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Breaking soliton equations and negative-order breaking soliton equations of typical and higher orders

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Abstract. We develop breaking soliton equations and negative-order breaking soliton equations of typical and higher orders. The recursion operator of the KdV equation is used to derive these models. We establish the distinct dispersion relation for each equation. We use the simplified Hirota's method to obtain multiple soliton solutions for each developed breaking soliton equation. We also develop generalized dispersion relations for the typical breaking soliton equations and the generalized negative-order breaking soliton equations. The results provide useful information on the dynamics of the relevant nonlinear negative-order equations.

Keywords. Breaking soliton equations; negative-order breaking soliton equations; recursion operator.

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1. Introduction

Solitons play important roles in nonlinear science and engineering applications. Solitons provide more insight into the relevant nonlinear science phenomena, thus leading to further scientific features. The earliest soliton equation was KdV equation which was derived by Korteweg and de Vries to model the evolution of shallow water waves in 1895. However, the concept of soliton was introduced by Zabusky and Kruskal in 1965 [1].

Many scientific and engineering applications such as optical fibres, fluid dynamics, plasma physics, ocean engineering, chemical physics etc. are described by nonlinear equations where soliton solutions may appear. Some of these nonlinear evolution equations are integrable which give multiple soliton solutions. The study of integrable equations, that possess sufficiently large number of conservation laws [1–16], plays a major role in solitary waves theory.

The (2+1)-dimensional breaking soliton equation [2] reads as

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0. \qquad (1)$$

This equation was used to describe the (2+1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis [1–3,16–19]. A class of overturning soliton solutions has been introduced in [2]. For $y = x$, and by integrating the resulting equation (1), the equation is reduced to the potential KdV equation. Moreover, eq. (1) was studied in [16] using the homogeneous balance principle followed by the simplified Hirota's method.

A recursion operator is an integrodifferential operator which maps the generalized symmetry of a nonlinear PDE to a new symmetry [6–8]. The existence of a recursion operator for any nonlinear evolution equation guarantees that this equation has infinitely many higher-order symmetries, which is a key feature of complete integrability [8]. Fokas [7] defines a partial differential equation as completely integrable if and only if it possesses infinitely many generalized symmetries. The existence of a sufficiently large number of conservation laws or symmetries guarantees complete integrability for this equation.

The recursion operator is an interesting topic of growing interest and is used in the construction of higher-dimensional equations. The recursion operator of any equation helps to find infinitely many symmetries of a given integrable equation. Olver [5] provided a method for the construction of infinitely many symmetries of evolution equations, and showed that the recursion operator maps a symmetry to a new symmetry.

Magri [4] studied the connection between conservation laws and symmetries from the geometric point of view, and proved that some systems admitted two distinct but compatible Hamiltonian structures, now known as bi-Hamiltonian system. For example, the KdV equation is a bi-Hamiltonian system given by

$$u_t = D_x\left(u_{xx} + \frac{1}{2}u^2\right) \tag{2}$$

and

$$u_t = \left(D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x\right)u. \tag{3}$$

These two operators are a Hamiltonian pair [4,6].

Numerous analytic and numerical methods, such as the Painlevé analysis, the inverse scattering method, the Bäcklund transformation method, the Hirota's bilinear method and the simplified Hirota's method [19–26] have been used to investigate the integrable models. The inverse scattering method and the Hirota's bilinear method are the commonly used methods to determine multiple soliton solutions.

Lou [3] presented a useful work to formally derive higher-dimensional integrable models using the hereditary symmetry. A hereditary symmetry $\Phi(u)$ is an operator-valued function of u which generates a hierarchy of evolution equations for which $\Phi(u)$ is a recursion operator [3]. The hereditary symmetry $\Phi(u)$ is a recursion operator of the following hierarchy of evolution equations:

$$u_t = \Phi^n u_x, \quad n = 0, 1, 2, \ldots. \tag{4}$$

It is obvious that this equation gives rise to (1+1)-dimensional equations.

The KdV equation is the pioneer model in soliton wave theory, that gives rise to solitons, given by

$$u_t + 6uu_x + u_{xxx} = 0, \tag{5}$$

with recursion operator Φ given by

$$\Phi(u) = -\partial_x^2 + 4u + 2u_x\partial_x^{-1}, \tag{6}$$

where ∂_x denotes the total derivative with respect to x and ∂_x^{-1} is its integration operator. The inverse operator ∂_x^{-1} is defined by

$$(\partial_x^{-1}f)(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t, \tag{7}$$

under the decaying condition at infinity. Note that $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$.

It is to be noted that if the function $u(x)$ is defined in space and time only, then $u(x) = \Phi(u_x)$ gives the KdV equation. However, if $u(x)$ is defined in one space and one time dimensions, then $u_t = \Phi(u_y)$ gives the breaking soliton equation (1).

In this work, we aim to establish a typical breaking soliton equation of higher order by using Φ^2, where Φ is the recursion operator of the KdV equation. Moreover, we plan to develop negative-order breaking soliton equations by using the recursion operator Φ and Φ^2 as well. Multiple soliton solutions will be derived for all derived models. We shall show that the dispersion relations for these models are distinct, and this in turn gives solutions with distinct physical structures.

2. Formulation of breaking soliton equation of higher order

We first give a brief discussion of the construction of the typical breaking soliton equation (1). Details can be found in [2].

2.1 *Typical breaking soliton equation*

It was stated before that this equation is obtained by using

$$u_t = \Phi(u_y). \tag{8}$$

Using (6) in (8) we get

$$u_t = -u_{xxy} + 4uu_y + 2u_x\partial^{-1}u_y. \tag{9}$$

Using the potential $u = v_x$ leads to the breaking soliton equation

$$v_{xt} + v_{xxxy} - 4v_xv_{xy} - 2v_{xx}v_y = 0. \tag{10}$$

This equation was studied thoroughly in [1–10]. When $y = x$, eq. (10) reduces to the well-known potential KdV equation.

2.2 *Breaking soliton equation of higher order*

In a similar manner, we shall develop a breaking soliton equation of higher order by using

$$u_t = \Phi^2(u_y). \tag{11}$$

To derive Φ^2, we use (6) to find

$$\begin{aligned}\Phi^2 &= (-\partial_x^2 + 4u + 2u_x\partial_x^{-1})(-\partial_x^2 + 4u + 2u_x\partial_x^{-1})\\ &= \partial_x^4 - 8u\partial_x^2 - 12u_x\partial_x + (16u^2 - 8u_{xx})\\ &\quad + (12uu_x - 2u_{xxx})\partial_x^{-1} + 4u_x\partial_x^{-1}u,\end{aligned} \tag{12}$$

where $\partial_x^{-1}(u_x\partial_x^{-1})$ was evaluated by integrating by parts. Substituting (12) into (11) gives the breaking soliton equation in higher order by

$$u_t = u_{xxxxy} - 8uu_{xxy} - 12u_xu_{xy} + (16u^2 - 8u_{xx})u_y + (12uu_x - 2u_{xxx})\partial_x^{-1}u_y + 4u_x\partial_x^{-1}uu_y. \quad (13)$$

We next use potential $u = v_x$ to obtain

$$v_{xt} = v_{xxxxxy} - 8v_xv_{xxxy} - 12v_{xx}v_{xxy} + (16v_x^2 - 8v_{xxx})v_{xy} + (12v_xv_{xx} - 2v_{xxxx})v_y + 4v_{xx}\partial_x^{-1}v_xv_{xy}, \quad (14)$$

which can be written as

$$\partial_x^{-1}v_xv_{xy} = -\frac{K}{4v_{xx}}, \quad (15)$$

where

$$K(x,t) = -v_{xt} + v_{xxxxxy} - 8v_xv_{xxxy} - 12v_{xx}v_{xxy} + (16v_x^2 - 8v_{xxx})v_{xy} + (12v_xv_{xx} - 2v_{xxxx})v_y. \quad (16)$$

Differentiating both sides of (14) and using (15) give the breaking soliton equation in higher order

$$K_x + 4v_xv_{xx}v_{xy} - \frac{K(x,t)v_{xxx}}{v_{xx}} = 0. \quad (17)$$

When $y = x$, eq. (14) reduces to the fifth-order KdV equation in a potential form.

2.3 *Multiple soliton solutions*

The typical breaking soliton equation (10) has been studied extensively in the literature, and multiple soliton solutions were obtained. In this section, we shall study the (2+1)-dimensional breaking soliton equation of higher order (14) given by

$$v_{xt} = v_{xxxxxy} - 8v_xv_{xxxy} - 12v_{xx}v_{xxy} + (16v_x^2 - 8v_{xxx})v_{xy} + (12v_xv_{xx} - 2v_{xxxx})v_y + 4v_{xx}\partial_x^{-1}v_xv_{xy}. \quad (18)$$

The simplified Hirota's method will be used to conduct this work. We first set the transformation

$$v(x,y,t) = R(\ln f(x,y,t))_x, \quad (19)$$

where the auxiliary function $f(x,y,t)$ is given by

$$f(x,y,t) = 1 + e^{k_ix+r_iy-c_it}, \quad i = 1,2,3. \quad (20)$$

Substituting (19) into (18) and solving we get

$$R = -2. \quad (21)$$

To determine the dispersion relation, we substitute

$$v(x,y,t) = e^{k_ix+r_iy-c_it}, \quad (22)$$

into the linear terms of eq. (18) to find that the dispersion relations are given by

$$c_i = -k_i^4r_i, \quad i = 1,2,3 \quad (23)$$

and as a result the wave variable is given by

$$\theta_i = k_ix + r_iy + k_i^4r_it, \quad i = 1,2,3. \quad (24)$$

Using (22), for $i = 1$, we get the single soliton solution

$$u(x,y,t) = \frac{-2k_1^2\, e^{k_1x+r_1y+k_1^4r_1t}}{(1 + e^{k_1x+r_1y+k_1^4r_1t})^2}, \quad (25)$$

obtained upon using the potential $u(x,y,t) = v_x(x,y,t)$.

For the two-soliton solutions we set the auxiliary function as

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \quad (26)$$

where a_{12} is the phase shift. Substituting (26) and (22) into (18) we obtain the phase shift as

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (27)$$

and hence we set the phase shifts as

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3. \quad (28)$$

Combining (26) and (27) and substituting the outcome into (22), we obtain the two-soliton solutions when $u(x,y,t) = v_x(x,y,t)$.

For the three-soliton solutions, we set

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \quad (29)$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \quad (30)$$

This shows that it is possible to obtain three-soliton solutions. The existence of three-soliton solutions often indicates the complete integrability of the equation under examination.

It is interesting to point out that the typical breaking soliton eq. (10) is integrable and gives multiple soliton solutions. However, looking at [1–2,17–19], we report that the dispersion relation of (10) is given by

$$c_i = k_i^2 r_i, \quad i = 1, 2, 3 \tag{31}$$

and the single soliton solution is given by

$$u(x, y, t) = \frac{-2k_1^2\, e^{k_1x+r_1y-k_1^2r_1t}}{(1 + e^{k_1x+r_1y-k_1^2r_1t})^2}. \tag{32}$$

Combining the results obtained for (10) and (18), then for the generalized case where

$$u_t = \Phi^n(u_y), \quad n \ge 1, \tag{33}$$

we can show that the dispersion relation is given by

$$c_i = (-1)^{n+1} k_i^{2n} r_i, \quad i = 1, 2, 3 \tag{34}$$

and the single soliton solution is given by

$$u(x, y, t) = \frac{-2k_1^2\, e^{k_1x+r_1y+(-1)^{n+1}k_1^{2n}r_1t}}{(1 + e^{k_1x+r_1y+(-1)^{n+1}k_1^{2n}r_1t})^2}, \quad n \ge 1. \tag{35}$$

3. Negative-order breaking soliton equation

Olver [5] proved a general theorem about recursion operators for symmetries of an evolution equation, where it was proved that such an operator creates a new symmetry generator when applied to a known symmetry generator. However, Verosky [10] extended the work of Olver in the negative direction to obtain a sequence of equations of increasingly negative orders. Recall that (8) indicates

$$u_t = \Phi u_y. \tag{36}$$

By the negative-order hierarchy, we refer to

$$u_t = \Phi^{-1} u_y, \tag{37}$$

i.e. the powers of Φ goes to the opposite direction [7–10]. In other words, the negative-order equation can be denoted by

$$\Phi u_t = u_y. \tag{38}$$

If the recursion operator is hereditary and admits an invertible implectic–simplectic separation, the negative-order equations also have multi-Hamiltonian structures [7–10].

3.1 *Breaking soliton equation of typical order*

Using the recursion operator for the KdV equation Φ as defined earlier by (6) in (38) we obtain

$$-u_{xxt} + 4uu_t + 2u_x\partial_x^{-1}u_t = u_y. \tag{39}$$

Using the potential

$$u(x, y, t) = v_x(x, y, t), \tag{40}$$

carries (39) into

$$-v_{xxxt} + 4v_xv_{xt} + 2v_{xx}v_t = v_{xy}, \tag{41}$$

a typical negative-order breaking soliton equation. In what follows, we shall follow our analysis presented above to determine soliton solutions for this equation.

We first set the transformation

$$v(x, y, t) = R(\ln f(x, y, t))_x, \tag{42}$$

where the auxiliary function $f(x, y, t)$ is given by

$$f(x, y, t) = 1 + e^{k_ix+r_iy-c_it}, \quad i = 1, 2, 3. \tag{43}$$

Substituting (42) into (41) and solving we find

$$R = -2. \tag{44}$$

To determine the dispersion relation, we substitute

$$v(x, y, t) = e^{k_ix+r_iy-c_it}, \tag{45}$$

into the linear terms of (41) to find that the dispersion relations are given by

$$c_i = \frac{r_i}{k_i^2}, \quad i = 1, 2, 3, \tag{46}$$

and hence the wave variable is given by

$$\theta_i = k_ix + r_iy - \frac{r_i}{k_i^2}t, \quad i = 1, 2, 3. \tag{47}$$

Equation (42), for $i = 1$, gives the single soliton solution

$$u(x, y, t) = \frac{-2k_1^2\, e^{k_1x+r_1y-(r_1/k_1^2)t}}{(1 + e^{k_1x+r_1y-(r_1/k_1^2)t})^2}, \tag{48}$$

obtained upon using the potential $u(x, y, t) = v_x(x, y, t)$.

For the two-soliton solutions we set the auxiliary function as

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{49}$$

where a_{12} is the phase shift. Substituting (49) and (42) into (41) we obtain the phase shift by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{50}$$

and hence we set the phase shifts by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3. \tag{51}$$

Combining (49) and (50) and substituting the outcome into (45), we obtain the two-soliton solutions when $u(x, y, t) = v_x(x, y, t)$.

For the three-soliton solutions, we set

$$\begin{aligned} f(x, y, t) = {} & 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ & + a_{13}\mathrm{e}^{\theta_1+\theta_3} + a_{23}\mathrm{e}^{\theta_2+\theta_3} \\ & + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{52}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{53}$$

This shows that it is possible to obtain the three-soliton solutions. The existence of three-soliton solutions often indicates the complete integrability of the equation under examination. Other methods, such as Painlevé analysis, are necessary to cofirm the integranility concept.

4. Negative-order breaking soliton equation of higher order

To determine negative-order breaking soliton equation of higher order we use

$$u_t = \Phi^2 u_y. \tag{54}$$

By the negative-order hierarchy, we refer to

$$u_t = \Phi^{-2} u_y, \tag{55}$$

i.e. the powers of Φ goes to the opposite direction [7–10]. In other words, the negative-order equation can be denoted by

$$\Phi^2 u_t = u_y. \tag{56}$$

If the recursion operator is hereditary and admits an invertible implectic–simplectic separation, the negative-order equations also have multi-Hamiltonian structures [7–10].

Recall that

$$\begin{aligned} \Phi^2 = {} & \partial_x^4 - 8u\partial_x^2 - 12u_x\partial_x + (16u^2 - 8u_{xx}) \\ & + (12uu_x - 2u_{xxx})\partial_x^{-1} + 4u_x\partial_x^{-1} u. \end{aligned} \tag{57}$$

Substituting (57) into (56) gives the breaking soliton equation in higher order by

$$\begin{aligned} u_{xxxxt} & - 8uu_{xxt} - 12u_xu_{xt} + (16u^2 - 8u_{xx})u_t \\ & + (12uu_x - 2u_{xxx})\partial_x^{-1}u_t \\ & + 4u_x\partial_x^{-1}uu_t = u_y. \end{aligned} \tag{58}$$

We next use the potential $u = v_x$ to obtain

$$\begin{aligned} -v_{xy} & + v_{xxxxxt} - 8v_xv_{xxxt} - 12v_{xx}v_{xxt} \\ & + (16v_x^2 - 8v_{xxx})v_{xt} + (12v_xv_{xx} - 2v_{xxxx})v_t \\ & + 4v_{xx}\partial_x^{-1}v_xv_{xt} = 0, \end{aligned} \tag{59}$$

or equivalently

$$\partial_x^{-1}v_xv_{xt} = -\frac{G}{4v_{xx}}, \tag{60}$$

where

$$\begin{aligned} G(x, y, t) = {} & -v_{xy} + v_{xxxxxt} - 8v_xv_{xxxt} - 12v_{xx}v_{xxt} \\ & + (16v_x^2 - 8v_{xxx})v_{xt} \\ & + (12v_xv_{xx} - 2v_{xxxx})v_t. \end{aligned} \tag{61}$$

Differentiating both sides of (59) and using (60) gives the breaking soliton equation in higher dimension

$$G_x + 4v_xv_{xx}v_{xt} - v_{xxx}\frac{G(x, y, t)}{v_{xx}} = 0. \tag{62}$$

4.1 *Multiple soliton solutions*

In this section, we shall study the (2+1)-dimensional breaking soliton equation of higher order (59) given by

$$\begin{aligned} v_{xy} = {} & v_{xxxxxt} - 8v_xv_{xxxt} - 12v_{xx}v_{xxt} \\ & + (16v_x^2 - 8v_{xxx})v_{xt} + (12v_xv_{xx} - 2v_{xxxx})v_t \\ & + 4v_{xx}\partial_x^{-1}v_xv_{xt}. \end{aligned} \tag{63}$$

Proceeding as before, we set the transformation

$$v(x, y, t) = R(\ln f(x, y, t))_x, \tag{64}$$

where the auxiliary function $f(x, y, t)$ is given by

$$f(x, y, t) = 1 + \mathrm{e}^{k_i x + r_i y - c_i t}, \quad i = 1, 2, 3. \tag{65}$$

Substituting (64) into (63) and solving we find

$$R = -2. \tag{66}$$

To determine the dispersion relation, we substitute

$$v(x, y, t) = \mathrm{e}^{k_i x + r_i y - c_i t}, \tag{67}$$

into the linear terms of (63) to find that the dispersion relations are given by

$$c_i = -\frac{r_i}{k_i^4}, \quad i = 1, 2, 3, \tag{68}$$

and as a result the wave variable becomes

$$\theta_i = k_i x + r_i y + \frac{r_i}{k_i^4} t, \quad i = 1, 2, 3. \tag{69}$$

Using (64), for $i = 1$, gives the single soliton solution

$$u(x, y, t) = \frac{-2k_1^2 \, e^{k_1 x + r_1 y + (r_1/k_1^4)t}}{(1 + e^{k_1 x + r_1 y + (r_1/k_1^4)t})^2}, \tag{70}$$

obtained upon using the potential $u(x, y, t) = v_x(x, y, t)$.

For the two-soliton solutions we proceed as before to get the phase shift by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3, \tag{71}$$

where we used the auxiliary function

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}. \tag{72}$$

The two-soliton solutions are obtained as presented earlier.

For the three-soliton solutions, we set

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + b_{123} e^{\theta_1 + \theta_2 + \theta_3}. \tag{73}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{74}$$

This shows that it is possible to obtain the three-soliton solutions. The existence of three-soliton solutions often indicates the complete integrability of the equation under examination.

Combining the results obtained for negative-order breaking soliton equation and the negative-order breaking soliton equation of higher order, then for the generalized case where

$$u_t = \Phi^n(u_y), \quad n \ge 1, \tag{75}$$

we can show that the dispersion relation is given by

$$c_i = (-1)^{n+1} \frac{r_i}{k_i^{2n}}, \quad i = 1, 2, 3, \tag{76}$$

and the single soliton solution is given by

$$u(x, y, t) = \frac{-2k_1^2 \, e^{k_1 x + r_1 y + (-1)^{n+1} (r_i/k_i^{2n}) t}}{(1 + e^{k_1 x + r_1 y + (-1)^{n+1} (r_i/k_i^{2n}) t})^2}, \quad n \ge 1. \tag{77}$$

5. Conclusions

We extended our previous works in [16–19] to establish a standard breaking soliton equation in higher order. Moreover, we developed two forms of negative-order breaking soliton equations. The established standard and negative-order breaking soliton equations were generalized. We obtained multiple soliton solutions for each model and generalized dispersion relations were also derived.

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