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Phenomenological approach to describe logistic growth and carrying capacity-dependent growth processes

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Abstract. In this communication, different classes of phenomenological universalities of carrying capacity-dependent growth processes have been proposed. The logistic as well as carrying capacity-dependent West-type allometry-based biological growths can be explained in this proposed framework. It is shown that logistic and carrying capacity-dependent West-type growths are phenomenologically identical in nature. However, there is a difference between them in terms of coefficients involved in the phenomenological descriptions. Involuted Gompertz function, used to describe biological growth processes undergoing atrophy or a demographic and economic system undergoing involution or regression, can be addressed in this proposed environment-dependent description. It is also found phenomenologically that the energy intake of an organism depends on carrying capacity whereas metabolic cost does not depend on carrying capacity. In addition, some other phenomenological descriptions have been examined in this proposed framework and graphical representations of variation of different parameters involved in the description are executed.

Keywords. Logistic growth; exogenous process; carrying capacity; metabolic cost.

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1. Introduction

Generally, microscopic and macroscopic approaches, with some of their limitations, are frequently used to describe temporal evolution of systems occurring in physics, biology, statistics and economics. A midway description of the system, termed as mesoscopic approach, may be helpful in different cases [1–3], but the coordination among different level of description may not be easy [4]. On the other hand, the proposal of different types of universality classes prepares the background for studying a dynamical system based on the phenomenological approach [4–10]. In each class, a characteristic parameter of different systems shares common sets of properties. The field of application and the nature of the variables involved are completely ignored in this approach. This new approach is used for the analysis of experimental datasets and cross-fertilization among different fields. In fact, this method indicates similarities to be detected among datasets in totally different fields of application and serves as a magnifying glass upon them, enabling one to extract all the available information in a generalized way. Therefore, a classification scheme in universality classes of broader categories of phenomenologies, belonging to different disciplines, is emerging as a useful tool for recognizing the characteristic feature of a system and cross-fertilization among different branches of science.

The main aim of the phenomenological approach [11] is the formulation of the family of classes (UN), which at the level $N = 0$ corresponds to autocatalytic processes. At the level $N = 1$ (U1), it produces the Gompertz law [12] which was first derived empirically. It has been used to describe most of the diversified growth phenomena, for more than a century. The class U2 includes the model of West and his collaborators [9,10]. Different types of applications of the classes U1 and U2 have been reported in diversified and unrelated fields; e.g. auxology [13], nonlinear elasticity

[14], oncology [15,16] and fracture dynamics [17]. It is also found useful to detect the phase of inflation growth [18]. In nonlinear problems, this approach shows the nonscaling invariance to be extracted by means of suitable redefined fractal-dimensioned variables [13]. An extension of this phenomenological approach in terms of complex variable(s), termed as complex univeralities of growth, is used to explain concurrent growth of phenotypic features of a system [19,20].

It is found that different systems proposed by Castorina *et al* [11] are capable of attaining carrying capacity of the corresponding system. The specific growth rate in phenomenological approach can be expressed in terms of power series of specific growth rate. The coefficients (b_n's) of the power series may be termed as phenomenological coefficients. Different parameters involved in their description are also dependent only on the values of phenomenological coefficients (b_n's) [11]. As an example, well-known West-type equation of ontogenetic growth model can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha_1 y^{3/4} - \alpha_2 y, \tag{1}$$

where α_1 and α_2 are parameters of eq. (1). The first term ($\alpha_1 y^{3/4}$) of eq. (1) represents the energy intake by the organism and the second term ($\alpha_2 y$) stands for the metabolic cost. According to Castorina *et al* [11], the energy intake of the West-type growth equation does not depend on environmental conditions. It is expected that the energy intake should be dependent on environmental conditions. Here the carrying capacity is controlled by the phenomenological coefficients indicating growth mechanisms of the system. The values of the phenomenological coefficients (b_n's) are determined by the nature of growth mechanism of the system [11]. As an example, the value of b_2 for different growing systems can be explained with the help of fractal-like distribution for energy transformation [7,9,10]. So, the values of the phenomenological coefficients are not determined by environmental constraints. Therefore, the carrying capacity does not depend on environmental constraints in these cases. The carrying capacity and the maximum attainable value ($y_{\max}$) of the dynamical variable y are synonymous in these cases. But there are different growing systems which cannot reach carrying capacity [21]. Another important growth equation addressed by Castorina *et al* [11] is the logistic growth equation, frequently used in population dynamics. The usual logistic growth equation can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \beta_1 y - \beta_2 y^2, \tag{2}$$

where β_1 and β_2 are parameters of eq. (2). They have tried to describe the logistic growth within the phenomenological class of West-type equation of biological growth. But it is not possible to capture the feature of logistic growth processes with the phenomenological approach proposed by Castorina *et al* [11]. Detailed discussion regarding it is done by Biswas and Poria [22]. One of the important factors in the classification scheme of the phenomenological universalities is the initial conditions. Therefore, new initial conditions may be considered to define a new class whose growth is affected by environmental constraints like carrying capacity. In the present communication, following the formalism developed by Castorina *et al* [11], a new type of phenomenological description based on proposed initial conditions, is considered to describe the temporal evolution of the system. In this description of the system, environment-dependent West-type biological growth and logistic-type growth pattern can be addressed in the same framework. It is established that there is no difference in the underlying phenomenological descriptions between the environment-dependent West-type and the logistic-type growth patterns. The only difference is in terms of the value of the coefficients involved in the phenomenological description. A switching between environment-dependent West-type and logistic-type growth patterns is shown in a proper way. It is also shown that the biological growth following West-type growth equation proposed by Castorina *et al* [11] is different from the proposed biological growth governed by environment-dependent West-type growth equation. It is also found that the energy intake of an organism depends on environmental conditions whereas the metabolic cost does not depend on environmental conditions. Involuted Gompertz function, used to analyse growth of a biological system undergoing atrophy or a demographic or economic system following involution or regression [23], can also be addressed from the proposed phenomenological point of view. In addition, some phenomenological descriptions in this framework have been explained with its minute features, along with graphical representation.

In a nutshell, the phenomenological framework proposed by Castorina *et al* [11] does not address logistic growth properly. It also does not describe involuted Gompertz growth function and the dependence of energy intake or metabollic cost of an organism on carrying capacity. These facts motivate us to address these issues with the help of phenomenological description. The paper is organized as follows: In §2, we propose a phenomenological description of carrying capacity-dependent growth processes. In this connection, the classification scheme of phenomenological universalities proposed by Castorina *et al* [11] is discussed in brief with different initial conditions. Different aspects of the proposed phenomenological classes are considered in §3. Logistic growth and involuted Gompertz growth are explained using phenomenological framework in this section. Dependence of energy intake of an organism on carrying capacity is also discussed in this section. Finally, we conclude with our results in §4.

2. The phenomenological description

In a generalized way, any growth phenomenon may be described by a simple relation as given below [11].

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \alpha(t)Y(t), \tag{3}$$

where $\alpha(t)$ represents specific growth rate of a given dynamical variable $Y(t)$. $Y(t)$ may vary with some other independent characteristic variable of the system. But, the evolution of the system with respect to time is considered in the phenomenological approach, treating others as constant. Therefore, an ordinary differential equation, instead of a partial one, will serve the purpose. Now, eq. (3) can be expressed in terms of nondimensional variables, by introducing three nondimensional variables $\tau = \alpha(0)(t - t_0)$, $y(\tau) = Y(t)/Y(0)$ and $a(\tau) = \alpha(t)/\alpha(0)$ (t_0 is a time which is less than t) as

$$\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} = a(\tau)y(\tau). \tag{4}$$

Now, the time variation of $a(\tau)$ is defined through the function,

$$\varphi(a) = -\frac{\mathrm{d}a}{\mathrm{d}\tau}. \tag{5}$$

The explicit form of $\varphi(a)$ will generate a variety of growth patterns. $\varphi(a)$ can be expressed in terms of a power series as

$$\varphi(a) = \sum_0^\infty b_n a^n. \tag{6}$$

From eqs (4) and (5), the following relation can be established:

$$\ln y = -\int \frac{a\mathrm{d}a}{\varphi} + \text{constant}. \tag{7}$$

Equation (7) can also be expressed in the following form:

$$\ln y = \int a\mathrm{d}\tau + \text{constant}. \tag{8}$$

The growth processes may be endogenous or exogenous by nature. These natures do not depend on environmental constraints. Both of them can be addressed with the help of this classification scheme of the phenomenological universalities of growth [20]. Equations (4) and (5) along with eq. (6) generate different types of universality classes in the growth processes. One of the important factors in this classification scheme is the initial conditions that may depend upon its environmental conditions like carrying capacity. The initial conditions proposed by Castorina *et al* [11] may be independent of the environmental conditions. Different terms of eq. (6) represent several growth mechanisms of the system [11]. But the growth processes are expected to be influenced by the environmental conditions. One such environmental condition may be the carrying capacity. Carrying capacity of a growth process may be defined as the number of individuals of a given species (or the value of the dynamic variable) that a given habitat can support without being permanently damaged [24,25]. If the population exceeds the carrying capacity in a population dynamics, then the population will crash due to poisoning or the resources required to meet the requirements of that species will be exhausted [26]. A general confusion among the researchers is observed in relation to the carrying capacity of the environment and the maximum value of the dynamic variable [$y_{\max}$]. The difference between these two is very interestingly demonstrated with the help of a thought experiment by Hui [27]. An environment's carrying capacity may be defined as its maximum indefatigably supportable load [28].

It is generally expected that the value of the dynamical variable (y) of a growing system will try to attain a saturation level [$y_{\max}$] that may or may not be different from the carrying capacity. As the value of the dynamical variable approaches $y_{\max}$ that may be different from the carrying capacity [21], the growth rate of a system decreases with time (or, with dynamical variable). The growth rate stops when dynamical variable attains $y_{\max}$ (or, carrying capacity). Therefore, higher value of carrying capacity of the environment may favour sustainable growth and the growth rate at a time may depend on the fractional value (of carrying capacity) attained by the dynamical variable. Considering these facts, it is justified to consider that the growth rate may have different types of functional dependence on the fractional value (of carrying capacity) attained by the dynamical variable. Keeping these facts in mind, we propose here the initial conditions as $y(0) = c$ and $a(0) = a_0 = q(1 - (c/K))$, where q and c are constant quantities and c must be less than K (to continue a growth process). We have proposed here one of simplest functional dependence of specific growth rate on the fractional value (of the carrying capacity) attained by the dynamical variable. There may be different types of functional dependence of the specific growth rate on the fractional value of the carrying capacity attained by the system. q can be treated as the ideal specific growth rate that may not be attained by the system. Ideal specific growth rate may be equal to (one of the) phenomenological coefficient(s). In reality, actual specific growth rate may be less than q. It may be due to some constraints imposed by its environment. In these proposed initial conditions, K is a constant representing the carrying capacity of the environment that may or may not be attained by the system [21]. In the following sections, we have considered $c = 1$ for the sake of simplicity. The other values of c will not alter the behaviour of different growth processes discussed in this communication.

The behaviour of the system corresponding to constant specific growth rate, i.e. $\varphi = 0$ ($b_n = 0$ for any n) is represented by the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \pm r y \tag{9}$$

where r is a constant.

When r is a positive constant, the system shows an exponential growth. The system shows an exponential decay when $\varphi = 0$ and a is equal to a constant (less than zero). When $\varphi = b_0$ and $q = b_0$, the system is governed by the following relation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - b_0 y. \tag{10}$$

The solution of eq. (10) is obtained as

$$y = \exp\left[-\frac{b_0\tau^2}{2} + b_0\left(1 - \frac{1}{K}\right)\tau\right]. \tag{11}$$

Now, for $b_0 = 0$, $q = b_1$ and $\varphi(a) = b_1 a$ with $n = 1$ in eq. (6), the growth pattern is governed by the following expression:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = b_1 y\left[\left(1 - \frac{1}{K}\right) - \ln y\right] \tag{12}$$

with the solution,

$$y = \exp\left[\left(1 - \frac{1}{K}\right)(1 - \exp(-b_1\tau))\right]. \tag{13}$$

The system corresponding to the phenomenological class represented by $\varphi = b_0 + b_1 a$ with all $b_n = 0$ for $n > 1$, follows

$$y = \exp\left[\left(\frac{b_0}{b_1^2} + \frac{a_0}{b_1}\right)(1 - \exp(-b_1\tau)) - \frac{b_0}{b_1}\tau\right]. \tag{14}$$

When $b_0 = b_1 = 0$ and $\varphi(a) = b_2 a^2$ with $n = 2$, the corresponding differential equation showing the growth pattern for the condition $q = b_2$ is expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = b_2\left(1 - \frac{1}{K}\right) y^{1-b_2} \tag{15}$$

with the solution

$$y = \left[b_2^2\left(1 - \frac{1}{K}\right)\tau + 1\right]^{1/b_2}. \tag{16}$$

When $b_0 = 0$ and $n = 2$, $\varphi(a) = b_1 a + b_2 a^2$, the behaviour of the system for the condition $q = b_1$ is governed by the differential equation given as

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{b_1}{b_2}(y^{1-b_2} + \gamma y^{1-b_2} - y), \tag{17}$$

where $\gamma = (b_2/K)(K - 1)$.

The solution of eq. (17) is

$$y = [1 + \gamma - \gamma \exp(-b_1\tau)]^{1/b_2}. \tag{18}$$

This is similar to the solution derived by Castorina *et al* [11], with a little difference in the value of γ. Table 1 represents a comparison of growth functions proposed in this paper with its known form [11] of different phenomenological classes.

3. Discussions

3.1 *Variation of growth features for different phenomenological classes*

In the phenomenological class defined by $\varphi = 0$, the system shows an exponential growth or decay based on the value of the constant quantity which is equal to a_0. The system exhibits an exponential decay when the constant quantity is less than zero. One of the systems in which this type of decay is found is the radioactive system. The system follows exponential growth when the value of the constant quantity is greater than zero. One such system is the autocatalyst driven system. Generally, a growing system is termed as endogenous when specific growth rate can be expressed explicitly as a function of y. If the specific growth rate is expressed explicitly as a function of time, the growing

Table 1. Comparison of growth equations related to different phenomenological classes.

Phenomenological class	Proposed growth function	Growth function found in [11]
$\varphi(a) = b_0$	$y = \exp\left[-\frac{b_0\tau^2}{2} + b_0\left(1-\frac{1}{K}\right)\tau\right]$	-
$\varphi(a) = b_1 a$	$y = \exp\left[\left(1-\frac{1}{K}\right)(1-\exp(-b_1\tau))\right]$	$y = \exp[1-\exp(-\tau)]$ for $b_1 = 1$
$\varphi(a) = b_0 + b_1 a$	$y = \exp\left[\frac{1}{b_1^2}(1-\exp(-b_1\tau)) - \frac{b_0}{b_1}\tau\right]$ for $K = b_1$ and $b_0 b_1 = 1$	-
$\varphi(a) = b_2 a^2$	$y = \left[b_2^2\left(1-\frac{1}{K}\right)\tau + 1\right]^{1/b_2}$ Linear growth for $b_2 = 1$	-
$\varphi(a) = b_1 a + b_2 a^2$	$y = \left[1+\gamma-\gamma\exp(-b_1\tau)\right]^{1/b_2}$ where $\gamma = \frac{b_2}{K}(K-1)$ Logistic growth for $b_2 = -1$ West-type growth for $b_2 = 0.25$	$y = [1+b-b\exp(-\tau)]^{1/b}$ for $b_1 = 1$ and $b_2 = b$ West-type growth for $b_2 = 0.25$

system is called exogenous [20]. As a is not function of τ or y in this case, the growth is not endogenous or exogenous by nature. Therefore, it can be treated as a spontaneous process from the phenomenological point of view.

For the phenomenological description corresponding to $\varphi = b_0$, the system exhibits a growth at the early stage when $b_0 > 0$. Thereafter, the system shows an exponential decay and remains asymptotic to the adimensional time axis. The growth rate or decay rate increases with the increase in K when b_0 remains constant. The opposite nature is found in the system when b_0 is less than zero. The same nature is exhibited by the system with the variation in b_0, treating K as a constant. The growth rate or decay rate increases with the increase in b_0 when K behaves like a constant. One interesting feature found in this case is that the duration of growth is controlled by the magnitude of K, and not by the magnitude of b_0. The findings are represented graphically in figure 1a. The maximum of the characteristic growth pattern occurs at $\tau = (1 - \frac{1}{K})$. The specific growth rate (a) cannot be expressed explicitly in terms of y. It can only be expressed explicitly in terms of τ. Therefore, it may be concluded from theoretical point of view that such systems may be exogenous by nature [20,29].

The system corresponding to the phenomenological class represented by $\varphi = b_1 a$, shows a growth very similar to Gompertz law of growth [12]. Figure 1b shows that the growth rate increases with the increase of b_1 when K behaves like a constant quantity. The time required to attain saturation level decreases significantly with the increase of b_1 in this case. The growth rate and time required to attain saturation level vary significantly with K, even if b_1 remains constant. The system behaves like an endogenous system [20]. It shows a linear behaviour with respect to time when $b_2 = 1$ in case of $\varphi = b_2 a^2$, as shown in figure 1c. It is also an example of the endogenous system.

Equation (14) can be expressed for $q = b_0$, $K = b_1$ and $b_0 b_1 = 1$ as

$$y = \exp\left[\frac{1}{b_1^2}(1-\exp(-b_1\tau)) - \frac{b_0}{b_1}\tau\right]. \tag{19}$$

This is termed as involuted Gompertz function [23] (shown in figure 1d) generally used to describe the growth process of a biological system undergoing atrophy. Such a situation is found in avian primary lymphoid organs – thymus and bursa of Fabricius as well as in thymus of mammalians [23]. It is also useful for a demographic and economic system undergoing involution or regression. The growth process corresponding to this class is exogenous [20].

Equation (17) can be expressed in the following way:

$$\frac{dy}{d\tau} = C_1 y^{\sigma} - C_2 y, \tag{20}$$

where

$$C_1 = \frac{b_1(1+\gamma)}{b_2}, \quad \sigma = 1 - b_2 \quad \text{and} \quad C_2 = \frac{b_1}{b_2}.$$

This is very similar to the expression proposed by West *et al* for the allometric growth of biological systems

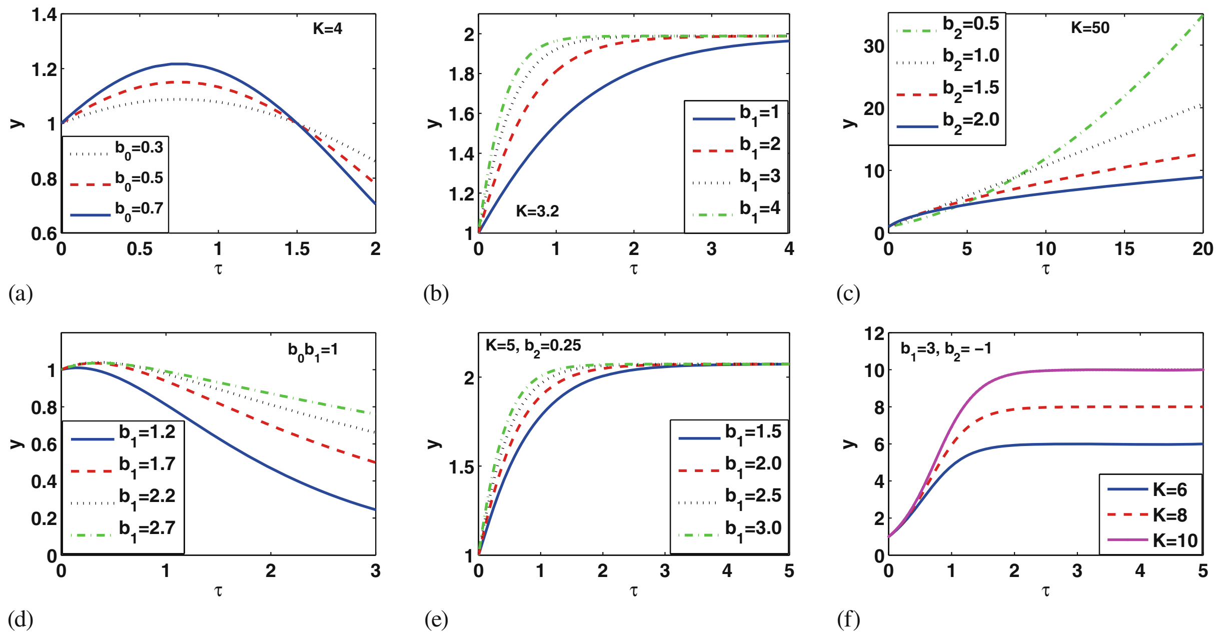

Figure 1. (**a**)–(**f**) are growth curves of different phenomenological classes. (**a**) For $\varphi = b_0$, the values of the parameter b_0 from the top to the bottom are 0.3, 0.5 and 0.7 for $K = 4$. (**b**) For $\varphi = b_1 a$, the values of the parameter b_1 from the top to the bottom are 4, 3, 2 and 1 for $K = 3.2$. (**c**) For $\varphi = b_2 a^2$, the values of the parameter b_2 from top to the bottom are 0.5, 1, 1.5 and 2 for $K = 50$. The curve, second from the top, represents a straight line. (**d**) For $\varphi = b_0 + b_1 a$ with $b_0 b_1 = 1$, the values of the parameter b_1 from the top to the bottom are 2.7, 2.2, 1.7 and 1.2. (**e**) For $\varphi = b_1 a + b_2 a^2$ with $b_2 = 0.25$ and $K = 5$, the values of the parameter b_1 from the top to the bottom are 3, 2.5, 2 and 1.5. All of them represent West-type biological growth. (**f**) For $\varphi = b_1 a + b_2 a^2$ with $b_1 = 3$ and $b_2 = -1$, the values of the parameter K from the top to the bottom are 10, 8 and 6. All of them represent the usual logistic growth.

when y represents the mass of the system and $b_2 = 0.25$ [7,10]. Figure 1e shows that the time required to attain saturation level decreases with the increase in b_1 when K is constant. The phenomenological class represented by eq. (17) shows endogenous growth process.

The first term of the West-type growth equation represents the energy intake of an organism and the second term represents the metabolic cost. The energy intake is expected to be dependent on environmental conditions such as carrying capacity. The expressions of C_1 and C_2 show that the energy intake depends on the carrying capacity whereas the metabolic cost does not depend on the carrying capacity.

Equation (18) is similar in nature to the expression given by Castorina *et al* [11] as

$$y = [1 + b - b\exp(-\tau)]^{1/b}. \tag{21}$$

The only difference is in terms of the parameters involved in the expression. Therefore, the specific growth rate (a) corresponding to eq. (17) is represented by

$$a = \frac{b_1(K-1)}{(1.25K - 0.25)\exp(b_1\tau) - 0.25(K-1)} \tag{22}$$

and the corresponding mass (when dynamic variable y represents mass) is expressed as

$$m = \left[1 + \frac{0.25(K-1)}{K}(1 - \exp(-b_1\tau))\right]^4. \tag{23}$$

This is similar in nature to the expression derived by Biswas *et al* [30]. It is found that the proposed biological growth leading to the environment-dependent West-type growth equation does not attain carrying capacity. There are different types of growing systems which cannot attain carrying capacity [21]. When $b_2 = -1$, eq. (17) can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = b_1 y\left(1 - \frac{y}{K}\right). \tag{24}$$

The solution of eq. (24) is expressed as

$$y = \frac{K}{1 + (K-1)\exp(-b_1\tau)}. \tag{25}$$

This is the logistic growth equation that is frequently used to describe different types of growth phenomena in diversified fields [31–34]. Figure 1f represents the variation of dynamic variable y with time in case of logistic growth for different values of K when b_1 and b_2 are constants. It is found that the time required to attain saturation level increases slowly with the increase in K. The specific growth rate corresponding to the logistic growth can be expressed in terms of adimensional time (τ) as

$$a = \frac{b_1(K-1)}{(K-1)+\exp(b_1\tau)}. \tag{26}$$

Therefore, the value of $b_2 = -1$ initiates the usual logistic growth whereas $b_2 = 0.25$ indicates West-type biological growth in the proposed phenomenological description.

3.2 *West-type biological growth and logistic growth*

The variation of specific growth rate with time corresponding to the logistic growth as well as the environment-dependent West-type biological growth is shown graphically in figures 2a–2e. In the case of logistic growth, it is found that specific growth rate changes with b_1 and K. Rate of this change increases with the increase of b_1 and K. The higher value of b_1 initiates higher initial specific growth rate, but it significantly lowers the time required to attain saturation level, as shown in figure 2a. The variation of K with time required to attain saturation level is not much significant (shown in figure 2b) as it is found in the case of b_1. Similar trend is found in the case of variation of b_1 for the environment-dependent West-type biological growth (shown in figure 2c). It is found in figure 2d that the effect of change of K on specific growth rate is negligible in this case. The specific growth rate corresponding to the West-type biological growth equation [11] with $\varphi = b_1 + b_2a^2$ and $y(0) = a(0) = 1$ can be expressed as

$$a = \frac{b_1}{(b_1+0.25)\exp(b_1\tau)+0.25}. \tag{27}$$

When $b_1 = 1$, it leads to the environment-independent West-type biological growth proposed by Castorina *et al* [11]. But the other values of b_1 ($b_1>0$) also lead to West-type biological growth equation that does not explicitly depend on carrying capacity. The rate of change of specific growth rate increases with the increase of b_1 in this case, as shown in figure 2e.

In the phenomenological class defined by $\varphi = a + ba^2$ with $y(0) = a(0) = 1$, as proposed by Castorina *et al* [11], the specific growth rate is given by

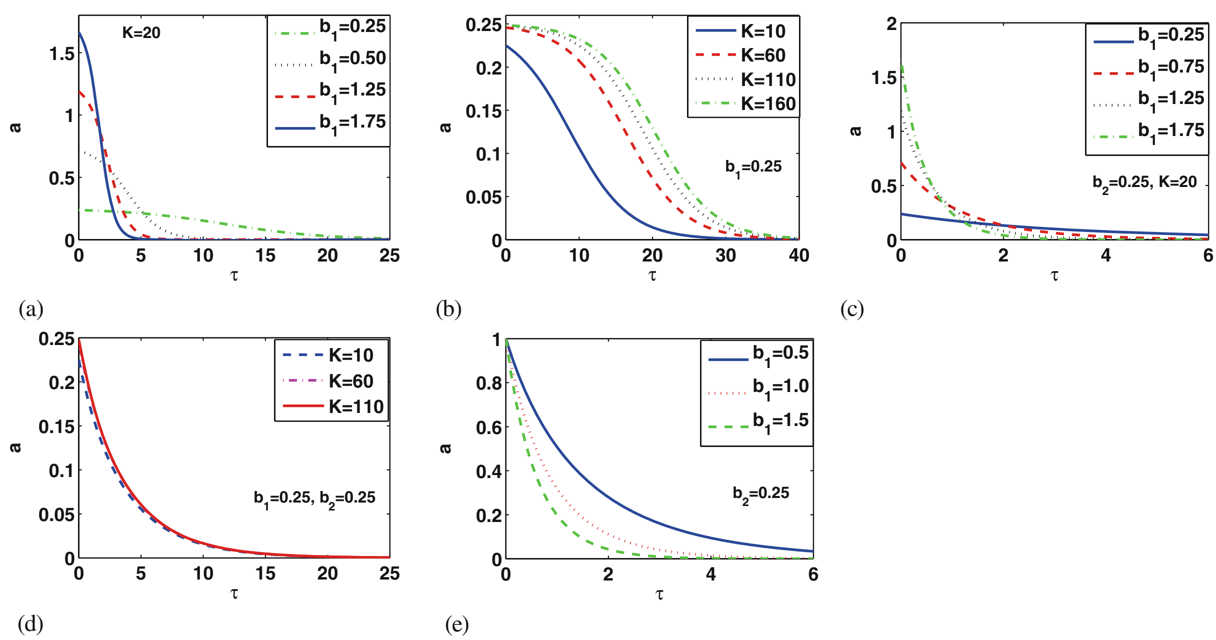

Figure 2. (**a**) and (**b**) are curves of specific growth rate corresponding to logistic growth ($b_2 = -1$). (**a**) The values of the parameter b_1 from the top to the bottom are 1.75, 1.25, 0.75 and 0.25 for $K = 20$. (**b**) The values of the parameter K from the top to the bottom are 160, 110, 60 and 10 for $b_1 = 0.25$. (**c**)–(**e**) are curves of specific growth rate corresponding to the West-type biological growth ($b_2 = 0.25$). (**c**) The values of the parameter b_1 from the top to the bottom are 1.75, 1.25, 0.75 and 0.25 for $K = 20$. (**d**) The values of the parameter K from the top to the bottom are 110, 60 and 10 for $b_1 = 0.25$. (**e**) The values of the parameter b_1 from the top to the bottom are 0.5, 1.0 and 1.5. The curve, second from the top, corresponds to the West-type biological growth proposed by Castorina *et al* [11].

$(1+(1/b))y^{-b}-(1/b)$. The maximum attainable value of y [$y_{\max}$] is given by $[1+b]^{1/b}$. Therefore, $y_{\max}$ and the specific growth rate do not depend on environmental conditions which may be defined in terms of carrying capacity. These are solely governed by one of the growth mechanism of the system (because Castorina *et al* have concluded that the term ba^2 represents one of the growth mechanism of the system [11]). Therefore, the growth processes described by Castorina *et al* [11] do not depend on environmental constraints such as carrying capacity. As a result, it is not possible to address logistic growth with the help of this phenomenological description [22].

In our proposal, the specific growth rate can be expressed as

$$\frac{b_1}{b_2}\left(1+\frac{b_2(K-1)}{K}\right)y^{-b_2}-\frac{b_1}{b_2}.$$

The maximum attainable value of y [$y_{\max}$] is given by $[1+\frac{b_2(K-1)}{K}]^{1/b_2}$. Therefore, the specific growth rate and $y_{\max}$ depend on the environmental conditions that are included in this approach through the parameter K (may be termed as carrying capacity). $y_{\max}$ is governed by b_2 as well as K. It is also possible that the carrying capacity may not be attained by the system [21]. The fact is controlled by b_2. As an example, the system is able to attain carrying capacity for $b_2=-1$.

Therefore, it can be concluded from the phenomenological point of view that two distinct classes of biological growth, following West-type growth equation, are possible to be observed in nature. One of them, proposed by Castorina *et al*, is capable of attaining the carrying capacity. It is totally governed by the phenomenological coefficients responsible for different growth mechanisms [11]. The other, proposed in this communication, does not attain the carrying capacity and is controlled by the growth mechanisms of the organism and the environmental condition, e.g. carrying capacity. Such a classification which is based on the initial growth rate, may or may not be governed by the environmental conditions. One of them is characterized by the condition $\varphi=b_1a+b_2a^2$ with $y(0)=a(0)=1$ (a special case for $b_1=1$ is proposed by Castorina *et al* [11]). In this type of growth, the functional dependence of specific growth rate (a) on y is as follows:

$$a=\frac{b_1(1-y^{b_2})+b_2}{b_1+(b_2-b_1)y^{b_2}}. \quad (28)$$

Different values of b_1 form several subclasses in this category. The other one is following the condition $\varphi=b_1a+b_2a^2$ with $y(0)=c$ and $a(0)=b_1(1-(c/K))$, as proposed in this communication. In addition, it is possible to address logistic growth with the help of the proposed initial conditions.

4. Conclusions

Logistic growth (carrying capacity-dependent growth) is very useful to explain a wide range of biological growth processes. Castorina *et al* [11] had attempted to explain logistic growth using phenomenological framework. However, it is not possible to address logistic growth with the help of phenomenological framework proposed by Castorina *et al* [11]. In this communication, we propose a new initial condition of phenomenological framework to address this issue. Different types of growing systems are considered in terms of new initial condition of phenomenological description. Truly linear behaviour has been identified in this study from the phenomenological point of view. It is found that involuted Gompertz function (used to describe the growth of a biological system undergoing atrophy or a demographic and economic system undergoing involution or regression) can be addressed in the proposed phenomenological description. The logistic growth and the environment-dependent West-type growth of a biological system have been represented in this proposed framework of the phenomenological universalities of growth. Another interesting feature is that there is no difference between these two types of growth in terms of phenomenological approach. The only difference is related to the value of coefficient in phenomenological description. It is also explained phenomenologically that the energy intake of an organism depends on the carrying capacity whereas the metabolic cost does not depend on the carrying capacity. Dependence of growth features on different parameters in each case is shown graphically. The key observation is that the logistic growth and the environment-dependent West-type growth are originated from the same phenomenological description with different values of phenomenological coefficient. The findings may be helpful to take deep insight in the mechanism involved in different growth processes in nature.

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