interacting nuclei is given as

$$V_{\text{total}}(\vec{R}) = V_{\text{N}}(\vec{R}) + V_{\text{C}}(\vec{R}) + V_{\text{rot}}(\vec{R}),$$

where V_{N} represents the strong (nuclear), V_{C} the electrostatic (Coulomb) and V_{rot} the rotational interactions respectively (we have ignored the rotational part in the present study). Double folding is a method for calculating the total interaction potential. This model calculates the real part of the optical potential with densities of two colliding nuclei and an effective nucleon–nucleon interaction. The nuclear part of the interaction potential has two terms: the direct term V_{ND} and the exchange term V_{NE}. The direct part of the interaction between the two nuclei is determined by the DFM as follows [12,13]:

$$V_{\mathrm{ND}}(\vec{R}, E_{\mathrm{P}}) = g(E_{\mathrm{P}}) \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \rho_1(\vec{r}_1) \times \rho_2(\vec{r}_2) v_{NN(D)}(s), \tag{1}$$

where $\vec{R}$ denotes the distance between the centres of mass of the colliding nuclei, the vector $\vec{s} = \vec{R} - \vec{r}_1 + \vec{r}_2$ corresponds to the distance between two specified interacting points of the projectile and the target whose radius vectors are $\vec{r}_{1(2)}$ respectively, $g(E_{\mathrm{P}}) = 1 - kE_{\mathrm{P}}$ is an energy-dependent coefficient as $E_{\mathrm{P}} = E_{\mathrm{lab}}/A_{\mathrm{P}}$ (E_{lab}, A_{P} are the laboratory energy and the nucleon number of the projectile) is the average energy of each projectile nucleon, $\rho_{1(2)}(\vec{r}_{1(2)})$ is the distribution of the centres of mass of the nucleons in the ground state of the projectile or the target nucleus and v_{NN} is the effective nucleon–nucleon interaction. It is slightly different for the exchange part. The exchange part of the interaction by the DFM is

$$V_{\mathrm{NE}}(\vec{R}, E_{\mathrm{P}}) = g(E_{\mathrm{P}}) \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \rho_1(\vec{r}_1; \vec{r}_1 + \vec{s}) \times \rho_2(\vec{r}_2; \vec{r}_2 - \vec{s}) v_{NN(E)}(s) \times \exp(i k_{\mathrm{rel}} \vec{s}/A_{\mathrm{red}}). \tag{2}$$

The exchange density determined using the density matrix expansion relations [14–16] is

$$\rho(\vec{r}; \vec{r} \pm \vec{s}) \approx \rho(\vec{r} \pm \vec{s}/2) \hat{j}_1(|k_{\mathrm{eff}}(\vec{r} \pm \vec{s}/2)| s),$$
$$\hat{j}_1(x) = 3[\sin(x) - x\cos(x)]/x^3, \tag{3}$$

where $\hat{j}_1(x)$ is the first degree spherical Bessel function and k_{eff}, the effective Fermi momentum has been extracted from the extended Thomas Fermi approximation (ETF app.) [16] as

$$k_{\mathrm{eff}}^2(\vec{r}) = \left(\frac{3\pi^2 \rho(\vec{r})}{2}\right)^{2/3} + \frac{5C_s}{3}\left(\frac{\vec{\nabla}\rho(\vec{r})}{\rho(\vec{r})}\right)^2 + \frac{5\nabla^2 \rho(\vec{r})}{36\rho(\vec{r})}. \tag{4}$$

C_s determines the strength of the Weizsäcker correction term to the kinetic energy density [14,18]. We use $C_s = 1/36$ that provides good results [17]. The wave number k_{rel} is added after considering the relative motion of the two colliding nuclei as follows:

$$k_{\mathrm{rel}}^2 = 2m_{\mathrm{n}} A_{\mathrm{red}}[E_{\mathrm{c.m.}} - V_{\mathrm{tot}}(R)]/\hbar^2, \tag{5}$$

where $A_{\mathrm{red}} = A_{\mathrm{P}} A_{\mathrm{T}}/(A_{\mathrm{P}} + A_{\mathrm{T}})$ is the reduced mass number and m_{n} is the bare nucleon mass.

The *NN* interaction used in the double folding model is widely considered in the literature for the finite-range approximation as M3Y potential (a sum of the Yukawa-type terms) for both the direct and the exchange terms [19] as

$$v_{D(E)}(s) = \sum_{i=1}^{3} G_{D(E)i} v_i(s),$$
$$v(s) = \frac{\exp[-a_i s]}{[a_i s]}. \tag{6}$$

It was widely known that the M3Y *NN* interaction must be density-dependent [19,20]. We use a generalized density dependence of the M3Y interaction introduced in refs [19,20]. It enters as a multiplier to the nuclear part of the interaction as follows:

$$F(\rho) = C\{1 + \alpha \exp(-\beta\rho) - \gamma\rho\}. \tag{7}$$

In place of the finite-range *NN* interaction, zero-range interaction was used in the early works [9] as follows:

$$v_{E\delta}(s) = J(E)\delta(s), \quad J(E) = G_{E\delta} g(E_{\mathrm{P}}). \tag{8}$$

The coefficients are given in table 1, for the Reid and the Paris M3Y interaction and the coefficients of the density-dependent M3Y nucleon–nucleon interaction are presented in table 2.

The Coulomb part has a form similar to eq. (1) for the direct interaction [21,22] as

$$V_{\mathrm{C}}(\vec{R}) = \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \rho_1(\vec{r}_1) \rho_2(\vec{r}_2) v_{\mathrm{C}}(s), \tag{9}$$

Table 1. The interaction coefficients of M3Y-Reid and M3Y-Paris.

Coefficient	Reid	Paris
G_{D1} (MeV)	7999	11062
G_{D2} (MeV)	−2134	−2537.5
G_{D3} (MeV)	0	0
G_{Ef1} (MeV)	4631.4	−1524.25
G_{Ef2} (MeV)	−1787.1	−518.75
G_{Ef3} (MeV)	−7.847	−7.847
$1/a_1$ (fm)	0.25	0.25
$1/a_2$ (fm)	0.4	0.4
$1/a_3$ (fm)	1.414	1.414
$G_{E\delta}$ (MeV fm^3)	−276	−592
k (MeV^{-1})	0.002	0.003

Table 2. The density-dependent M3Y *NN* interaction coefficients.

DD label	Interaction	C	α	β (fm^{-3})	γ (fm^{-3})
0	D Independent	1	0.0	0.0	0.0
1	DDM3Y1	0.2963	3.7231	3.7348	0.0
2	CDM3Y1	0.3429	3.0232	3.5512	0.5
3	CDM3Y2	0.3346	3.0357	3.0685	1.0
4	CDM3Y3	0.2985	3.4528	2.6388	1.5
5	CDM3Y4	0.3052	3.2998	2.3180	2.0
6	CDM3Y5	0.2728	3.7367	1.8294	3.0
7	CDM3Y6	0.2685	3.8033	1.4099	4.0
8	BDM3Y1	1.2521	00	0.0	1.7452

where $\rho_{1(2)}(\vec{r}_{1(2)})$ are the charge densities of the projectile and the target nuclei and $v_C(s)$ is the Coulomb interaction considered usually in the form of the point–point Coulomb potential between two point charges (Z_1, Z_2) with good approximation as follows:

$$V_C(r) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{r}. \tag{10}$$

In this article the wave functions of the two colliding nuclei are used in place of their density distributions. Thus, we use the quantum mechanical equation for the density denoted by $\rho = |\psi\psi^*|$. Therefore, the density distributions for the direct and the Coulomb parts will be as follows:

$$|\psi_1(\vec{r}_1)\psi_1^*(\vec{r}_1)||\psi_2(\vec{r}_2)\psi_2^*(\vec{r}_2)| \tag{11}$$

and the density distributions for the exchange part will be as follows:

$$\left|\psi(\vec{r};\vec{r}\pm\vec{s})\psi^*(\vec{r};\vec{r}\pm\vec{s})\right| \approx \left|\psi(\vec{r}\pm\vec{s}/2)\psi^*(\vec{r}\pm\vec{s}/2)\right| \times \hat{j}_1(|k_{\text{eff}}(\vec{r}\pm\vec{s}/2)|s). \tag{12}$$

Therefore, we must obtain wave functions of the two nuclei for calculating the interaction potential. This subject is investigated in the next section.

3. The wave equation

The many-body forces are more easily introduced within the hyperspherical formalism. Therefore, the Schrödinger equation is rewritten for a system of N fixed identical particles as

$$\left\{-\frac{1}{2m}\sum_{i=1}^{N-1}\nabla_{r_i}^2 + \sum_{i,j\succ i} V(r_{ij}) - E\right\}\psi(r_{ij}) = 0, \tag{13}$$

where $r_{ij} = r_j - r_i$ denotes a set of relative coordinates of the particles and $V(r_{ij})$ is the interaction potential between the particles in terms of the relative distance of the pair in the two-body subsystem.

The relative distance can be explained by the Jacobi coordinate transformation as

$$\xi_i = \sqrt{\frac{i}{i+1}}\left(r_{i+1} - \frac{1}{i}\sum_{j=1}^{i} r_j\right), \quad i = 1, 2, \ldots, N-1. \tag{14}$$

It can be seen that the solution of the N-particle system is composed of $N-1$ Jacobi coordinates. First the centre of mass of two arbitrary particles is considered. Therefore, the location of the third particle is considered to be relative to the centre of mass of two arbitrary particles and similarly for all particles. Indeed, each Jacobi contacts the centre of mass of the subsystem for the remaining particles. If the potential between the particles only depends on the relative distance between them, it can be written in terms of the hyper-radius x. In this case it is named as the hypercentral potential. The hyper-radius is, thus, the relative distance of the particles from each other and from the centre-of-mass [23] as

$$x = (\xi_1 + \xi_2 + \cdots + \xi_{N-1})^{1/2}. \tag{15}$$

According to the centre-of-mass coordinates, the relation for R is $R = (1/N)\sum_i^N r_i$ and the Jacobi coordinates can be written as

$$\begin{aligned}
\xi_1^2 &= \frac{r_{12}^2}{2} = \frac{1}{2}(r_1 - r_2)^2 = 2(R - r_2)^2\\
\xi_2^2 &= \tfrac{1}{6}(r_1 + r_2 - 2r_3)^2 = \tfrac{3}{2}(R - r_3)^2\\
&\vdots\\
\xi_{N-1}^2 &= \frac{1}{N(N-1)}[r_1 + r_2 + \cdots + r_{N-1} - (N-1)r_N]^2\\
&= \frac{N}{N-1}(R - r_N)^2.
\end{aligned} \tag{16}$$

Using the above equation, the hyper-radius is

$$x = \left[\sum_{i=1}^{N-1}\xi_i^2\right]^{1/2} = \left[\sum_{i=1}^{N-1}\frac{i+1}{i}(R - r_{i+1})^2\right]^{1/2}$$

$$\begin{aligned}
\Rightarrow x^2 &= \frac{1}{N}(r_{12}^2 + r_{23}^2 + \cdots + r_{N1}^2)\\
&= (r_1 - R)^2 + \cdots + (r_N - R)^2.
\end{aligned} \tag{17}$$

Therefore, x^2 is the relative square distance of the particles from each other and the centre-of-mass coordinate.

For the hyperspherical coordinates and D-dimensional space, the Laplacian is [24–26]

$$\sum_{i=1}^{N-1} \nabla^2_{\xi_i} = \left(\frac{d^2}{dx^2} + \frac{D-1}{x}\frac{d}{dx} + \frac{\Gamma^2(\Omega)}{x^2} \right), \tag{18}$$

where $\Gamma^2(\Omega)/x^2$ is a generalization of the centrifugal barrier which involves the angular coordinates $Y_\gamma(\Omega_{\xi_1}, \Omega_{\xi_2}, \ldots, \varphi_1, \varphi_2, \ldots)$. They are called hyperspherical harmonics (HH), and form a complete orthogonal basis. φ_i is the hyperangle, for instance $\varphi_1 = \arctan(\xi_1/\xi_2)$. Also, $\Gamma^2(\Omega)$ is the grand orbital operator that its eigenvalues are as follows [23–28]:

$$\Gamma^2(\Omega) = -\gamma(\gamma + D - 2). \tag{19}$$

γ is the grand angular quantum number, $\gamma = 2n + l_{\xi_1} + l_{\xi_2} + \cdots$. In this equation, n is any non-negative integer and $l_{\xi_1}, l_{\xi_2}, \ldots$ are the angular momenta associated with the relative Jacobi coordinates ξ_1 and ξ_2, etc. HH contains the usual spherical harmonics with the angular momentum of $l_{\xi_1}, l_{\xi_2}, \ldots$ as well as known functions of the hyperangle φ. In other words, except for the hyper-radius, the other variables that describe the position of the point in the hyperspace can be conveniently parametrized as $D-1$ angles which are collectively referred to as Ω_{D-1}.

Finally, based on the above assumptions, the radial Schrödinger equation in the D-dimensional space is [23–28]

$$\left\{ \frac{-1}{2m}\frac{1}{x^{D-1}}\left[\frac{d}{dx}x^{D-1}\frac{d}{dx}\right] + \frac{\Gamma^2(\Omega)}{x^2} + V(x) \right\} \psi(x) = E\psi(x). \tag{20}$$

Investigation of a nucleus using the D-dimensional Schrödinger equation means that we assume a nucleus with its nucleons to be a multidimensional system. The dimension is related to the number of particles in the centre of mass coordinate or $D = 3N - 3$ (N is the number of particles).

3.1 *Solution of the hyper-radial part*

In this paper $^{16}O + ^{209}Bi$ interaction is considered because the projectile and the target have relatively simple structures. Also, they both have closed shell spherical nuclei that are well understood.

As stated in the previous section, the wave functions of the two nuclei are required for calculating the interaction potential between two nuclei. Thus, the D-dimensional equation should be solved to obtain the wave functions of the two nuclei.

We assume that the internal interaction of the nucleons with each other is the Yukawa potential in each nucleus. Therefore, the hyper-radial Schrödinger equation with this potential must be solved separately for each nucleus. The potential placed in the D-dimensional equation is dependent on the hyper-radius as follows:

$$V(x) = -v_0 \frac{e^{-\alpha x}}{x}. \tag{21}$$

The dependence on hyper-radius x means, in general, that the potential has an N-body character. It is dependent only on the relative distance of the particles and it is invariant for any rotation in D-dimensional space. The nature of the potential is two-body, but it is dependent on the relative distance between the available particles. Herein, this potential is dependent on the relative distance between the N particles. In this way, and according to the eigenvalues of the centrifugal barrier, eq. (20) becomes

$$\left\{ \frac{-1}{2m}\frac{1}{x^{D-1}}\left[\frac{d}{dx}x^{D-1}\frac{d}{dx}\right] - \frac{\gamma(\gamma+D-2)}{x^2} - v_0\frac{e^{-\alpha x}}{x} \right\} \psi(x) = E\psi(x). \tag{22}$$

To simplify the equation, the following approximation is required [29,30]:

$$\frac{1}{x} \approx 2\alpha \frac{e^{-\alpha x}}{(1 - e^{-2\alpha x})}. \tag{23}$$

And we take changing the variables as follows:

$$y = 1 - e^{-2\alpha x}. \tag{24}$$

Substituting these into eq. (22) gives

$$y(1-y)\frac{d^2\psi(y)}{dy^2} - y\frac{d\psi(y)}{dy} - \left[\left(\frac{(D-1)(D-3)}{4} + \gamma(\gamma+D-2) \right)\frac{1}{y} - \frac{m(E_{n,l}-2)}{4\alpha^2}\frac{y}{1-y} - \frac{mv_0}{\alpha} \right]\psi(y) = 0. \tag{25}$$

A solution for this wave equation is [31]

$$\psi(y) = y^\mu (1-y)^\nu f(y). \tag{26}$$

Rewriting eq. (25) with this function gives

$$y(1-y)f''(y) + [2\mu(1-y) - 2\nu y - y]f'(y) + \left[\mu(\mu-1)\frac{(1-y)}{y} - 2\mu\nu + \nu(\nu-1)\frac{y}{(1-y)} - \mu + \nu\frac{y}{(1-y)} - \left(\frac{(D-1)(D-3)}{4} + \gamma(\gamma+D-2) \right)\frac{1}{y} + \frac{m(E-2)}{4\alpha^2}\frac{y}{(1-y)} + \frac{mV_0}{\alpha} \right] f(y) = 0. \tag{27}$$

Equation (27) is a hypergeometric equation with solution $f(y) = {}_2F_1(a, b, c; y)$. We have solved this hypergeometric equation; the coefficients are calculated as follows

$$a = \mu + \nu + \sqrt{\frac{-m(E-2)}{4\alpha^2}},$$

$$b = \mu + \nu - \sqrt{\frac{-m(E-2)}{4\alpha^2}}, \quad c = 2\mu, \tag{28}$$

$$\mu = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + (D-1)(D-3) + \frac{\gamma(\gamma + D - 2)}{4}},$$

$$\nu = i\sqrt{\frac{m(E-2)}{4\alpha^2}}.$$

Using these coefficients, the radial wave function of a nucleus with a Yukawa internal interaction is

$$\psi(x) = N(1 - e^{-2\alpha x})^{\mu} e^{-2\alpha \nu x} {}_2F_1(a, b, c; 1 - e^{-2\alpha x}), \tag{29}$$

where N is the normalization constant.

The wave function of each nucleus with a Yukawa internal interaction can be calculated by changing the number of nucleus particles and, consequently, the dimensions of Hilbert space ($D = 3N - 3$) using the above relation.

In the present work, we considered the $^{16}O + {}^{209}Bi$ interaction. Using the wave function obtained for each nucleus, we numerically calculated the DFM integral. The nuclear, Coulomb and total interaction potentials were obtained for $^{16}O + {}^{209}Bi$ and the numerical results are shown in figures 1–3.

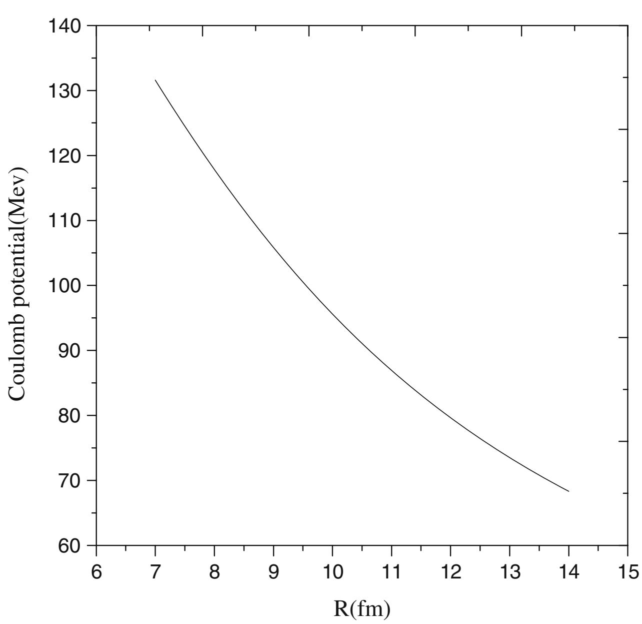

Figure 1. The Coulomb interaction potential for $^{16}O + {}^{209}Bi$.

4. The scattering problem

The scattering problem is studied using the partial wave method and the scattering cross-section is obtained based on the asymptotic behaviour of the wave function. It is necessary to consider that a plane wave is scattered by a scattering centre. The asymptotic form of the scattered wave function is [32,33]

$$\psi(r, \theta) \underset{r \to \infty}{\longrightarrow} e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{30}$$

The angle-dependent part of this wave is called the scattering amplitude. It is the amplitude of an outgoing

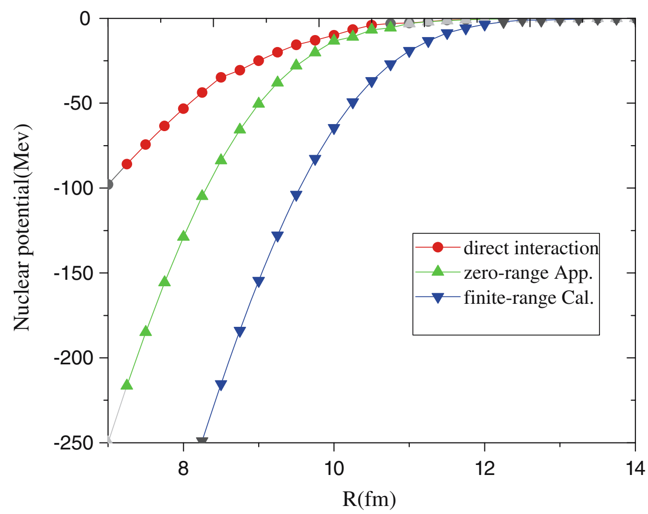

Figure 2. The nuclear interaction potential for $^{16}O + {}^{209}Bi$ obtained from the DFM using the D-dimensional space calculations.

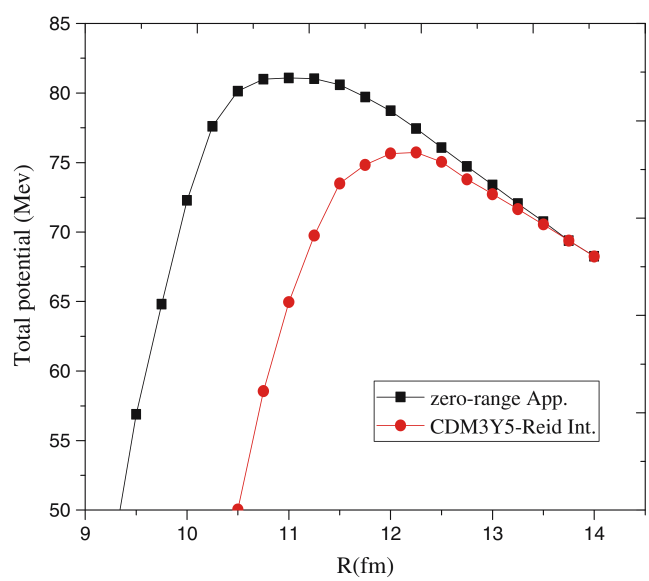

Figure 3. The total interaction potential for $^{16}O + {}^{209}Bi$ with zero-range approximation and CDM3Y5-Reid density-dependent finite-range calculation.

spherical wave relative to the incoming plane wave in a stationary-state scattering process. In the partial wave expansion, the scattering amplitude is represented as a sum over the partial waves [32,33]

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta), \tag{31}$$

where $f_l(k)$ is the partial amplitude and $P_l(\cos\theta)$ is the Legendre polynomial. The partial amplitude can be expressed using the scattering phase shift δ_l as

$$f_l(k) = \frac{\mathrm{e}^{2i\delta_l} - 1}{2ik} = \frac{\mathrm{e}^{i\delta_l}\sin\delta_l}{k}. \tag{32}$$

By this definition, the scattering amplitude in terms of the phase shift can be written as

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\mathrm{e}^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{33}$$

In general, the scattering amplitude is complex and is related indirectly to the energy. The differential cross-section can be obtained from the calculation of the outward flux of particles which are scattered through a spherical surface $r^2\mathrm{d}\theta$ for a large r divided by the incident flux and by the element of a solid angle $\mathrm{d}\theta$. Therefore, the differential scattering cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2. \tag{34}$$

Thus, the phase shift must be calculated first, as seen in the previous relations. We define a relation for the phase shift in the next section using the asymptotic behaviour of the hypergeometric function.

4.1 *The differential cross-section*

We have considered collision of two nuclei near the Coulomb barrier. As was mentioned, the phase shift is needed to obtain the cross-section. To illustrate this problem, it is necessary to explain how to obtain phase shift. From the classical mechanics, the two-body motion with masses m_1 and m_2 considered in the centre of mass coordinate system is equivalent to the motion of one particle with the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

and the relative radius-vector r in the central field $V(r)$ [33]. So, the hyper-radial Schrödinger equation (eq. (22)) with the reduced mass for $^{16}\mathrm{O} + {}^{209}\mathrm{Bi}$ collision in the following form is used:

$$\left\{\frac{-1}{2\mu}\frac{1}{x^{D-1}}\left[\frac{\mathrm{d}}{\mathrm{d}x}x^{D-1}\frac{\mathrm{d}}{\mathrm{d}x}\right] + \frac{\Gamma^2(\Omega)}{x^2} + V(x)\right\}\psi(x) = E\psi(x). \tag{35}$$

It is assumed that the two nuclei are influenced by their internal potential (Yukawa) and the total interaction potential at the Coulomb barrier obtained from the DFM. The hyper-radius is related to both nuclei here. It should be noted that according to the subjects mentioned in §3, calculations are performed in the centre of mass coordinate system and the potential depends only on the relative distance of the particles.

Therefore, the obtained wave function of eq. (35) is similar to the obtained wave function in §2 but with the reduced mass

$$\psi(x) = N(1-\mathrm{e}^{-2\alpha x})^{\mu}\mathrm{e}^{-2\alpha\nu x}{}_2F_1(a,b,c;1-\mathrm{e}^{-2\alpha x}). \tag{36}$$

This is the scattering wave function.

Next, the asymptotic behaviour of the obtained wave function is studied and the phase shift is calculated using the properties of hypergeometric functions.

For studying the asymptotic behaviour of the hypergeometric part of the wave function, we need the following two hypergeometric function properties [34]:

$$\begin{aligned}{}_2F_1(a,b,c;x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\\ &\times{}_2F_1(a,b;a+b-c+1;1-x)\\ &+(1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\\ &\times{}_2F_1(c-a,c-b;c-a-b+1;1-x),\end{aligned} \tag{37a}$$

$${}_2F_1(a,b,c;0) = 1. \tag{37b}$$

The hypergeometric part of the total wave function in eq. (36) can be written according to eq. (37a) as follows:

$$\begin{aligned}{}_2F_1(a,b,c;1-\mathrm{e}^{-2\alpha x}) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\\ &\times\ {}_2F_1(a,b;a+b-c+1;\mathrm{e}^{-2\alpha x})\\ &+(\mathrm{e}^{-2\alpha x})^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\\ &\times\ {}_2F_1(c-a,c-b;c-a-b+1;\mathrm{e}^{-2\alpha x}).\end{aligned} \tag{38}$$

According to eq. (37b) the asymptotic form of this function is given as

$$\begin{aligned}{}_2F_1(a,b,c;x)\xrightarrow[x\to\infty]{} &\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right.\\ &\left.+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\mathrm{e}^{4\nu\alpha x}\right\}.\end{aligned} \tag{39}$$

The above equation can be written [6,35] (δ is a real number) as

$$ {}_2F_1(a,b,c;x)\underset{x\to\infty}{\longrightarrow}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \times(\mathrm{e}^{i\delta}+\mathrm{e}^{(4\alpha\nu x-i\delta)}). \quad (40)$$

With $\nu = ik'$ ($k' = \sqrt{\mu(E-2)/4\alpha^2}$), the asymptotic form of the total hyper-radial wave function can be written as (see eq. (36)) [6,35]

$$\begin{aligned}\psi(x)\underset{x\to\infty}{\longrightarrow} & N\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \times(1-\mathrm{e}^{-2\alpha x})^{\mu}\mathrm{e}^{-2\alpha\nu x}(\mathrm{e}^{i\delta}+\mathrm{e}^{-i(4\alpha k' r+\delta)})\\ &= N\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \times(\mathrm{e}^{-i(2k'\alpha x-\delta)}+\mathrm{e}^{i(2k'\alpha x-\delta)})\\ &= N\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\cos(2k'\alpha x-\delta)\\ &= N\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\sin\left(\delta-2k'\alpha x+\frac{\pi}{2}\right).\end{aligned} \quad (41)$$

Now, it is helpful to compare the former equation and the general boundary condition of the scattering state wave function ($\varphi(r\to\infty) = 2\sin(kr-(\pi/2)\,l+\delta_l)$). With this comparison and the last relation obtained in eq. (41), a relationship can be written for the phase shift as follows [6,35]:

$$\delta_l = \frac{\pi}{2}(l+1) + \arg\Gamma(c-a-b) - \arg\Gamma(c-a) - \arg\Gamma(c-b). \quad (42)$$

This is the relationship that we are looking for. We have calculated differential scattering cross-section by this relation and according to eqs (33) and (34). The numerical results are presented in figure 4.

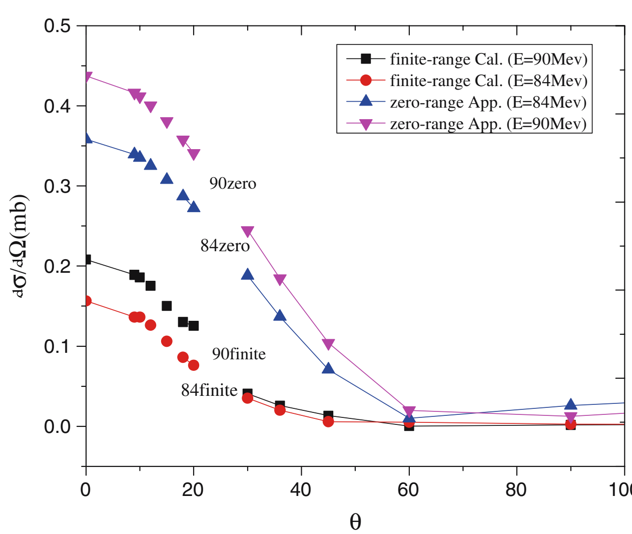

Figure 4. The differential scattering cross-sections at 84 and 90 MeV laboratory bombarding energies for ^{16}O+^{209}Bi.

5. Conclusions

The interaction potential for ^{16}O + ^{209}Bi by the DFM was calculated. The double folding model was defined using the integration on the nucleon density distributions of the two colliding nuclei and *NN* interaction potential. In the present study, we have used the wave functions of the colliding nuclei instead of their densities in the DFM integral. To calculate the wave function of each nucleus, the differential equation in the *D*-dimensional Hilbert space based on Jacobi transformation coordinates was solved. The exchange part of the interaction was taken to be of finite range and the results have been compared with the zero-range approximation. The density dependence of the *NN* interaction was accounted for. The numerical results of the nuclear, Coulomb and total interaction potential were presented in figures 1–3 using the obtained wave functions. In figure 1 the Coulomb interaction potential is shown. Figure 2 shows the difference between the direct and the exchange parts of the nuclear interaction for zero-range approximation and finite-range calculation. From this figure, we see that the direct part has larger values and tends to have a smaller slope toward zero. The slope is greater for the exchange part and the finite-range values tend toward zero slower than the zero-range values. In figure 3, the total interaction potential calculated with the zero-range approximation is compared with the finite-range calculation. In this figure, it is seen that the zero-range has a larger Coulomb barrier. The height of the Coulomb barrier for zero-range in $R = 11$ fm is 81.09 MeV and for the finite-range in $R = 12.25$ fm is 75.72 MeV. These values are in good agreement with the values of the previous works [36]. The differential scattering cross-section for the finite and zero-range calculations near the Coulomb barrier (84 and 90 MeV) were studied in figure 4. From this figure we see that the differential cross-section has larger values for zero-range in both energies while for the finite-range the values are closer together and the diagram has a greater slope in $E = 90$ MeV. The results of the finite-range calculation are in acceptable agreement with the results of ref. [36].

The results of our folding analysis shows that the wave functions of the two nuclei can also produce satisfactory results for nucleus–nucleus interaction using the DFM. The use of the different forms of the *NN* interaction (M3Y-Reid or Paris, zero-range or finite-range, density-dependent or independent) affects the results.

Acknowledgements

The authors thank the referee for the invaluable suggestions that have greatly helped them to improve this paper.

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