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Space–time transformation for the propagator in de Broglie–Bohm theory

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Abstract. A linear space–time transformation proposed to calculate the propagator in the de Broglie–Bohm theory, is viewed as an expansion of the guiding wave function over the velocity space. It is shown that the quantum evolution is preserved in its semiclassical scheme through this change. The case of variable-frequency harmonic oscillator is presented as an example.

Keywords. Propagator; de Broglie–Bohm theory; space–time transformation.

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1. Introduction

Quantum mechanics is described by different approaches. Among them we can quote the standardized approach or the Copenhagen interpretation, that introduces the probabilistic nature of quantum mechanics. The Heisenberg uncertainty relations come from the interaction between the measuring apparatus and which is measured and other parallel approaches such as interpretation Bohm. This latter is a causal theory which attempts to give a physical meaning to the concept of trajectory and at the same time, to explain the nature of the quantum phenomena. Accordingly, the particle dynamics is described by a deterministic trajectory submitted to a quantum potential at the origin of nonlocality of the quantum phenomenon. This point of view allows me to consider that any physical system in a definite quantum state $\Psi(x,t)$ is governed by laws generalizing that of classical mechanics as follows. If the classical system is described by the Lagrangian

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2-V(x,t), \tag{1}$$

then the quantum causal trajectory will be governed by the quantum Hamilton–Jacobi equation

$$\frac{\partial S(x,t)}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+V(x,t)+V_q(x,t)=0, \tag{2}$$

where the quantum potential $V_q(x,t)$ (at the origin of nonlocality of the quantum phenomenon) is generated by the pilot wave

$$\Psi(x,t)=\sqrt{\rho(x,t)}\exp\left[\frac{i}{\hbar}S(x,t)\right]$$

as

$$V_q(x,t)=-\frac{\hbar^2}{2m\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}. \tag{3}$$

This pilot wave satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x,t)\right]\Psi(x,t) \tag{4}$$

and $\rho(x,t)=|\Psi(x,t)|^2$ represents the probability density which is preserved according to

$$\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial}{\partial x}[\rho(x,t)v(x,t)]=0 \tag{5}$$

with

$$v(x,t)=\frac{1}{m}\frac{\partial S}{\partial x}. \tag{6}$$

$\rho(x,t)v(x,t)$ is then the current of the particles where $v=\mathrm{d}x/\mathrm{d}t$ is the velocity at the point (x,t). If $\hbar\to 0$, $V_q(x,t)\to 0$ and $S(x,t)$ is a solution of classical Hamilton–Jacobi equation, then it is called the quantum action.

The de Broglie–Bohm theory is relevant as a coherent interpretation of quantum mechanics and its concept of trajectory is useful in many practical aspects of the study of quantum systems. It has been reliably proved in most tests of various areas of physics such as quantum interference, Aharanov–Bohm effect and measurement process in quantum mechanics [1]. It has various modern applications [2]. The quantum processes are visualized in barrier penetration and tunnelling effect [3–6], two-slit experiment [7], molecular dynamics [8], dynamics in nonlinear systems and scattering [9–11]. It is used to approximate the solution of Schrödinger equation by using Newton equations of motion in a potential corrected by the initial quantum fluctuation potential. Using the information of the initial value of the quantum state and the first corrected quantum trajectory, this solution can then be approximated step by step. It justifies the dynamical origin of the quantum relaxation [12–16] and gives a context of observation of the arrival time and times of flight, as the time needed to connect any two points along a quantum trajectory is a well-defined quantity. Quantum scattering problem in the de Broglie–Bohm theory has been presented in [17–21] where the rules of scattering probabilities adapting the concept of quantum trajectories are established. A numerical simulation of scattering by a single nucleus has been given in [22], while in [23] the phenomena of atom–surface scattering as well as neutron diffraction by slits are considered. In parallel, there is also a formalism of the quantum mechanics which uses classical description combined with the strange superposition principle, known as Feynman path integral formalism. This well-known approach is a suitable language of quantization. Compared to the Feynman formalism, the solutions of the de Broglie–Bohm equations have been developed only in special cases.

In this paper, our aim is to adapt the space–time technique frequently used in Schrödinger equation and Feynman path integral by using the semiclassical solution of the de Broglie–Bohm equation. We consider the probability density in the form of Gaussian wave packet and we develop this last in the vicinity of a classical trajectory. This is basically an approximate approach which we call the semiclassical method. Then, the propagator is deduced via a composition formula of a generalized Fourier transform [24,25]. Following this approach, the quantum propagator is viewed as an expansion of the guiding wave function over the velocity space [26,27]. This method was used to determine the Feynman propagator in all cases of time independent of quadratic Hamiltonians and also in the case of Kanai–Caldirola oscillator [28,29]. The case of variable frequency oscillator treated already via the formalism of canonical transformations in the Feynman approach [30] is considered.

In §2, we shall give a short outline on the semiclassical method in the de Broglie–Bohm theory [28], the quantum propagator is viewed as an expansion of the guiding wave function over the velocity space. Section 3 is dedicated to the presentation of the technique of space–time transformations. The method is adapted to calculate the propagator in de Broglie–Bohm theory. It is shown that the quantum evolution is preserved in its semiclassical scheme via this change. The case of variable-frequency harmonic oscillator is presented as an example. In §4, we provide concluding remarks.

2. Semiclassical method in de Broglie–Bohm theory

If the characteristic dimensions of the problem are very large compared to the wavelengths of the waves accompanying the particles, then the physical system can be described in a satisfactory way by the semiclassical method. In this case, the situation is similar to the passage from the wave optics to geometrical optics and the propagation equation is reduced to that of the eikonal. The de Broglie–Bohm theory based on the trajectory concept, contains this reduction implicitly and the semiclassical method can be applied in a natural way.

It is easy to show that when $\hbar \to 0$ and if a wave function has initially at $t = 0$, the expectation values of the position and momentum as x_0 and p_0 then it is sharply concentrated near these values as

$$\Psi(x,0) = (2\pi a_0^2)^{-1/4} \exp\left[\frac{ip_0 x}{\hbar} - \frac{(x-x_0)^2}{4a_0^2}\right]. \quad (7)$$

Then after a time t the wave function will be concentrated at $X(t)$ and $p(t)$. If it is limited to a semiclassical evolution, $X(t)$ and $p(t)$ are related to x_0 and p_0 by classical canonical transformation generated by the classical action. This motivates the choice for the semiclassical solution of the de Broglie–Bohm problem as follows:

$$\Psi(x,t) = \sqrt{\rho(x,t)} \exp\left[\frac{i}{\hbar} S(x,t)\right]$$

with the following probability density

$$\sqrt{\rho(x,t)} = (2\pi a^2(t))^{-1/4} \exp\left[-\frac{(x-X(t))^2}{4a^2(t)}\right], \quad (8)$$

where $X(t)$ is the classical trajectory solution of the Euler–Lagrange equation

$$\ddot{X} + \frac{1}{m}\frac{\partial V}{\partial x}[X(\tau),\tau] = 0 \quad (9)$$

and $a(t)$ is an arbitrary function related to quantum potential (3) which then will have the Gaussian form

$$V_q(x,t) = -\frac{\hbar^2}{2m\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2} = -\frac{\hbar^2}{2m}\left[\frac{(x-X(t))^2}{4a^4(t)} - \frac{1}{2a^2(t)}\right]. \quad (10)$$

Now, plugging expression (8) in eq. (5) one obtains

$$\frac{\partial v}{\partial x} = v\frac{[x-X(t)]}{a^2(t)} - \frac{\dot{a}(t)}{a^3(t)}[x-X(t)]^2 - \frac{\dot{X}(t)}{a^2(t)}[x-X(t)] + \frac{\dot{a}(t)}{a(t)}, \quad (11)$$

which can be directly integrated giving the general solution

$$v(x,t) = \frac{\dot{a}(t)}{a(t)}[x-X(t)] + \dot{X}(t). \quad (12)$$

It turns out that we are interested by the semiclassical solution. Let us proceed like before and consider only small deviations from the classical path $X(t)$. Then the velocity $v(x,t)$, the potential $V(x,t)$ and the action $S(x,t)$ can be written as

$$v(x,t) = \frac{\dot{a}(t)}{a(t)}[x-X(t)] + \dot{X}(t) + \cdots \quad (13)$$

$$V(x,t) = V(X(t),t) + \frac{\partial V}{\partial X}(X(t),t)[x-X(t)] + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(X(t),t)[x-X(t)]^2 + \cdots \quad (14)$$

$$S(x,t) = S(X(t),t) + \frac{\partial S}{\partial X}(X(t),t)[x-X(t)] + \frac{1}{2}\frac{\partial^2 S}{\partial X^2}(X(t),t)[x-X(t)]^2 + \cdots . \quad (15)$$

From eq. (6), we deduce

$$S(x,t) = S(X(t),t) + m\dot{X}(t) + \frac{m\dot{a}(t)}{2a(t)}[x-X(t)]^2 + \cdots . \quad (16)$$

$S(X(t),t) \equiv S[X]$ is the action along the classical path. Inserting (10), (14) and (16) in (2) and equalizing the power orders of $[x-X(t)]$ gives the following equations:

$$\dot{S}[X(t),t] = \frac{m}{2}\dot{X}^2(t) - V[X(t),t] - \frac{\hbar^2}{4ma^2(t)}, \quad (17)$$

$$\ddot{a}(t) + \left(\frac{1}{m}\frac{\partial^2 V}{\partial x^2}[X(t),t]\right)a(t) = \frac{\hbar^2}{4m^2a^3(t)}, \quad (18)$$

where $X(t)$ is a solution of (9). It is clear that the contributions proportional to $\hbar^2$ in eqs (17) and (18) are due to the quantum potential (10). This represents the non-locality of the quantum phenomenon. We impose the following initial conditions [26,28]:

$$X(0) = x_0, \quad \dot{X}(0) = v_0, \quad a(0) = a_0, \quad \dot{a}(0) = 0 \quad \text{and} \quad S[X(0),0] = mx_0v_0. \quad (19)$$

This choice guarantees that initially the velocity of the quantum trajectory of the particle is equal to that of the classical one and the effect of the quantum potential had not yet propagated. The problem is then reduced to seek $X(t)$ which is the classical solution, and respectively, $a(t)$ and $S[X(t),t]$ which satisfy eqs (18), (17) with the initial conditions (19). Then the solution of the de Broglie–Bohm problem will be written as

$$\Psi(x,t) = [2\pi a^2(t)]^{-1/4} \times \exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)} - \frac{1}{4a^2(t)}\right)[x-X(t)]^2\right] \times \exp\left[\frac{im\dot{X}(t)}{\hbar}[x-X(t)] + \frac{imv_0x_0}{\hbar}\right] \times \exp\left[\frac{i}{\hbar}\int_0^t \mathrm{d}t'\left(\frac{1}{2}m\dot{X}^2(t') - V\big[X(t')\big] - \frac{\hbar^2}{4ma^2(t')}\right)\right]. \quad (20)$$

Now, it would be interesting to see how one could connect solution (20) to the usual propagator of quantum mechanics. To this end, let us present the method already developed by Bernstein [25] for the localized wave function of short local wavelength in the limit $\hbar \to 0$. The propagator is defined by the following integral equation:

$$\Psi(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}x_0 K(x,x_0,t)\Psi(x_0,0), \tag{21}$$

where

$$\Psi(x,0) = [2\pi a_0^2]^{-1/4} \times \exp\left[-\frac{[x-x_0]^2}{4a_0^2} + \frac{imv_0 x}{\hbar}\right]. \tag{22}$$

Let us now define the normalized quantity

$$\Phi(v_0,x,t) = [2\pi a_0^2]^{1/4}\Psi(v_0,x,t), \tag{23}$$

which satisfies the completeness relation:

$$\int_{-\infty}^{+\infty} \mathrm{d}v_0 \Phi^*(v_0,x,t)\Phi(v_0,x',t) = \left(\frac{2\pi\hbar}{m}\right)\delta(x-x'). \tag{24}$$

Using the latter equation and the conservation of the probability, we can easily deduce the expression of the propagator in the following interesting form [24,25]:

$$K(x,t,x_0,0) = \frac{m}{2\pi\hbar} \times \int \mathrm{d}v_0 \Phi(v_0,x,t)\Phi^*(v_0,x_0,0). \tag{25}$$

The solution of the de Broglie–Bohm problem (20) gives

$$K(x,t,x_0,0) = \int_{-\infty}^{+\infty} \mathrm{d}v_0 \Theta_{v_0}(x,x_0,X(t),\dot{X}(t)) \times \exp\left(\frac{i}{\hbar}S_{v_0}[X(t)]\right) \tag{26}$$

with

$$S_{v_0}[X(t)] = \int_0^t \left[\frac{1}{2}m\dot{X}^2(\tau) - V(X(\tau),\tau)\right]\mathrm{d}\tau \tag{27}$$

and

$$\Theta_{v_0}\big(x,x_0,X(t),\dot{X}(t)\big) = \frac{m}{2\pi\hbar}\left(\frac{a(t)}{a_0}\right)^{-1/2} \times \exp\left\{\left(\frac{im\dot{a}(t)}{2\hbar a(t)} - \frac{1}{4a^2(t)}\right)(x-X(t))^2 + \frac{im\dot{X}(t)}{\hbar}[x-X(t)] - \frac{i}{\hbar}\int_0^t \frac{\hbar^2}{4ma^2(\tau)}\mathrm{d}\tau\right\}. \tag{28}$$

We recall that in this proof, we have used the semi-classical limit, the wave function localized with a short wavelength and results of standard quantum mechanics because we seek a propagator which satisfies the superposition principle and this last is specific to standard quantum mechanics.

Instead of varying the initial velocity of the classical path, we introduce its final extremity which permits to recover a certain affinity with the Feynman path integral approach. Then by replacing the velocity v_0 by position $X(t)$ the propagator is written as

$$K(x,x_0,t) = \int_{-\infty}^{+\infty} \mathrm{d}X(t)\left(\frac{\mathrm{d}v_0}{\mathrm{d}X(t)}\right)\Theta_{v_0(X(t))} \times (x,x_0,X(t),\dot{X}(t)) \times \exp\left[\frac{i}{\hbar}S_{v_0(X(t))}[X(t)]\right], \tag{29}$$

$X(t)$ being the final classic path extremity.

It is obvious that this classical path is incompatible with the Feynman quantum evolution because the extremity $X(t)$ is free. Therefore, $X(\tau)$ is a fictitious classical path that permits the simulation of the quantum evolution *à la* Feynman by means of its free extremity $X(t)$. In other terms

$$\int_{\substack{X(0)=x_0\\ \dot{X}(0)=v_0}} \mathrm{d}X(t)\left(\frac{\mathrm{d}v_0}{\mathrm{d}X(t)}\right)\Theta_{v_0(X(t))}(x,x_0,X(t),\dot{X}(t)) \times \exp\left[\frac{i}{\hbar}S[X(t)]\right] \sim \int_{x(0)=x_0}^{x(t)=x} Dx(t) \times \exp\left[\frac{i}{\hbar}S[X(t)]\right]. \tag{30}$$

This formula presents a remarkable resemblance with the semiclassical one obtained via the Hamilton–Jacobi method where the ponderation factor $\Theta_{v_0(X(t))} \times(x,x_0,X(t),\dot{X}(t))$ plays the role of Jacobian of the transformation between the momentum variables [31]. This choice of variable brings us closer to the Feynman method. It is true that in this proof we have used the semi-classical limit, the wave function localized with a short wavelength and results of standard quantum mechanics.

In what follows we are interested by the preceding developments in the case of the variable-frequency harmonic oscillator. First, we present the technique of space–time transformation frequently used in quantum mechanics, to bring the problem to that of the harmonic oscillator with constant frequency, which will be calculated by the semiclassical method described previously.

3. Space–time transformation in the de Broglie–Bohm theory

We consider the following space–time transformation $(x, t) \to (y, s)$ proposed in [28]:

$$x = \lambda(s)y \quad \text{and} \quad s(t) = \int^t \mathrm{d}\tau \mu(\tau), \tag{31}$$

where $\lambda(s)$ and $\mu(\tau) > 0$ are arbitrary functions. Moreover, we impose on this change that it describes the de Broglie–Bohm physics model. In other words, according to the probability interpretation, we have

$$\mathrm{d}P = \rho(x, t)\mathrm{d}x \quad \text{and} \quad \mathrm{d}\bar{P} = \bar{\rho}(y, s)\mathrm{d}y, \tag{32}$$

where $\mathrm{d}\bar{P}$ is the new probability interpretation (with the same physical interpretation) and $\bar{\rho}(y, s)$ its density in the new coordinates. From the equality $\mathrm{d}P = \mathrm{d}\bar{P}$, we deduce

$$\rho[x(y, s), t(y, s)] = \frac{\bar{\rho}(y, s)}{\lambda(s)}. \tag{33}$$

It is easy to show that

$$v = \mu(\lambda' y + \lambda \bar{v})$$

with

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

and

$$\bar{v} = \frac{\mathrm{d}y}{\mathrm{d}s} \quad \text{is the new velocity,} \tag{34}$$

where we have used

$$\frac{\partial}{\partial t} = \mu\left(\frac{\partial}{\partial s} - \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}s}y\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial x} = \frac{1}{\lambda}\frac{\partial}{\partial y}. \tag{35}$$

Now, we replace v defined in (34) and ρ defined in (33) by their expressions in eq. (5). Using (35), we find that $\bar{\rho}(y, s)$ verifies the following probability conservation equation:

$$\frac{\partial \bar{\rho}(y, s)}{\partial s} + \frac{\partial}{\partial y}[\bar{\rho}(y, s)\bar{v}(y, s)] = 0. \tag{36}$$

Now, let us see what happend to the evolution eq. (2). Remaining in the spirit of the de Broglie–Bohm theory, we must consider the following change $(v, S) \to (\bar{v}, \bar{S})$ conditioned by

$$v(x, t) = \frac{1}{m}\frac{\partial S}{\partial x} \quad \text{and} \quad \bar{v}(y, s) = \frac{1}{m}\frac{\partial \bar{S}}{\partial y}. \tag{37}$$

From eqs (31), (34) and (37) we identify

$$\frac{\partial S}{\partial y} = \mu\lambda^2\frac{\partial \bar{S}}{\partial y} + \mu m\lambda\lambda' y. \tag{38}$$

A natural choice of

$$\mu\lambda^2 = 1 \tag{39}$$

gives

$$S = \bar{S} + \frac{m\lambda'}{2\lambda}y^2. \tag{40}$$

The new evolution equation becomes

$$\frac{\partial \bar{S}(y, s)}{\partial s} + \frac{1}{2m}\left(\frac{\partial \bar{S}}{\partial y}\right)^2 + \bar{V}_{\mathrm{eff}}(y, s) + \bar{V}_q(y, s) = 0, \tag{41}$$

where the new quantum potential maintains its usual form

$$\bar{V}_q(y, s) = V_q\big[x(y, s), t(y, s)\big] = -\frac{\hbar^2}{2m\sqrt{\bar{\rho}}}\frac{\partial^2\sqrt{\bar{\rho}}}{\partial y^2}. \tag{42}$$

The potential is corrected by the space–time transformation giving rise to an effective potential $\bar{V}_{\mathrm{eff}}(y, s)$ given by

$$\bar{V}_{\mathrm{eff}}(y, s) = \frac{1}{\mu}V(x(y, s), t(y, s)) + \frac{m}{2}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\lambda'}{\lambda}\right) - \left(\frac{\lambda'}{\lambda}\right)^2\right]y^2. \tag{43}$$

Now, it is clear that this new wave function

$$\bar{\Psi}(y, s) = \sqrt{\bar{\rho}(y, s)}\exp\left[\frac{i}{\hbar}\bar{S}(y, s)\right] \tag{44}$$

satisfies the following Schrödinger equation:

$$i\hbar\frac{\partial \bar{\Psi}(y, s)}{\partial s} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \bar{V}_{\mathrm{eff}}(y, s)\right]\bar{\Psi}(y, s) \tag{45}$$

and the relation between the old and new wave functions is given according to (33) and (40) by

$$\Psi(x, t) = \sqrt{\frac{1}{\lambda(s)}}\exp\left[\frac{i}{\hbar}\frac{m\lambda'(s)}{2\lambda(s)}y^2\right]\bar{\Psi}(y, s). \tag{46}$$

From (23), (46) we have

$$\Phi(v_0, x, t) = [2\pi a_0^2]^{1/4}\sqrt{\frac{1}{\lambda(s)}} \times \exp\left[\frac{i}{\hbar}\frac{m\lambda'(s)}{2\lambda(s)}y^2\right]\bar{\Psi}(y, s), \tag{47}$$

$$\bar{\Phi}(\bar{v}_0, y, s) = \left[2\pi\left(\frac{a_0}{\lambda(0)}\right)^2\right]^{1/4}\bar{\Psi}(y, s), \tag{48}$$

where

$$v_0 = \mu(0)(\lambda'(0)y(0) + \lambda(0)\bar{v}_0),$$

$$\mathrm{d}v_0 = \mu(0)\lambda(0)\mathrm{d}\bar{v}_0,$$

and from the equation of the propagator (25)

$$K(x_2, t, x_1, 0) = \frac{m}{2\pi\hbar} \times \int \mathrm{d}v_0 \Phi(v_0, x_2, t)\Phi^*(v_0, x_1, 0), \quad (49)$$

we deduce

$$K(x_2, t, x_1, 0) = \sqrt{\frac{1}{\lambda(s)}} \times \exp\left[\frac{i}{\hbar}\frac{m\lambda'(s)}{2\lambda(s)}y_2^2\right]\tilde{K}(y_2, s, y_1, 0) \times \sqrt{\frac{1}{\lambda(0)}}\exp\left[-\frac{i}{\hbar}\frac{m\lambda'(0)}{2\lambda(0)}y_1^2\right], \quad (50)$$

where $\tilde{K}(y_2, s, y_1, 0)$ is the new propagator given by

$$\tilde{K}(y_2, s, y_1, 0) = \left[\frac{m}{2\pi\hbar}\int \mathrm{d}\bar{v}_0\bar{\Phi}(\bar{v}_0, y_2, s)\bar{\Phi}^*(\bar{v}_0, y_1, 0)\right]. \quad (51)$$

Let us now consider a simple application of this technique in the case of time-dependent frequency variable oscillator

$$V(x, t) = \frac{m\omega^2(t)x^2}{2}. \quad (52)$$

The classical path $X(t)$ verifies the following differential equation:

$$\ddot{X} + \omega^2(t)X(t) = 0. \quad (53)$$

From eqs (43), (31), (39) and (52), the effective potential looks like

$$\bar{V}_{\mathrm{eff}}(y, s) = \frac{1}{2}m[\Omega^2(s) + \lambda^4\omega^2(s)]y^2 \quad (54)$$

with

$$\Omega^2(s) = \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\lambda'}{\lambda}\right) - \left(\frac{\lambda'}{\lambda}\right)^2\right] = \lambda^3\ddot{\lambda}.$$

We set $\Omega^2(s) + \lambda^4\omega^2(s) = \omega_0^2$, where ω_0 is a constant. We obtain the equation

$$\ddot{\lambda} + \lambda\omega^2(s) = \frac{\omega_0^2}{\lambda^3} \quad (55)$$

which is the well-known auxiliary equation [30,32]. Using eq. (50) one derives the propagator $K(x_2, t, x_1, 0)$ as

$$K(x_2, t, x_1, 0) = \sqrt{\frac{1}{\lambda(s)\lambda(0)}} \times \exp\frac{im}{2\hbar}\left[\frac{\lambda'(s)}{\lambda(s)}y^2 - \frac{\lambda'(0)}{\lambda(0)}y_1^2\right] \times \tilde{K}(y_2, s, y_1, 0), \quad (56)$$

where $\tilde{K}(y_2, s, y_1, 0)$ is the propagator of the constant frequency oscillator which can be calculated using the semiclassical de Broglie–Bohm semiclassical method [28]

$$\tilde{K}(y_2, s, y_1, 0) = \sqrt{\frac{m\omega_0}{2\pi i\hbar\sin\omega_0 s}}\exp\left\{\frac{im\omega_0}{2\hbar\sin\omega_0 s} \times [(y^2 + y_0^2)\cos\omega_0 s - 2yy_0]\right\}. \quad (57)$$

Consequently, the propagator of the variable frequency harmonic oscillator $K(x_2, t, x_1, 0)$ is

$$K(x_2, t, x_1, 0) = \sqrt{\frac{m\omega_0}{2\pi i\hbar\lambda(s)\lambda(0)\sin\omega_0 s}} \times \exp\frac{im}{2\hbar}\left[\frac{\lambda'(s)}{\lambda(s)}y^2 - \frac{\lambda'(0)}{\lambda(0)}y_1^2\right] \times \exp\left\{\frac{im\omega_0}{2\hbar\sin\omega_0 s}[(y^2 + y_0^2) \times \cos\omega_0 s - 2yy_0]\right\}. \quad (58)$$

This result coincides with [30,32] obtained by the technique of path integrals, using the generalized canonical transformations and time-dependent invariants.

It is remarkable that this linear space–time transformation, $x = \lambda(s)y$ and $s(t) = \int^t \mathrm{d}\tau\,\mu(\tau)$, leaves invariant the shape of the wave packet and quantum evolution following the scheme of Bohm. It is true that this result depends on the semiclassical approximation. Indeed for the problem studied, the used linear transformation simplifies the calculation and also guarantees this invariance. This aspect of invariance can be verified for all quadratic systems without restriction. However, for the general case of any potential the calculation is feasible via a general coordinate transformation but this semiclassical approximation cannot give exactly that invariance because the quantum corrections would be truncated.

At last, we give the correspondence between the original system and the transformed system following the linear transformation

$$\begin{array}{ll}
(x,t) & (y,s) \\
X_{\mathrm{cl}}(t)\colon \ddot{X}+\omega^2(t)X(t)=0 & Y_{\mathrm{cl}}(s)\colon Y''(s)+\omega_0^2 Y(s)=0 \\
 & \ddot{\lambda}+\lambda\omega^2(s)=\dfrac{\omega_0^2}{\lambda^3} \\
v^{\Psi}_{\mathrm{Bohm}}(x,t)\colon v(x,t)=\dfrac{1}{m}\dfrac{\partial S}{\partial x} & \bar{v}^{\bar{\Psi}}_{\mathrm{Bohm}}(y,s)\colon \bar{v}(y,s)=\dfrac{1}{m}\dfrac{\partial \bar{S}}{\partial y} \\
\rho(x,t)\colon \dfrac{\partial\rho(x,t)}{\partial t}+\dfrac{\partial}{\partial x}[\rho(x,t)v(x,t)]=0 & \bar{\rho}(y,s)\colon \dfrac{\partial\bar{\rho}(y,s)}{\partial s}+\dfrac{\partial}{\partial y}[\bar{\rho}(y,s)\bar{v}(y,s)]=0 \\
V_q(x,t)=-\dfrac{\hbar^2}{2m\sqrt{\rho}}\dfrac{\partial^2\sqrt{\rho}}{\partial x^2} & \bar{V}_q(y,s)=-\dfrac{\hbar^2}{2m\sqrt{\bar{\rho}}}\dfrac{\partial^2\sqrt{\bar{\rho}}}{\partial y^2} \\
a(t)\colon \ddot{a}(t)+\omega^2(t)a(t)=\dfrac{\hbar^2}{4m^2a^3(t)} & \tilde{a}(s)\colon \tilde{a}''(s)+\omega_0^2\tilde{a}(s)=\dfrac{\hbar^2}{4m^2\tilde{a}^3(s)} \\
\dfrac{\partial S(x,t)}{\partial t}+\dfrac{1}{2m}\left(\dfrac{\partial S}{\partial x}\right)^2+V(x,t)+V_q(x,t)=0 & \dfrac{\partial\bar{S}(y,s)}{\partial s}+\dfrac{1}{2m}\left(\dfrac{\partial\bar{S}}{\partial y}\right)^2+\bar{V}_{\mathrm{eff}}(y,s)+\bar{V}_q(y,s)=0 \\
 & \bar{V}_{\mathrm{eff}}(y,s)=\tfrac{1}{2}m\omega_0^2y^2 \\
\Psi(x,t)=\sqrt{\rho(x,t)}\exp\left[\dfrac{i}{\hbar}S(x,t)\right] & \bar{\Psi}(y,s)=\sqrt{\bar{\rho}(y,s)}\exp\left[\dfrac{i}{\hbar}\bar{S}(y,s)\right] \\
i\hbar\dfrac{\partial\Psi(x,t)}{\partial t}=\left[-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}+V(x,t)\right]\Psi(x,t) & i\hbar\dfrac{\partial\bar{\Psi}(y,s)}{\partial s}=\left[-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial y^2}+\bar{V}_{\mathrm{eff}}(y,s)\right]\bar{\Psi}(y,s) \\
K(x,t,x_0,0) & \tilde{K}(y_2,s,y_1,0)
\end{array} \tag{59}$$

4. Conclusion

In this paper, we have applied space–time transformation to the de Broglie–Bohm theory. We have shown that according to this transformation, the quantum evolution is preserved in its semiclassical framework, based on the generalized Fourier transform. The expression of this de Broglie–Bohm propagator is related to some semiclassical Feynman propagator as a sum over final extremity. The method is applied to the variable-frequency harmonic oscillator and is converted to that with constant frequency using this linear space–time transformation. The method can be generalized to the general space–time transformation in which the geometric characteristics of coordinates are relevant.

References

[1] C Philippidis, D Bohm and R Kaye, *Nuovo Cimento B* **71**, 75 (1982)

[2] R E Wyatt, *Quantum dynamics with trajectories* (Springer, 2005)

[3] J Hirschfelder, A C Christoph and W E Palke, *J. Chem. Phys.* **61**, 5435 (1974)

[4] C Dewdney and B J Hiley, *Found. Phys.* **12**, 27 (1982)

[5] R T Skodje, H W Rohrs and J VanBuskirk, *Phys. Rev. A* **40**, 2894 (1989)

[6] C L Lopreore and R E Wyatt, *Phys. Rev. Lett.* **82**, 5190 (1999)

[7] C Philippidis, C Dewdney and B Hiley, *Nuovo Cimento B* **52**, 15 (1979)

[8] E Gindensperger, *Utilisation des trajectoires quantiques dans des processus dynamiques moléculaires*, Ph.D. Dissertation (Université Toulouse III, 2003)

[9] E Gindensperger, C Meier, J A Beswick and M C Heitz, *J. Chem. Phys.* **116**, 10051 (2002)

[10] A S Sanz, F Borondo and S Miret-Artés, *J. Chem. Phys.* **120**, 8794 (2004)

[11] A S Sanz, F Borondo and S Miret-Artés, *Phys. Rev. B* **69**, 115413 (2004)

[12] A Valentini and H Westman, *Proc. R. Soc. A* **461**, 253 (2005)

[13] C Efthymiopoulos and G Contopoulos, *J. Phys. A* **39**, 1819 (2006)

[14] A Bennett, *J. Phys. A* **43**, 195304 (2010)

[15] S Colin and W Struyve, *New J. Phys.* **12**, 043008 (2010)

[16] M D Towler, N J Russell and A Valentini, arXiv:1103.1589 (2011)

[17] M Daumer, Bohmian mechanics and quantum theory: An apraisal, *Boston Study Phil. Sci.*, edited by J T Cushing, A Fine and S Goldstein, Vol. 84 (1996)

[18] M Daumer, D Dürr, S Goldstein and N Zanghi, *Lett. Math. Phys.* **38**, 103 (1996)
[19] M Daumer, D Dürr, S Goldstein and N Zanghi, *J. Stat. Phys.* **88**, 967 (1996)
[20] D Dürr, S Goldstein, S Teufel and N Zanghi, *Physica A* **279**, 416 (2000)
[21] D Dürr, S Goldstein, T Moser and N Zanghi, *Commun. Math. Phys.* **266**, 665 (2006)
[22] M Meisinger, *Coulomb scattering in Bohmian mechanics*, Diploma Thesis (University of Innsbruck, 2006)
[23] A S Sanz, F Borondo and S Miret-Artés, *J. Phys.: Condens. Matter* **14**, 6109 (2002)
[24] G Hazak and I B Bernstein, *Phys. Fluids* **26**, 684 (1983)
[25] I B Bernstein, *Phys. Rev. A* **32**, 1 (1985)
[26] A B Nassar, J M F Bassalo, P T S Alencar, L S G Cancela and M S D Cattani, *Phys. Rev. E* **56**, 1230 (1997)
[27] A B Nassar, J M F Bassalo, P T S Alencar, J L M Lops, J I F de Oliveira and M S D Cattani, *Braz. J. Phys.* **32**, 812 (2002)
[28] A Tilbi, T Boudjedaa, M Merad and L Chetouani, *Phys. Scr.* **75**, 474 (2007)
[29] M A Vandyck, *J. Phys. A* **27**, 1743 (1994)
[30] L Chetouani and L Guechi, *Phys. Rev. A* **40**, 1157 (1989)
[31] A O Barut and N Zanghi, *Phys. Rev. Lett.* **52**, 2009 (1984)
[32] A K Dhara and S V Lawnde, *Phys. Rev. A* **30**, 560 (1984)