



Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics

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Abstract. Using the variational method and supersymmetric quantum mechanics we calculated, in an approximate way, the eigenvalues, eigenfunctions and wave functions at the origin of the Cornell potential. We compared results with numerical solutions for heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$.

Keywords. Quarkonium; variational method; supersymmetric quantum mechanics.

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1. Introduction

Since the discovery of J/ψ in 1974 [1,2], the study of heavy quarkonia has been very valuable in hadron physics because they involve the non-perturbative aspect of quantum chromodynamics (QCD) and there are many experimental data about them [3–5]. From a theoretical perspective, heavy quarkonia have been studied using several models [6,7], with the non-relativistic potential models standing out due to their simplicity, where quark interaction is modelled using potential energy in the usual Schrödinger equation.

This potential picture can be justified for heavy quarkonia by using non-relativistic QCD (NRQCD) with a potential that can be expressed as a one-over quark mass expansion, where the Cornell potential is a good first term [8–11]. The Cornell potential [12,13] was one of the first potentials proposed to describe the interaction between heavy quarks. It corresponds to a Coulomb potential plus a linear confinement part. Therefore, the Cornell potential considers general properties of quark interactions as asymptotic freedom and confinement. Although there is a large amount of literature related to quark potentials, we would like to list refs [7,12–19]. Although this list is incomplete, these references are a good starting point on this topic.

The Schrödinger equation with the Cornell potential does not provide exact analytical solutions. Although it

can be solved using numerical methods [20,21], it is interesting to obtain analytical solutions (at least approximate ones) that offer the possibility of different applications. For example, in many hadron physics applications in light-cone frame, it is common to see the use of an analytical ansatz inspired by harmonic oscillator wave function in constant time frame in order to achieve the light front wave function used in calculations (for example, see [22–24]). We believe that it could be interesting to start from a more realistic wave function, for example, associated with the Cornell potential.

In this work, we solved, in an approximate way, the Schrödinger equation with the Cornell potential using a procedure that corresponds to an adaptation of the method suggested in [25,26], which considered the usual variational methods with supersymmetric quantum mechanics (SUSY QM). Some additional examples using SUSY QM and the variational method can be found in [27–29].

SUSY QM [30] was born at the beginning of the 1980s in studies about breaking supersymmetry in theories of quantum fields with extra dimensions [31], and is a technique that allows us to get isospectral potentials for the Schrödinger equation. The isospectral potential corresponds to the supersymmetric partner potential of the original, with the particularity that the ground state of the original potential is not present in the spectrum of the associated isospectral potential. Therefore,

the ground state of the supersymmetric partner potential is related to the first excited state of the original potential. This procedure can be repeated to get successive potentials, the ground states of which are related by some standard transformations in SUSY QM to the different states of the original potential; hence, SUSY QM can be used to build an infinite family of isospectral potentials. Thus, if we use the variational method to get solutions for the ground state of different supersymmetric partners of the Cornell potential, we can obtain the spectrum and wave functions for heavy quarkonium. It is important to mention that the standard variational method has been used to study heavy quarkonium properties by considering different phenomenological quark potentials [32,33].

The procedure described in the previous paragraph was used in this work to get approximations to eigenvalues and eigenfunctions for the Schrödinger equation with the Cornell potential, and we used it to study heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$. Additionally, we paid special attention to the wave function at the origin (WFO), which is an important quantity involved in several decay rates of heavy quarkonium. Although the procedure described in this paper can be used in general to study radial and angular excitations, we restricted our study to only to S states.

This paper consists of four sections. In §2 we summarize the main elements of SUSY QM used in this work. Section 3 is dedicated to obtaining approximate calculations of energies, wave functions and WFO for heavy quarkonium using the variational method and SUSY QM, and in §4 we discuss our results and conclusions.

2. Basics of SUSY QM

In this section we summarize the main elements of SUSY QM used in the following sections to calculate heavy meson properties using the Cornell potential. For more details, we suggest ref. [30].

We consider the Schrödinger equation for the ground state with an eigenvalue equal to zero (this can be done without losing generality, because the potential can be redefined by adding a constant term equal to minus ground-state energy). Thus, the wave function ψ_0 obeys

$$H_1\psi_0(x) = -\frac{\hbar^2}{2m} \frac{d^2\psi_0(x)}{dx^2} + V_1(x)\psi_0(x) = 0. \quad (1)$$

Then,

$$V_1(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \quad (2)$$

The Hamiltonian H_1 can be factorized as

$$H_1 = A^\dagger A,$$

where

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \quad \text{and} \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x).$$

With this, we observed that for a known V_1 , the superpotential W satisfies the Riccati equation

$$V_1(x) = -\frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} + W^2(x).$$

The solution for $W(x)$ in terms of the ground-state wave function is

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)}. \quad (3)$$

Additionally, with operators A and A^\dagger it is possible to build a new Hamiltonian H_2 given by

$$H_2 = AA^\dagger,$$

and this new Hamiltonian can be expressed as

$$H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x),$$

where

$$V_2(x) = \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} + W^2(x).$$

Potentials $V_1(x)$ and $V_2(x)$ are known as supersymmetric partner potentials, and they have several interesting properties (see [30]). It is important to point out that eigenvalues and eigenfunctions of H_1 and H_2 are related to

$$E_n^{(2)} = E_{n+1}^{(1)}; \quad E_0^{(1)} = 0 \quad (4)$$

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A \psi_{n+1}^{(1)} \quad (5)$$

and

$$\psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger \psi_n^{(2)}. \quad (6)$$

We pay special attention to the relationship in the spectrum of H_1 and H_2 because, with the exception of the ground state of H_1 (that did not appear in H_2), additional levels are the same in both Hamiltonians, i.e., potentials V_1 and V_2 are isospectrals except for $E_0^{(1)}$.

Similarly, starting from H_2 and its ground state, we can build Hamiltonian H_3 , isospectral to H_2 (in the same sense that H_1 and H_2 are isospectrals), and if we

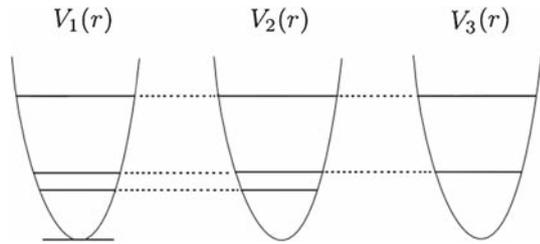


Figure 1. Schematic representation of potential V_1 and its first two supersymmetric partners V_2 and V_3 with its corresponding spectrum. The ground state in V_1 is not present in V_2 , ground state of V_2 is not present in V_3 and so forth. The shape of each potential is different, but this issue is disregarded to explain that the ground state of one potential is not present in its supersymmetric partner.

repeat this procedure, it is possible to obtain a family of isospectral potentials where, as can be seen in figure 1, the ground state of V_2 is related to the first excited level of V_1 , the ground state of V_3 is related to the first excited level of V_2 and the second level of V_1 , and so forth in order to achieve different levels of the original potential V_1 .

According to the previous paragraph, the ground state of H_2 corresponds to the first excited level of H_1 . This fact is especially important because the common variational method in quantum mechanics is a good tool to obtain approximate values for the ground states in the Schrödinger equation. Therefore, we can use this simple procedure to get approximate solutions for ground states for different supersymmetric partner potentials by using the variational method, and we can use SUSY QM relationships to get approximate solutions for different levels of a potential of interest V_1 , such as the Cornell potential.

3. Solutions for the Cornell potential with variational methods and SUSY QM

In this section we use the procedure described in §2. This was suggested in [25,26] and we adapted it to calculate energies, wave functions and WFO for S states in quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$ using the Cornell potential.

Consider $U_{nl}(r) = r R_{nl}(r)$, which satisfies

$$-\frac{1}{2\mu} \frac{d^2 U_{nl}(r)}{dr^2} + V_{\text{eff}}(r)U_{nl}(r) = E_n U_{nl}(r), \quad (7)$$

where $\hbar = 1$, μ is the reduced mass for heavy quarkonium considered, and

$$V_{\text{eff}}(r) = V(r) + \frac{l(l+1)}{2\mu r^2}.$$

Although the procedure we considered can be used to study properties of different radial and angular excitations, in this work we only consider the case $l = 0$ (S states). Thus, the effective potential is equal to the Cornell potential,

$$V_{\text{eff}}(r) = V(r) = -\frac{\kappa}{r} + \beta r, \quad (8)$$

where the parameters involved are [12,13,34,35]

$$\kappa = 0.52; \quad \beta = \frac{1}{(2.34)^2} \text{GeV}^2$$

and the quark masses are

$$\mu_c = 1.84 \left[\frac{\text{GeV}}{c^2} \right] \quad \text{and} \quad \mu_b = 5.18 \left[\frac{\text{GeV}}{c^2} \right].$$

With this potential it is possible to gain a mass spectrum for quarkonia by using

$$m_n(\bar{Q}_1 \bar{Q}_2) = m_{Q_1} + m_{Q_2} + E_n + \Delta, \quad (9)$$

where m_{Q_1} and m_{Q_2} are masses of the quarks and the antiquarks inside the quarkonia considered, E_n is the eigenvalue associated with (8), and Δ is a constant that must be added to the Cornell potential in (8).

For the variational method we consider the trial wave function

$$U(r) = N r^\gamma e^{-ar^b}. \quad (10)$$

This trial wave function can be used to obtain the ground state in the Cornell potential and in its supersymmetric partners to calculate the successive levels in Cornell potential. In this wave function, γ takes the values 1, 2, 3,... depending on the calculation of the ground state of potentials V_1, V_2, \dots (according to §2). Changing this parameter in this way is important, because to gain approximations for the wave functions of the Cornell potential for different levels, it will be necessary to apply successive transformations defined by (6), and this choice turns out to be the only possibility to gain finite WFOs with a trial wave function like (10). Parameters a and b are variational parameters and N is the normalization constant given by

$$N = \sqrt{\frac{(2a)^{(1+2\gamma)/b} b}{\Gamma((1+2\gamma)/b)}}.$$

To calculate the ground state for the Cornell potential we use (7) and the trial wave function with $\gamma = 1$. Thus, the expectation value of the energy is

$$E = -\frac{1}{2\mu} \int_0^\infty U(r) \frac{d^2}{dr^2} U(r) dr + \int_0^\infty \left(-\frac{\kappa}{r} + \beta r \right) U^2(r) dr.$$

By using $U(r)$ given in (10), we get an expectation value of the energy that depends on parameters a and b ($E(a, b)$), and by minimizing this, an approximate value for the energy and values for the parameters a and b are obtained. In this case, we can call the parameters associated with the ground state a_0 and b_0 .

Then, WFO is calculated for the ground state. In this case we consider two approaches that are equivalent when working with the exact solutions, but they have different values when using approximate wave functions. In this case we use $R(r) = U(r)/r$.

The first approach considered in WFO calculations, and henceforth called Method 1, is based on a well-known expression (valid for S states only) that relates WFO to expectation values for the first derivative of the potential

$$|\Psi(0)|^2 = \frac{\mu}{2\pi} \left\langle \frac{dV(r)}{dr} \right\rangle.$$

For S states

$$\Psi(r) = \frac{1}{\sqrt{4\pi}} R(r).$$

Therefore,

$$|R(0)|^2 = 2\mu \left\langle \frac{dV(r)}{dr} \right\rangle.$$

The second approach is simply to take $r \rightarrow 0$ in the wave function. This is what we have called Method 2 in the following. According to this, WFO for a ground state can be found directly from $R(r) = (U(r)/r)$ and (10) (with $\gamma = 1$ for the ground state), and we get

$$|R(0)|^2 = N^2 = \frac{(2a_0)^{(1+2\gamma)/b_0} b_0}{\Gamma((1+2\gamma)/b_0)}.$$

Once we finish with the ground state, we calculate the first radial excitation in the Cornell potential. Note that using variational method to approximate the excited states is not a simple task because it is important to ensure that the trial eigenfunctions are orthogonal. In this paper, we solve the problem of the first radial excitation by using the procedure discussed in §2. Then, we explain how to use SUSY QM and the variational method to solve $2S$ states.

Previously, eq. (10) was used with $\gamma = 1$ to get solutions for the ground state, thus using the common variational method, energy values for the $1S$ states are obtained, and we fit the parameter in the wave functions. Then ‘0’ index in the variational parameters indicates that these parameters are associated with the ground state. With this trial wave function (related to

the ground state of the Cornell potential) we obtain the superpotential

$$W_{21}(r) = -\frac{1}{\sqrt{2\mu}} \frac{U'(r)}{U(r)} = \frac{-1 + a_0 b_0 r^{b_0}}{\sqrt{2\mu} r}.$$

The index 21 in W indicates that starting from solutions of potential V_1 (Cornell in this paper), we can build a potential V_2 (an approximate supersymmetric partner for the Cornell potential)

$$V_2(r) = [W_{21}(r)]^2 + \frac{1}{\sqrt{2\mu}} \left(\frac{dW_{21}(r)}{dr} \right)$$

$$V_2(r) = \left(\frac{2 + a_0 b_0 r^{b_0} (-3 + a_0 b_0 + a_0 b_0 r^{b_0})}{2\mu r^2} \right).$$

Next, the variational method is used to get an approximate value for the ground-state energy of potential V_2 . In this case, the trial wave function has the shape (10) with $\gamma = 2$, and the energy is related to the first excited state of V_1 .

For V_2 , the energy expectation value depends on a_0 , b_0 (fixed in previous steps when we calculate the ground state of V_1) and a and b , which must be fixed once this expectation value has been minimized.

It is important to note that in our discussion for SUSY QM we consider a ground state with eigenvalue equal to zero; therefore, the ground state of energy for V_2 represents ΔE_2 , and consequently the energy for the first excited state is

$$E_1 = E_0 + \Delta E_2.$$

Thereafter, the second excited level in the Cornell potential is obtained. As we have an approximate solution for V_2 , we can build W_{32} and obtain its supersymmetric partner V_3 . If we find the ground state of this new potential using the variational method using (10) with $\gamma = 3$, we can find the energy of the second excited state of the original potential by using

$$E_2 = E_1 + \Delta E_3,$$

and so forth. Table 1 shows the energy values calculated with the method used in this paper, and we compare it with an exact numerical solution obtained using a MATHEMATICA program called mathschroe.nb [20].

Here we would like to show how we obtained the wave function for the excited levels in the Cornell potential. If we have a solution for the ground state of the potential V_2 , which we call for example $\psi_0^{(2)}$, additionally as we know W_{21} (we used it to build V_2),

Table 1. Energy values (in GeV) for heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$. Column ‘Exact’ shows the solution that corresponds to numerical calculations using mathschroe.nb with step $h = 0.00001$ and the column ‘Ours’ shows the energies calculated in this work.

E_n	$c\bar{c}$		$b\bar{b}$		$b\bar{c}$	
	Exact	Ours	Exact	Ours	Exact	Ours
1s	0.2575	0.2578	-0.1704	-0.1702	0.1110	0.1113
2s	0.8482	0.8096	0.4214	0.3579	0.6813	0.6324
3s	1.2720	1.1427	0.7665	0.5612	1.0686	0.9065

it is possible to get a wave function for the first excited state of V_1 using the equation

$$\psi_1^{(1)} \sim A_{21}^\dagger \psi_0^{(2)}.$$

In principle, for exact normalized solutions, (6) gives the right normalization, but as we are working with approximate solutions we prefer to normalize each wave function at the end, and therefore in previous expressions we used symbol ‘ \sim ’.

Operator A_{21}^\dagger transforms the solution $\psi_0^{(2)}$ for the ground state of V_2 in a solution for the first excited state of V_1 , and thus the wave function of the first excited state of V_1 is obtained.

$$\psi_1^{(1)} \sim \left(\frac{-1}{\sqrt{2\mu}} \partial r + W_{21}(r) \right) \psi_0^{(2)}.$$

In a similar way, we can build the wave function for the second excited state of the Cornell potential starting from the ground state of V_3

$$\psi_2^{(1)} \sim A_{21}^\dagger A_{32}^\dagger \psi_0^{(3)}$$

$$\psi_2^{(1)} \sim \left(\frac{-1}{\sqrt{2\mu}} \partial r + W_{21}(r) \right) \left(\frac{-1}{\sqrt{2\mu}} \partial r + W_{32}(r) \right) \psi_0^{(3)}.$$

Figure 2 shows the radial density of probabilities calculated with the method discussed here that joins SUSY QM and variational methods, and the results are compared with the result obtained numerically using mathschroe.nb. The wave functions are used to calculate WFO, and tables 2, 3 and 4 show a summary of our results obtained by using Method 1 and Method 2.

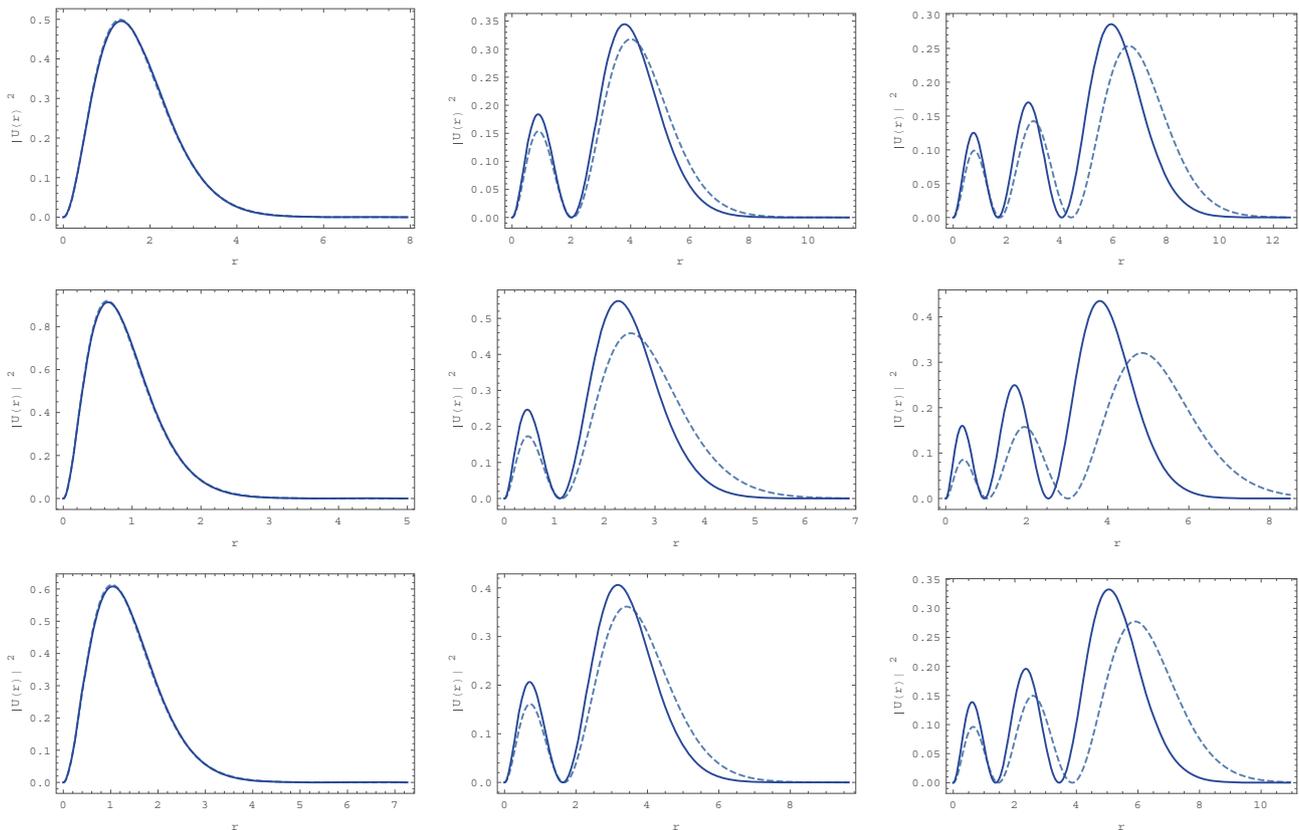


Figure 2. Density functions of radial probability. The continuous line corresponds to the numerical and the dashed line corresponds to the method used in this paper. The first column corresponds to the ground states, the second to the first excited state and the third to the second excited state. The upper row corresponds to $c\bar{c}$, the middle to $b\bar{b}$ and the lower row to $b\bar{c}$.

Table 2. Mass spectrum (in GeV) for heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$. Columns ‘Exact’ and ‘Ours’ represent the same as those in table 1, and the column called ‘Exp’ corresponds to experimental values according to [36].

$m_{Q_1 Q_2}$	$c\bar{c}$			$b\bar{b}$			$b\bar{c}$		
	Exp.	Exact	Ours	Exp.	Exact	Ours	Exp.	Exact	Ours
1s	3.097	3.097	3.097	9.460	9.350	9.350	6.275	6.261	6.291
2s	3.686	3.688	3.649	10.023	9.941	9.878	–	6.861	6.812
3s	4.040	4.112	3.963	10.355	10.287	10.081	–	7.249	7.087

Table 3. Comparison of WFO of the first three energy values in heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$. The values calculated with the numerical solutions are given in the column Method 1 (abbreviated by M1).

$ R(0) ^2$	$c\bar{c}$		$b\bar{b}$		$b\bar{c}$	
	Exact	M1	Exact	M1	Exact	M1
1s	1.4591	1.4384	14.1294	13.9824	3.1950	3.1486
2s	0.9304	0.8160	5.7033	4.1764	1.7712	1.4551
3s	0.7936	0.6781	4.2917	2.6210	1.4509	1.1171

Table 4. Comparison of WFO of the first three energy values in heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$. The values calculated with the numerical solution are given in column Method 2 (abbreviated by M2).

WFO $ R(0) ^2$	$c\bar{c}$		$b\bar{b}$		$b\bar{c}$	
	Exact	M2	Exact	M2	Exact	M2
1s	1.4591	1.2897	14.1294	13.0031	3.1950	2.8380
2s	0.9304	0.6631	5.7033	3.3871	1.7712	1.1601
3s	0.7936	0.5186	4.2917	1.8017	1.4509	0.8048

4. Conclusion and discussion

We used a procedure to solve, in an approximate way, the Schrödinger equation with the Cornell potential using a variational method and SUSY QM. This is phenomenologically interesting, because the Cornell potential can describe some properties of heavy quarkonium, for example its masses, as shown in table 2. Therefore, it is useful to have analytical wave functions, as the ones provided in this paper, which can be useful in additional phenomenological applications in mesonic physics.

The results in table 1 show the values for the energies of the first three excited states, which are close to the exact computation, especially for the ground and the first excited states. The same happens with the wave functions shown in figure 2, where the numerical and

approximate wave functions are almost the same for the ground state and very close to the first excited state, but when we consider higher radial excitations both wave functions are different.

We also calculated WFO. To do so, we considered two methods that are equivalent when working with exact wave functions, but as shown in tables 2 and 3, we obtained different results when using approximate wave functions. Method 1, based on calculations of the expectation values of the first derivative of the potential, delivered a better result than Method 2, which was based on putting $r \rightarrow 0$ directly. Considering that $R(r) = U(r)/r$, as $U(r \rightarrow 0) \sim 0$, to use $r \rightarrow 0$ directly could cause problems as we had an expression $0/0$, and with this it was possible to understand the difference in $R(0)$ by using Methods 1 or 2. This shows that the best choice to evaluate

WFO is Method 1 if there is an approximate wave function.

Our results suggest that the approximate method discussed can produce results in agreement with numerical exact solutions for lower states, and in disagreement with higher excited levels. This is not surprising, because we solved the Schrödinger equation in an approximate way (we used the variational method) with an approximate potential (we obtained supersymmetric partner potential starting from variational trial functions). Therefore, even if we start with a good trial wave function of the ground state, for higher radial excitations the method begins to produce results which are inconsistent with the numerical solutions. However, for lower states this approach provides good results. Undoubtedly, if we use trial wave functions with several parameters we can improve our results, but as we have shown, this method works well for lower states, and it could be helpful to use it as a complement with a method that works for higher excitations, such as WKB.

This approach gives us analytical expressions for wave functions close to numerical solutions and they are orthogonal. So this can be used to calculate other heavy quarkonium properties.

Finally, we would like to return to what we mentioned at the beginning of this paper. In many applications of hadronic physics in light-cone frame, it is common to see the use of an analytical ansatz inspired in harmonic oscillator wave function in a constant time frame, and this is utilized to get the light front wave function used in calculations (for examples, see [22–24]). We believe this could be an interesting start of a more realistic wave function, for example, associated with the Cornell potential as a method that joins SUSY QM and the variational method discussed in this article.

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