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Hypersurface-homogeneous Universe filled with perfect fluid in $f(R, T)$ theory of gravity

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Abstract. The exact solutions of the field equations with respect to hypersurface-homogeneous Universe filled with perfect fluid in the framework of $f(R, T)$ theory of gravity (Harko *et al, Phys. Rev. D* **84**, 024020 (2011)) is derived. The physical behaviour of the cosmological model is studied.

Keywords. $f(R, T)$ gravity; hypersurface-homogeneous Universe; perfect fluid.

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1. Introduction

Recent years have witnessed the emergence of the idea of an accelerating Universe and due to some results observed [1–4], it is now established that the Universe is accelerating. The prime factor in this setting is supposed to be dark energy (DE) which dominates the current cosmic picture. According to different estimations, dark energy (DE) occupies 73% of the energy of the Universe, while dark matter, about 23% and the usual baryonic matter occupies about 4%. In view of the late-time acceleration of the Universe and the existence of dark energy and dark matter, several modified theories of gravitation have been proposed as alternative to Einstein's theory. By modifying the geometrical part of Einstein–Hilbert action of general relativity, we obtain the modified gravity. Modified gravity is of great importance because it can successfully explain the rotation curve of galaxies and the motion of galaxy clusters in the Universe. There are various modified gravities, namely $f(R)$, $f(G)$, $f(R, G)$, $f(T)$ and $f(R, T)$ theories of gravity. Noteworthy amongst them are the $f(R)$ theory of gravity formulated by Nojiri and Odintsov [5]. The $f(R)$ theory of gravity helps to explore cosmic expansion. It is the simplest modification to general relativity as the Einstein–Hilbert action for this gravity contains an arbitrary function of the Ricci scalar $f(R)$. In this theory, a general function of Ricci scalar as $f(R)$ is replaced by R in the Einstein–Hilbert action, first discussed by Buchdahl [6].

The problem of dark matter can also be addressed by using viable $f(R)$ gravity models. Among the other modified theories, a theory of scalar-Gauss–Bonnet gravity, the so-called $f(G)$ [7,8] and a theory of $f(T)$ gravity [9,10], where T is the torsion, have been proposed to explain the accelerated expansion of the Universe. The $f(R, T)$ gravity, proposed by Harko *et al* [11], is the modification of $f(R)$ theory, where T dependence is induced by quantum effects or exotic non-ideal matter configurations. The coupling between matter and geometry in this gravity results in the non-geodesic motion of test particles and an extra acceleration is always present.

Myrzakulov [12] presented point-like Lagrangians for $f(R, T)$ gravity. The $f(R, T)$ gravity model that satisfies the local tests and transition of matter from the dominated era to the accelerated phase was considered by Houndjo [13]. Adhav [14] has obtained Bianchi type-I cosmological model in $f(R, T)$ gravity. Bianchi type-III and Kaluza–Klein cosmological models in $f(R, T)$ gravity have been discussed by Reddy *et al* [15,16]. Singh and Shri Ram [17] discussed the dynamics of anisotropic Bianchi type-III bulk viscous string model with magnetic field. Bianchi type-II string cosmological model in the presence of magnetic field in the context of $f(R, T)$ theory of gravity has been studied by Sharma and Singh [18]. Several researchers [19–36] studied the various cosmological models in $f(R, T)$ theory of gravity.

This work is motivated by the above research works, and the purpose of this paper is to study hypersurface-homogeneous space-time cosmological model in the frame of the newly established extension of the standard general relativity known as the $f(R, T)$ theory of gravity. In this paper, we study hypersurface-homogeneous space-time cosmological model filled with perfect fluid matter source in $f(R, T)$ gravity proposed by Harko *et al* [11]. In §1, a brief introduction of $f(R, T)$ gravity is given. The field equations in metric versions of $f(R, T)$ gravity are given in §2. In §3, an explicit form of field equations in $f(R, T)$ gravity by using the particular form of $f(T) = \lambda T$ is obtained. Further solutions of the field equations using the special law of variation for Hubble's parameter (proposed in [37]) are obtained in §4. Section 5 deals with the discussion of physical and kinematical properties of the model and some discussion and concluding remarks are given in §6.

2. Gravitational field equations of $f(R, T)$ gravity

The $f(R, T)$ theory of gravity is the modification of General Relativity (GR). The field equations are derived from a variational Hilbert–Einstein-type principle.

The action of modified gravity

$$S = \frac{1}{16\pi}\int \sqrt{-g} f(R,T)\mathrm{d}^4x + \int \sqrt{-g} L_m \mathrm{d}^4x, \quad (1)$$

where $f(R, T)$ is an arbitrary function of the Ricci scalar R and trace of the stress–energy tensor T of the matter T_{ij} ($T = g^{ij}T_{ij}$) and L_m is the matter Lagrangian density.

The stress–energy tensor of the matter is defined as

$$T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}}. \quad (2)$$

Assuming that the Lagrangian density L_m of the matter depends only on the metric tensor components g_{ij} and not on its derivatives, we obtain

$$T_{ij} = g_{ij}L_m - \frac{\delta(L_m)}{\delta g^{ij}}. \quad (3)$$

The $f(R, T)$ gravity field equations are obtained by varying the action S with respect to the metric tensor components g_{ij},

$$\begin{aligned} &f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + f_R(R,T) \\ &\times(g_{ij}\nabla^i\nabla_i - \nabla_i\nabla_j) = 8\pi T_{ij} - f_T(R,T)T_{ij} \\ &\qquad - f_T(R,T)\Theta_{ij}, \end{aligned} \quad (4)$$

where

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\delta g^{ij}\partial g^{\alpha\beta}}. \quad (5)$$

Here

$$f_R = \frac{\delta f(R,T)}{\delta R},$$

$$f_T = \frac{\delta f(R,T)}{\delta T},$$

$$\Theta_{ij} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ij}},$$

∇_i is the covariant derivative and T_{ij} is the standard matter energy–momentum tensor derived from the Lagrangian L_m. One should note that when $f(R, T) = f(R)$, eq. (4) reduces to the field equations of $f(R)$ gravity.

Contracting eq. (4), it gives the relation between Ricci scalar R and the trace T of the energy–momentum tensor as follows:

$$\begin{aligned} &f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) \\ &\quad = (8\pi - f_T(R,T))T - f_T(R,T)\Theta \end{aligned} \quad (6)$$

with

$$\Theta = g^{ij}\Theta_{ij}.$$

Using matter Lagrangian L_m, the stress–energy tensor of the matter is given by

$$T_{ij} = (p+\rho)u_i u_j - p g_{ij}, \quad (7)$$

where $u^i = (0, 0, 0, 1)$ denotes the four-velocity vector in co-moving coordinates which satisfies the condition $u^i u_i = 1$. The problem of the perfect fluids described by energy density ρ, pressure p and matter Lagrangian can be taken as $L_m = -p$, as there is no unique definition of the matter Lagrangian.

The variation of stress–energy of a perfect fluid has the following expression:

$$\Theta_{ij} = -2T_{ij} - p g_{ij}. \quad (8)$$

On the physical nature of the matter field, the field equations also depend on the tensor Θ_{ij}. Several theoretical models corresponding to different matter contributions for $f(R, T)$ gravity are possible. However, Harko *et al* [11] gave three classes of these models as follows:

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases}. \quad (9)$$

In this paper, we have focussed on the first model $f(R, T) = R + 2f(T)$, where $f(T)$ is an arbitrary function of the trace of the stress–energy tensor of matter of the form $f(T) = \lambda T$ where λ is a constant. For this choice, the gravitational field equations of $f(R, T)$ gravity becomes

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} - 2f'(T)T_{ij} -2f'(T)\Theta_{ij} + f(T)g_{ij}, \quad (10)$$

where the prime denotes a derivative with respect to the argument. If the matter source is a perfect fluid then the field equations (in view of eq. (8)) becomes

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} + 2f'(T)T_{ij} + [2pf'(T) + f(T)]g_{ij}. \quad (11)$$

3. Cosmological model and field equations

The general solutions of Einstein's field equations for a perfect fluid distribution satisfying a barotropic equation of state have been investigated by Stewart and Ellis [38] for the hypersurface-homogeneous space-time. We consider the hypersurface-homogeneous space-time of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - A^2(t)\mathrm{d}x^2 - B^2(t) \times [\mathrm{d}y^2 + \Sigma^2(y, K)\mathrm{d}z^2], \quad (12)$$

where A and B are functions of t only and $\Sigma(y, K) = \sin y, y, \sinh y$ respectively for $K = 1, 0, -1$.

A method to build exact solutions of field equations in the case of the metric (12) in the presence of perfect fluid have been developed by Hajj-Boutros [39] and exact solutions of the field equations are obtained which add to the rare solutions not satisfying the barotropic equation of state. Hypersurface-homogeneous bulk viscous fluid cosmological models with time-dependent cosmological term have been discussed by Chandel *et al* [40]. The exact solutions of the field equations for hypersurface-homogeneous space-time after assuming anisotropy of the fluid (dark energy) are obtained by Katore and Shaikh [41] in a scalar–tensor theory of gravitation. Katore and Shaikh [42] presented a class of solutions of Einstein's field equations describing two-fluid models of the Universe in hypersurface-homogeneous space-time.

We choose the function $f(T)$ of the trace of the stress–energy tensor of the matter so that

$$f(T) = \lambda T, \quad (13)$$

where λ is a constant [11].

Using co-moving coordinates and eqs (7), (8) and (13), the $f(R, T)$ gravity field equations, (11), for metric (12) can be written as

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = (8\pi + 3\lambda)p - \rho\lambda, \quad (14)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = (8\pi + 3\lambda)p - \rho\lambda, \quad (15)$$

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = -(8\pi + 3\lambda)\rho + p\lambda, \quad (16)$$

where a dot hereinafter denotes ordinary differentiation with respect to cosmic time t only.

4. Isotropization and the solution

The isotropy of the expansion can be parametrized after defining the directional Hubble's parameters and the average Hubble's parameter of the expansion. The directional Hubble's parameters in x, y, z directions for the hypersurface-homogeneous metric defined in (12) may be defined as follows:

$$H_x = \frac{\dot{A}}{A} \text{ and } H_y = H_z = \frac{\dot{B}}{B}. \quad (17)$$

The mean Hubble's parameter, H, is given by

$$H = \frac{\dot{R}}{R} = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right), \quad (18)$$

where R is the mean scale factor and $V = R^3 = AB^2$ is the spatial volume of the Universe.

The mean anisotropy parameter of the expansion A_m is defined as

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \quad (19)$$

in the x, y, z directions, respectively. The mean anisotropic parameter of the expansion A_m has a very crucial role in deciding whether the model is isotropic or anisotropic. It is the measure of deviation from isotropic expansion, the Universe expands isotropically when $A_m = 0$.

Let us introduce the dynamical scalars, such as expansion parameter θ and the shear σ^2 as usual:

$$\theta = 3H, \quad (20)$$

$$\sigma^2 = \frac{3}{2}A_m H^2. \quad (21)$$

The field equations (14)–(16) are a system of three independent equations containing four unknowns, viz. A, B, ρ, p. The solution of field equations generated by applying law of variation of Hubble's parameter was first proposed by Berman [37] in the FRW model, which yields a constant value of deceleration parameter. The deceleration parameter measures the rate at which expansion of the Universe is slowing down.

Here we can consider the constant deceleration parameter model defined by

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \text{constant}, \tag{22}$$

where R is the mean scale factor. Here the constant is taken as negative because $f(R, T)$ theory of gravity is about the accelerated expansion of the Universe.

The solution of (22) is

$$R = (AB^2)^{1/3} = \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{1/(1+q)}, \tag{23}$$

where $\alpha_1 \neq 0$ and α_2 are integration constants. Equation (23) implies that the condition of accelerated expansion is $1 + q > 0$ (as the scale factor R cannot be negative and we know that if $q > 0$ then $\mathrm{d}R/\mathrm{d}t$ is slowing down and if $q < 0$ then $\mathrm{d}R/\mathrm{d}t$ is speeding up).

As the field equations (14)–(16) are highly nonlinear, we use (23) and a physical condition that expansion scalar θ is proportional to shear scalar σ which gives

$$A = B^n (n \neq 1). \tag{24}$$

According to Thorne [43], the observations of the velocity red-shift relation for extragalactic sources suggest that Hubble expansion of the Universe is isotropic within about 30% range approximately (Kantowski and Sachs [44], Kristian and Sacks [45]) and red-shift studies place the limit $(\sigma/H) \leq 0.3$, on the ratio of shear σ to Hubble H in the neighbourhood of our galaxy today. The physical significance of this condition for perfect fluid and barotropic EoS in a more general case has been discussed by Collins *et al* [46]. Many researchers [47–50] use this condition to find exact solutions of cosmological models.

Using eq. (24), we solve the field equations (14)–(16) and obtain scale factors as

$$A = \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{3n/(1+q)(n+2)}, \tag{25}$$

$$B = \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{3/(1+q)(n+2)}. \tag{26}$$

The metric (12) with the help of (25) and (26) can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{6n/(1+q)(n+2)} \mathrm{d}x^2 - \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{6/(1+q)(n+2)} \times[\mathrm{d}y^2 + \Sigma^2(y, K)\mathrm{d}z^2]. \tag{27}$$

The model (27) represents an anisotropic hypersurface-homogeneous cosmological model with negative constant deceleration parameter model of the Universe in the presence of perfect fluid within the framework of $f(R, T)$ theory of gravity.

5. Physical properties of the model

Equation (27) represents a hypersurface-homogeneous Universe, with perfect fluid matter source, in $f(R, T)$ theory of gravity which is physically significant for the discussion of the early stages of evolution of the Universe. The different kinematical and physical parameters which are important for discussing the physics of the cosmological model (27) are given below.

The spatial volume V is defined as

$$V = \left[(q+1)(\alpha_1 t + \alpha_2)\right]^{3/(1+q)}. \tag{28}$$

Figure 1 indicates the behaviour of spatial volume with respect to time. It indicates that the spatial volume is zero at $t = -(\alpha_2/\alpha_1)$ and it increases as t increases. Thus, the Universe starts evolving with zero volume at $t = -(\alpha_2/\alpha_1)$ and expands with cosmic time t. This shows the late-time accelerated expansion of the Universe as $1 + q > 0$.

From (18), the average generalized Hubble's parameter H has the value given by

$$H = \frac{\alpha_1}{(1+q)(\alpha_1 t + \alpha_2)}. \tag{29}$$

The Hubble's parameter is infinite at $t = 0$ and approaches zero as $t \to \infty$.

From (19), (20) and (21), the dynamical scalars are given by

$$\text{Expansion scalar } \theta = \frac{3\alpha_1}{(1+q)(\alpha_1 t + \alpha_2)}. \tag{30}$$

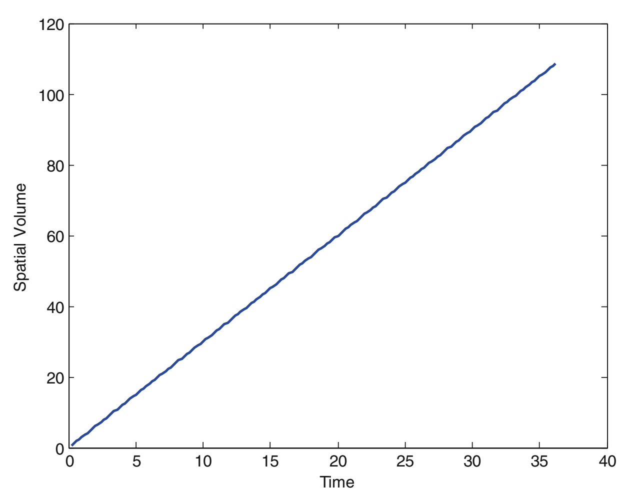

Figure 1. Spatial volume vs. time when $q = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 0$.

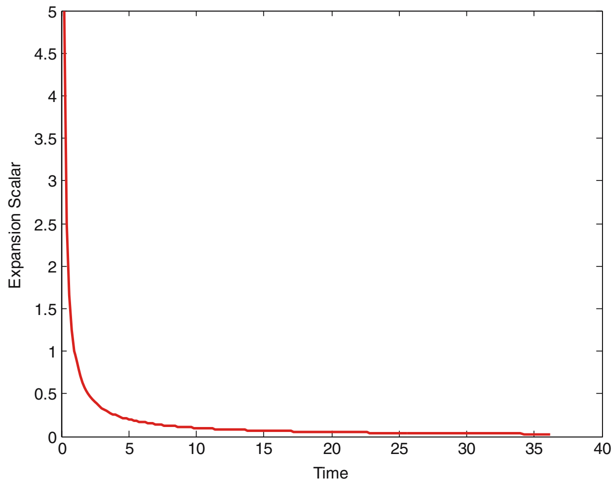

Figure 2. Expansion scalar vs. time when $q = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 0$.

We observe that the scalar expansion diverges at the initial epoch, i.e. $t = 0$; while for large value of t, the expansion scalar vanishes (shown in figure 2).

$$\text{Anisotropic parameter } A_m = \frac{3(n^2+2)}{(n+2)^2}. \tag{31}$$

As A_m is constant, the mean anisotropic parameter is uniform throughout the evolution of the Universe.

$$\text{Shear scalar } \sigma^2 = \frac{9}{2}\frac{\alpha_1^2(n^2+2)}{\left[(1+q)(n+2)(\alpha_1 t+\alpha_2)\right]^2}. \tag{32}$$

The shear scalar vanishes, for large value of t, which is shown in figure 3 and hence the shape of the Universe remains unchanged during the evolution.

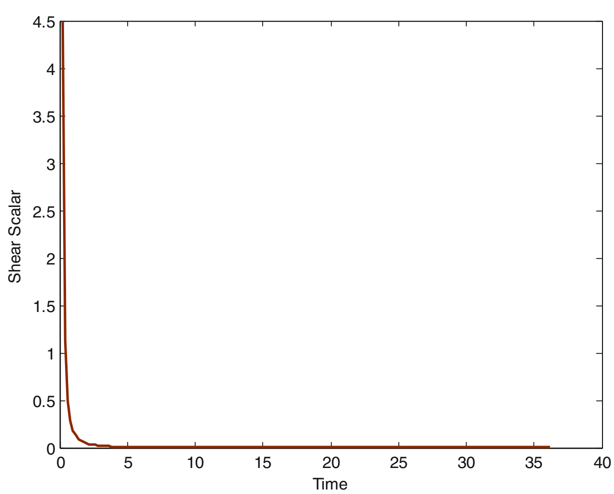

Figure 3. Shear scalar vs. time when $n = 0.5$, $q = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 0$.

The physical parameters of the cosmological model (27), energy density ρ and pressure p are obtained as

$$\begin{aligned}\rho = {} & \frac{\lambda^2}{(8\pi+3\lambda)(8\pi+4\lambda)\left[(1+q)(n+2)(\alpha_1 t+\alpha_2)^2\right]} \\ & \times \left\{ \frac{3\alpha_1^2[n^2(2-q)-q(3n+2)+1]}{\lambda} + \frac{6\alpha_1^2(1-qn-2q-n)-18n\alpha_1^2}{8\pi+2\lambda} \right\} \\ & - \frac{18n\alpha_1^2 \times 9\alpha_1^2}{(8\pi+3\lambda)\left[(1+q)(n+2)(\alpha_1 t+\alpha_2)\right]^2} \\ & - \frac{K}{(8\pi+3\lambda)\left[(1+q)(\alpha_1 t+\alpha_2)\right]^{\frac{6}{(n+2)(1+q)}}}. \end{aligned} \tag{33}$$

From (33), it is observed that the energy density ρ is a decreasing function of time and it approaches zero as time tends to infinity. This behaviour is clearly depicted in figure 4 as a representative case with appropriate choice of constants of integration and other physical parameters using reasonably well-known situations.

$$\begin{aligned}p = {} & \frac{\lambda}{(8\pi+3\lambda)\left[(1+q)(n+2)(\alpha_1 t+\alpha_2)^2\right]} \\ & \times \left\{ \frac{3\alpha_1^2[n^2(2-q)-q(3n+2)+1]}{\lambda} + \frac{6\alpha_1^2(1-qn-2q-n)-18n\alpha_1^2}{8\pi+2\lambda} \right\}. \end{aligned} \tag{34}$$

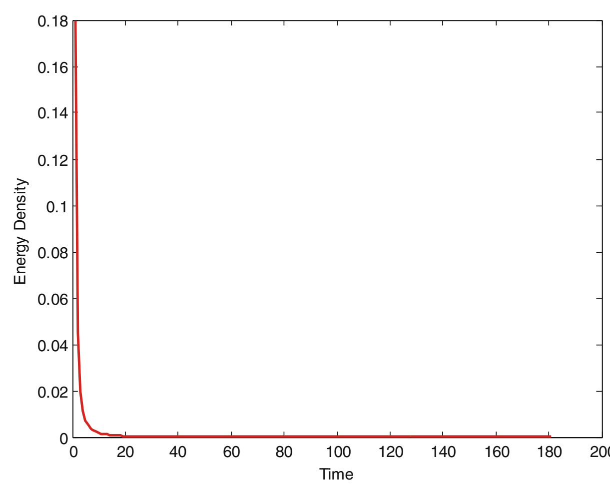

Figure 4. Energy density vs. time when $n = 0.5$, $q = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\lambda = 0.01$.

6. Discussion and conclusion

Equations (25) and (26) give solutions of hypersurface-homogeneous cosmological model in $f(R, T)$ theory of gravity. It is observed that the model (27) has no singularity at $t = -(\alpha_2/\alpha_1)$ when $n > 0$. It is also observed that at $t = -(\alpha_2/\alpha_1)$, the proper volume will be zero, whereas when $t \to \infty$, the spatial volume becomes infinitely large. At $t = -(\alpha_2/\alpha_1)$ the expansion scalar θ and shear scalar σ tend to infinity whereas when $t \to \infty$, expansion scalar θ and shear scalar σ tend to zero. The energy density ρ is a decreasing function of time. The Hubble's parameter H decreases with the increase of time. Also, as $1 + q > 0$, the models represent accelerating Universe. It is interesting to note that the results obtained resembles the results from the investigations of Samanta [26].

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