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Symmetry and conservation law structures of some anti-self-dual (ASD) manifolds

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Abstract. The ASD systems and manifolds have been studied via a number of approaches and their origins have been well documented. In this paper, we look at the symmetry structures, variational symmetries and related concepts around the associated conservation laws for a number of such manifolds.

Keywords. Symmetries and conservation laws; anti-self-dual manifolds.

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1. Introduction and background

The dispersionless integrable systems in 3+1 dimensions do not admit soliton solutions and there is no associated Riemann–Hilbert problem where the corresponding Lie group is finite-dimensional. However, such systems may be described in terms of ASD quaternion–Kahler four-manifolds, or equivalently ASD Einstein manifolds, are locally determined by one scalar function subject to Przanowski's equation (see [1,2] and references therein). In this case, the unknown in the equations is a potential of the metric on some four-manifolds ([3]) – this makes the dispersionless systems more geometric and may be of importance in the description of shock formations (see Manakov and Santini [4] and Dunajski [5]). If the Ricci-flat condition is imposed on top of the anti-self-duality, the work of Plebanski [6] implies the existence of a local coordinate system (x, y, z, w) and a function u on an open set such that any ASD Ricci-flat metric, g, is locally of the form

$$\mathrm{d}s^2 = 2(\mathrm{d}z\mathrm{d}y+\mathrm{d}w\mathrm{d}x-u_{xx}\mathrm{d}z^2-u_{yy}\mathrm{d}w^2+2u_{xy}\mathrm{d}w\mathrm{d}z), \tag{1}$$

where $u(x, y, z, w)$ satisfies the second heavenly equation (sHE)

$$u_{wx} - u_{zy} + u_{xx}u_{yy} - u_{xy}^2 = 0. \tag{2}$$

In [5], the above equations are studied subject to the existence of a conformal Killing vector X satisfying

$$Q_X g = pg, \tag{3}$$

for some function p and Q is the Lie derivative. The relationship between the HE and the Monge–Ampere equation is discussed in Husain [7] and Malykh and Sheftel [8]. Recently, Doubrov and Ferapontov [9] introduced the general HE arising in the study of normal forms of the integrable four-dimensional Monge–Ampere equation. In [10], Manakov and Santini describe reductions of the HE, based on certain self-similar transformations.

As a second case, we refer to the results of the phenomenal work by Plebanski who, in a number of papers, e.g., [6], showed that the Einstein field equations leading to anti-self-dual (ASD) metrics are reducible to the elliptic complex Monge–Ampere equation (CMA) or, after some reductions and substitutions, to the first or second 'heavenly equations' or the Boyer–Finley equations. For example, the complex manifold with a Kähler metric

$$\mathrm{d}s^2 = u_{i\bar{k}}\mathrm{d}z^i\mathrm{d}\bar{z}^k \tag{4}$$

with a metric potential $u(z^1, z^2, \bar{z}^1, \bar{z}^2)$ that satisfies the CMA equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = 1. \tag{5}$$

Versions of (5) are sometimes referred to as the ‘first heavenly equation’ and, via a number of transformations, can be reduced to the Boyer–Finley (BF) equation which has been studied by many people including Calderbank and Todd [11] and Martina *et al.* [12].

Recently, Nutku and Sheftel [13] have, in detail, considered transformations of (5) and the corresponding metric (4) based on some solutions of the BF.

(i) A solution is given by

$$w = \ln\left[\frac{(p + a(z))b'(z)}{1 + |b(z)|^2}\right]^2, \tag{6}$$

where $a(z)$ and $b(z)$ are arbitrary holomorphic functions. With $b = z$, $\mathcal{R}(a)$ constant and $\mathcal{I}(a) = \alpha$ and some detailed transformations (see [13]), the metric (1) becomes

$$\begin{aligned}\mathrm{d}s^2 =& \frac{4r^4}{r^4 + \alpha^2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\\ &+ \frac{r^4 + \alpha^2}{r^2}(\mathrm{d}t + (1 + \cos\theta)\mathrm{d}\phi)^2\end{aligned} \tag{7}$$

which looks like an Eguchi–Hanson [14] metric

$$\begin{aligned}\mathrm{d}s^2 =& \frac{r^4}{r^4 - \alpha^2}\mathrm{d}r^2 + \frac{1}{4}r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\\ &+ \frac{r^4 - \alpha^2}{4r^2}(\mathrm{d}t + \cos\theta\mathrm{d}\phi)^2.\end{aligned} \tag{8}$$

(ii) A non-invariant solution

$$w = \ln(p^2 + p(z + \bar{z}) + |z|^2 - 2\ln(1 + |z|^2), \tag{9}$$

of the BF equation leads to the metric

$$\begin{aligned}\mathrm{d}s^2 =& \frac{r^2}{r^4 + \cot^2((1/2)\theta)\sin^2\phi}\\ &\left(2r\mathrm{d}r + \cot((1/2)\theta)\sin\phi\mathrm{d}\phi + \frac{\cos\phi}{2\sin^2((1/2)\theta)}\mathrm{d}\theta\right)^2\\ &+r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\\ &+\frac{r^4 + \cot^2((1/2)\theta)\sin^2\phi}{r^2}\left[\mathrm{d}t + (1 + \cos\theta)\mathrm{d}\phi\right.\\ &\left.+\frac{(r^2 - \cot((1/2)\theta)\sin\phi)\sin\phi\mathrm{d}\theta - (r^2\cos\phi + \cot^2((1/2)\theta)\sin^2\phi)\sin\theta\mathrm{d}\phi}{2\sin^2((1/2)\theta)(r^4 + \cot^2((1/2)\theta)\sin^2\phi)}\right]^2.\end{aligned} \tag{10}$$

We present some salient features of an Euler–Lagrange system of differential equations. Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz. [15–17],

$$E^\beta(t, v, v_{(1)}, \ldots, v_{(r)}) = 0, \qquad \beta = 1, \ldots, m. \tag{11}$$

A conservation law of (11) is a solution of the equation given by

$$D_i T^i = 0, \tag{12}$$

on the solutions of (11), where D_i is the total derivative operator given by

$$D_i = \frac{\partial}{\partial t^i} + v_i^\alpha\frac{\partial}{\partial v^\alpha} + v_{ij}^\alpha\frac{\partial}{\partial v_j^\alpha} + \cdots, \qquad i = 1, \ldots, n$$

and $T = (T^1, \ldots, T^n)$ is a conserved vector of (11).

In this work, we assume that X is a Lie point symmetry operator, i.e., ξ and η are functions of t and v and are independent of derivatives of v. The Euler–Lagrange equations, if they exist, associated with (11) is the system $\delta L/\delta v^\alpha = 0$, $\alpha = 1, \ldots, m$, where the Euler–Lagrange operator $\delta/\delta v^\alpha$ is given by

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m.$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(t, v, v_{(1)}, \ldots, v_{(r)})\mathrm{d}t$$

defined over Ω. If we include point-dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries X are given by a Killing-type equation

$$XL + LD_i\xi_i = D_i f_i. \tag{13}$$

Corresponding to each X, a conserved vector $T = (T^1, \ldots, T^n)$ is obtained via Noether’s theorem.

For a space–time manifold, g, if s is the arclength variable, let the manifold be defined by a two-form

$$\mathrm{d}s^2 = u_{ik}\mathrm{d}y^i\mathrm{d}z^k. \tag{14}$$

The isometries/Killing vectors are the one-to-one, onto maps that leave invariant the distance function defined by the metric. It is well known (see [17], for e.g.) that the Killing vectors X are given by

$$Q_X g = 0 \tag{15}$$

and the conformal Killing vector X satisfies

$$Q_X g = pg, \tag{16}$$

for some function p, where Q is the Lie derivative. The geodesic equations are a system of second-order ordinary differential equations (ODEs) which, it turns out, are the Euler–Lagrange equations obtained via a Lagrangian

$$L = u_{ik}(y^i)'(z^k)', \tag{17}$$

where $'$ is the derivative with respect to s. Some details may be found in [18] and [19–22].

Another method for determining conservation laws utilizes 'multipliers' that nullify the Euler operation on the product of a multiplier and the equation. That is, if there exist multipliers $Q = Q^\beta(t, v, v_{(1)}, \ldots)$,

$$\frac{\delta}{\delta v}[Q^\beta(t,v,v_{(1)},\ldots,)E^\beta(t,v,v_{(1)},\ldots,v_{(r)})] = 0, \tag{18}$$

there exists a vector $T = (T^1, \ldots, T^n)$ such that

$$Q^\beta(t,v,v_{(1)},\ldots,)E^\beta(t,v,v_{(1)},\ldots,v_{(r)}) = D_i T^i. \tag{19}$$

On the solutions of a differential equation (11), the vector T is conserved. Here, no Lagrangian is required and there is freedom in the choice of the order of multipliers which arise naturally in Noether's theorem. For variational system, the multipliers are symmetries of eq. (11).

2. ASD manifolds

2.1 *Reductions of the heavenly equation*

We, first, present a procedure to determine exact solutions of the second heavenly equation.

A basis of the Lie point symmetry algebra of (2) is

$$\mathcal{X}_1 = x\partial_u,$$
$$\mathcal{X}_2 = xf^5(z)\partial_u,$$
$$\mathcal{X}_3 = 2w\partial_y - xy\partial_u,$$
$$\mathcal{X}_4 = w\partial_w + y\partial_y + u\partial_u,$$
$$\mathcal{X}_5 = 2w\partial_w - x\partial_x + y\partial_y - u\partial_u,$$
$$\mathcal{X}_6 = -2f^6(z)\partial_y + x^2 f^6_z\partial_u,$$
$$\mathcal{X}_7 = 6f^7(z)\partial_w + 6f^7_z\partial_y - x^2 f^7_{zz}\partial_u,$$
$$\mathcal{X}_8 = 6f^2(z,w)\partial_z - 6\left(x\int f^2_{zz}\mathrm{d}w + yf^2_z\right)\partial_y + 6(yf^2_w + xf^2_z)\partial_x - 6\int f^2_z \mathrm{d}w\partial_w + \left(x^3\int f^2_{zzz} + y(y^2 f^2_{ww} + 3x(yf^2_{zw} + xf^2_{zz}))\right)\partial_u.$$

(a) In $\mathcal{X}_8$, if $f^2 = -\frac{1}{6}z$, we get the scaling symmetry generator $\mathcal{X} = w\partial_w - x\partial_x + y\partial_y - z\partial_z$ leading to the invariants and transformed variables

$$X = xw, \quad Y = y/w, \quad Z = zw, \quad U = u,$$
$$U = U(X, Y, Z),$$

so that (2) becomes

$$U_X + XU_{XX} - YU_{XY} + ZU_{XZ} - U_{YZ} + U_{XX}U_{YY} - U^2_{XY} = 0, \tag{20}$$

which admits the symmetry generator $\mathcal{X}^1_8 = 2\partial_X + Y^2\partial_U$ and the scaling generator $\mathcal{X}^2_8 = X\partial_X + Y\partial_Y + 3U\partial_U$.

Via $\mathcal{X}^1_8$, it can be shown that (20) reduces to the canonical equation

$$\bar{U}_{YZ} + \frac{3}{2}Y^2 = 0,$$

where $\bar{U} = U - \frac{1}{2}XY^2$ ($\bar{U} = \bar{U}(Y, Z)$). It can be shown, thus, that a solution of the heavenly equation is

$$u = -\frac{1}{2}\frac{zy^3}{w^2} + \frac{1}{2}\frac{xy^2}{w}. \tag{21}$$

Similarly, (20) may be transformed using $\mathcal{X}^2_8$, i.e., $\alpha = Y/X$, $Z = Z$ and $\bar{U} = U/X^3$ with $\bar{U} = \bar{U}(\alpha, Z)$. That is,

$$2\alpha^2\bar{U}_{\alpha\alpha} - (z\alpha + 1)\bar{U}_{\alpha Z} - 7\alpha\bar{U}_\alpha + 3z\bar{U}_Z + 15\bar{U} + 4\bar{U}^2_\alpha = 0.$$

(b) For an alternative reduction of (20), one may take a linear combination of $\mathcal{X}_7$ and $\mathcal{X}_8$ with $F^7 = \frac{1}{6}z$ and $-\frac{1}{6}t$, respectively.

(c) $\mathcal{X}_4 + \mathcal{X}_5$ leads to similarity variables $\alpha = tx^3$, $\beta = yx^2$, $z = z$ and $U = u$ with $U = U(\alpha, \beta, z)$.

(d) A 'travelling wave' reduction takes the form

$$-cu_{\alpha\alpha} - u_{\beta\gamma} + u_{\alpha\alpha}u_{\beta\beta} - u^2_{\alpha\beta} = 0,$$

where $u = u(\alpha, \beta, \gamma)$, $\alpha = x - ct$, $\beta = y - kt$ and $\gamma = z - mt$, c, k and m are constants ('wave speeds' in the direction of x, y and z, respectively). The reduced PDE may then be analysed further using a Lie symmetry reduction.

Thus, a large class of reductions and invariant, exact solutions of (2) are obtainable.

2.2 *Lie and Noether symmetries and Killing vectors*

In general, the Lagrangian of the geodesic equations is given by

$$L = z'y' + w'x' - u_{xx}z'^2 - u_{yy}w'^2 + 2u_{xy}w'z',$$

where $'$ is the derivative with respect to the arc-length variable s. The Euler–Lagrange equations are

$$-w'' - w'^2u_{xyy} + 2w'z'u_{xxy} - z'^2u_{xxx} = 0,$$
$$-z'' - w'^2u_{yyy} + 2w'z'u_{xyy} - z'^2u_{xxy} = 0,$$
$$-y'' - 2w''u_{xy} - w'^2(u_{yyz} + 2u_{xyw}) + 2z''u_{xx} + z'^2u_{xxz} + 2y'z'u_{xxy}$$

$$\begin{aligned}&-2w'(y'u_{xyy}-z'u_{xxw}+x'u_{xxy})+2x'z'u_{xxx}=0,\\&-x''+2w''u_{yy}+w'^2u_{yyw}+2w'z'u_{yyz}+2w'y'u_{yyy}\\&-2z''u_{xy}-2z'^2u_{xyz}\\&+2w'x'u_{xyy}-2y'z'u_{xyy}-z'^2u_{xxw}-2x'z'u_{xxy}=0.\end{aligned} \tag{22}$$

For $u=-\frac{1}{2}\frac{zy^3}{w^2}+\frac{1}{2}\frac{xy^2}{w}$ given in (21), the Euler–Lagrange (geodesic) equations are given by

$$\begin{aligned}&\frac{w'^2}{w}+w''=0,\\&-z''+\frac{2w'z'}{w}+\frac{3zw'^2}{w^2}=0\\&-y''-\frac{2w''y}{w}+\frac{5w'^2y}{w^2}-\frac{2w'y'}{w}=0\\&-x''+2w''\left(\frac{x}{w}-\frac{3yz}{w^2}\right)+w'^2\left(-\frac{x}{w^2}+\frac{6yz}{w^3}\right)\\&\qquad-\frac{6w'z'y}{w^2}-\frac{6w'y'z}{w^2}-\frac{2}{w}(z''y-w'x'+y'z')=0.\end{aligned} \tag{23}$$

The Noether symmetry generators X which satisfy eq. (13) is given by

$$\begin{aligned}X_1&=2s\partial_s+w\partial_w+x\partial_x+y\partial_y+z\partial_z,\\X_2&=\left(-\frac{\sqrt{7}}{3}+\frac{2}{3}\right)z^{2+\sqrt{7}}\partial_w+yz^{1+\sqrt{7}}\partial_x,\\X_3&=\left(\frac{2}{3}+\frac{\sqrt{7}}{3}\right)z^{2-\sqrt{7}}\partial_w+yz^{1-\sqrt{7}}\partial_x,\\X_4&=-(2+\sqrt{7})wz^{1+\sqrt{7}}\partial_x+z^{\sqrt{7}}\partial_y,\\X_5&=wz^{-1-\sqrt{7}}(\sqrt{7}-2)\partial_x+z^{-\sqrt{7}}\partial_y,\\X_6&=z\partial_x,\quad X_7=sz\partial_x,\quad X_8=\partial_s,\end{aligned} \tag{24}$$

where X_1, X_7, X_8 are not Killing vectors and X_7 is a non-zero gauge ($f\neq 0$) variational symmetry.

Each of these or a linear combination lead to a conservation law for the geodesic equations by Noether's theorem above. For e.g., from X_8 we get the conserved quanitity

$$T_8=\frac{w'^2(xw-3yz)-2yww'z'-w^2(z'y'+x'w')}{w^2}.$$

The algebra of Killing vectors based on the equivalent metric is a subalgebra of the algebra above, i.e., $Q_X g=0$. These are generated by

$$X_2,\quad X_3,\quad X_4,\quad X_5,\quad X_6.$$

3. ASD manifolds – The second case

We first consider a basic illustrative example using the multiplier approach. The well-known flat Minkowski manifold is governed by the metric, in polar coordinates, by

$$\mathrm{d}s^2=-\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2, \tag{25}$$

which admits a well-known 10-dimensional algebra of Killing vectors, viz., $SO(3)\otimes\mathbb{R}^4$. It is shown in [23] that the algebra of variational symmetries of the geodesic equations via the Lagrangian

$$L=t'^2-r'^2-r^2\theta'^2-r^2\sin^2\theta\phi'^2 \tag{26}$$

admits a 17-dimensional algebra including the Killing vectors. In fact, the multiplier approach to determine conserved vectors lead to the following list of 17 multipliers (first derivative-dependent), with $Q=(Q^r,Q^\theta,Q^\phi,Q^t)$, viz.,

$$\begin{aligned}&\left(-t\sin\theta\sin\phi,-t\sin\phi\cos\theta,\frac{-t\cos\phi}{r\sin\theta},r\sin\theta\sin\phi\right),\\&\left(-t\sin\theta\cos\phi,-t\cos\phi\cos\theta,\frac{t\sin\phi}{r\sin\theta},r\sin\theta\sin\phi\right),\\&\left(-t\cos\phi,\frac{t\sin\theta}{r},0,r\cos\theta\right),\quad(\cos\theta,-\sin\theta/r,0,0),\\&\left(\sin\theta\sin\phi,\sin\phi\cos\theta/r,\frac{\cos\phi}{r\sin\theta},0\right),\quad\left(\sin\theta\cos\phi,\cos\phi\cos\theta/r,-\frac{\sin\phi}{r\sin\theta},0\right),\\&\left(s\sin\theta\sin\phi,s\sin\phi\cos\theta/r,\frac{s\cos\phi}{r\sin\theta},0\right),\quad\left(s\sin\theta\cos\phi,s\cos\phi\cos\theta/r,-\frac{s\sin\phi}{r\sin\theta},0\right),\\&(s\cos\theta,-s\sin\theta/r,0,0),\\&(0,-\cos\phi,\cot\theta\sin\phi,0),\quad(0,\sin\phi,\cot\theta\cos\phi,0),\\&(0,0,1,0),\quad(0,0,0,s),\\&(0,0,0,1),\quad(-r',-\theta',-\phi',t'),\\&(sr-r's^2,-\theta's^2,-\phi's^2,st-t's^2),\quad(r-2sr',-2s\theta',-2s\phi',t-2st')\end{aligned} \tag{27}$$

We note that the 35-dimensional algebra of Lie point symmetry generators are discussed in [24].

The space–time of constant curvature following the de-Sitter metric also admits a 10-dimensional algebra of isometries inequivalent to the above, viz., $SO(1,4)$. Furthermore, only one additional variational symmetry is obtained. This is easily verifiable using the multiplier approach.

Further well-known manifolds and their symmetry structures can be found in [23]. The multiplier approach covers all the Killing vectors, Noether symmetries including non-zero gauge ones which are not picked up by software packages to determine variational symmetries.

3.1 *Variational symmetries and Killing vectors*

In this subsection, we classify the symmetries of the manifolds that arise from the metrics due to some new ASD solutions of the Einstein equations.

In (1) above, the Lagrangian of the geodesic equations is given by

$$\begin{aligned} L = {} & \frac{4r^4}{r^4+\alpha^2}r'^2 + r^2\theta'^2 + r^2\sin^2\theta\phi'^2 \\ & + \frac{r^4+\alpha^2}{r^2}(t'^2 + 2(1+\cos\theta))t'\phi' \\ & + (1+\cos\theta))^2\phi'^2). \end{aligned}$$

The Euler–Lagrange equations are

$$\begin{aligned} & -\frac{1}{r^3}\big[2\big(-2(\alpha^2-r^4)r'(t'+(1+\cos(\theta))\phi')-r(\alpha^2+r^4) \\ & \qquad \times(\sin(\theta)\theta'\phi' - t'' - (1+\cos(\theta))\phi'')\big)\big] = 0, \\ & \frac{16r^7r'^2}{(\alpha^2+r^4)^2} - \frac{16r^3r'^2}{\alpha^2+r^4} + 2r\theta'^2 + 2r\sin(\theta)^2\phi'^2 \\ & \quad + 4r(t'+(1+\cos(\theta))\phi')^2 \\ & \quad - \frac{2(\alpha^2+r^4)(t'+(1+\cos(\theta))\phi')^2}{r^3} - \frac{8r^4r''}{\alpha^2+r^4} = 0, \\ & -4rr'\theta' + r^2\sin(2\theta)\phi'^2 \\ & \quad - \frac{2(\alpha^2+r^4)\sin(\theta)\phi'(t'+(1+\cos(\theta))\phi')}{r^2} \\ & \quad - 2r^2\theta'' = 0, \\ & \frac{1}{r^3}2\cos(\theta/2)[4\cos(\theta/2)r'((\alpha^2-r^4)t' \\ & \quad + (\alpha^2(1+\cos(\theta))-2r^4)\phi')+r(2(\alpha^2+r^4) \\ & \quad \times\sin(\theta/2)t'\theta'+4(\alpha^2(1+\cos(\theta))+r^4) \\ & \quad \times\sin(\theta/2)\theta'\phi'-2\cos(\theta/2)((\alpha^2+r^4)t'' \\ & \quad + (\alpha^2(1+\cos(\theta))+2r^4)\phi''))] = 0. \end{aligned} \tag{28}$$

The algebra of point variational symmetries, $X = \xi\partial_s + \tau\partial_t + \rho\partial_r + A\partial_\theta + B\partial_\phi$, can be obtained from (13). However, this is a cumbersome approach; the multiplier method provides all the appropriate information. The multipliers are (28),

$$\begin{aligned} & (1,0,0,0), \qquad (0,0,0,1), \\ & \left(-\frac{\cos\phi}{\tan((1/2)\theta)}, 0, \sin\phi, \frac{\cos\phi}{\tan\theta}\right), \\ & \qquad (-t', -r', -\theta', -\phi'), \\ & \left(\frac{\sin\phi}{\tan((1/2)\theta)}, 0, \cos\phi, -\frac{\sin\phi}{\tan\theta}\right). \end{aligned} \tag{29}$$

It turns out that each leads to a variational symmetry and the algebra is generated by

$$\begin{aligned} & \partial_t,\ \ \partial_\phi,\ \ \partial_s,\ \ -\frac{\cos\phi}{\tan((1/2)\theta)}\partial_t + \sin\phi\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi, \\ & \frac{\sin\phi}{\tan((1/2)\theta)}\partial_t + \cos\phi\partial_\theta - \frac{\sin\phi}{\tan\theta}\partial_\phi. \end{aligned} \tag{30}$$

It can be shown that all of the above, excluding the translation in s are isometries of the manifold.

3.2 *Variational symmetries and Killing vectors for cases of metrics with ultrahyperbolic signature*

One choice of the metric here [13] is

$$\mathrm{d}s^2 = w_p(4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z} - \mathrm{d}p^2) - \frac{1}{w_p}\big[\mathrm{d}t + i(w_z\mathrm{d}z - w_{\bar{z}}\mathrm{d}\bar{z})\big]^2, \tag{31}$$

in which case, the Einstein field equations reduce to the hyperbolic Boyer–Finley (hBF) equation

$$w_{z\bar{z}} - (\mathrm{e}^w)_{pp} = 0, \tag{32}$$

for which a solution is

$$w = \ln\left[\frac{(p+a(z))b'(z)}{b(z)+\bar{b}(\bar{z})}\right]^2. \tag{33}$$

This leads to an ASD Ricci-flat metric with ultrahyperbolic signature. A form of the solution is obtained by regarding z and $\bar{z}$ as real null coordinates u and v. Then there are two metrics of which one is

$$\begin{aligned} \mathrm{d}s_1^2 = {} & (2p+a+b)\left[\frac{4f'g'}{(f+g)^2}\mathrm{d}u\mathrm{d}v - \frac{1}{(p+a)(p+b)}\mathrm{d}p^2\right] \\ & + \frac{(p+a)(p+b)}{(2p+a+b)}\bigg[\mathrm{d}t + \left(\frac{2g'}{f+g} - \frac{b'}{p+b} - \frac{g''}{g'}\right)\mathrm{d}v \\ & - \left(\frac{2f'}{f+g} - \frac{a'}{p+a} - \frac{f''}{f'}\right)\mathrm{d}u\bigg]^2, \end{aligned} \tag{34}$$

where $a = a(u)$, $b = b(v)$, $f = f(u)$ and $g = g(v)$.

For $f = u$, $g = v$, $a = u$ and $b = v$, the only isometry is ∂_t with a single additional variational symmetry ∂_s.

If a and b are equal to constants with $f = u$, $g = v$, we obtain a large number of vector fields. The least cumbersome approach yielding the complete list is obtained via the multiplier approach. We choose $a = b = 1$; the multipliers for the geodesic equations being the Euler–Lagrange equations arising as a consequence of the Lagrangian, $Q = (Q^u, Q^v, Q^p, Q^t)$, are

$$\begin{aligned}
Q^1 &= \left(-\frac{su\sqrt{u+v}e^{(1/4)t}}{\sqrt{p}}, 0, \frac{\sqrt{p}sve^{(1/4)t}}{\sqrt{u+v}}, -2\frac{\sqrt{u+v}se^{(1/4)t}}{\sqrt{p}}\right),\\
Q^2 &= \left(\frac{s\sqrt{u+v}e^{(1/4)t}}{\sqrt{p}}, 0, \frac{\sqrt{p}se^{(1/4)t}}{\sqrt{u+v}}, 0\right),\\
Q^3 &= \left(0, -\frac{sv\sqrt{u+v}e^{-(1/4)t}}{\sqrt{p}}, \frac{\sqrt{p}sue^{-(1/4)t}}{\sqrt{u+v}}, 2\frac{\sqrt{u+v}se^{-(1/4)t}}{\sqrt{p}}\right),\\
Q^4 &= \left(0, \frac{s\sqrt{u+v}e^{-(1/4)t}}{\sqrt{p}}, \frac{\sqrt{p}se^{-(1/4)t}}{\sqrt{u+v}}, 0\right),\\
Q^5 &= \left(-\frac{u\sqrt{u+v}e^{(1/4)t}}{\sqrt{p}}, 0, \frac{\sqrt{p}ve^{(1/4)t}}{\sqrt{u+v}}, -2\frac{\sqrt{u+v}e^{(1/4)t}}{\sqrt{p}}\right),\\
Q^6 &= \left(\frac{\sqrt{u+v}e^{(1/4)t}}{\sqrt{p}}, 0, \frac{\sqrt{p}e^{(1/4)t}}{\sqrt{u+v}}, 0\right),\\
Q^7 &= \left(0, -\frac{v\sqrt{u+v}e^{-(1/4)t}}{\sqrt{p}}, \frac{\sqrt{p}ue^{-(1/4)t}}{\sqrt{u+v}}, 2\frac{\sqrt{u+v}e^{-(1/4)t}}{\sqrt{p}}\right),\\
Q^8 &= \left(0, \frac{\sqrt{u+v}e^{-(1/4)t}}{\sqrt{p}}, \frac{\sqrt{p}e^{-(1/4)t}}{\sqrt{u+v}}, 0\right),\\
Q^9 &= (0, (u+v)e^{-(1/2)t}, 0, -2e^{-(1/2)t}),\\
Q^{10} &= \left(-\frac{1}{2}u^2, \frac{1}{2}v^2, 0, -(u+v)\right),\\
Q^{11} &= (u, v, 0, 0), \quad Q^{12} = (-1, 1, 0, 0),\\
Q^{13} &= ((u+v)e^{(1/2)t}, 0, 0, 2e^{(1/2)t}),\\
Q^{14} &= (0, 0, 0, 1),\\
Q^{15} &= (-u', -v', -p', -t'),\\
Q^{16} &= (-su', -sv', (p+1) - sp', -st'). \qquad (35)
\end{aligned}$$

The isometries of the manifold, it can be shown, form a 10-dimensional algebra corresponding to the vector fields arising from multipliers Q^5 to Q^{14}. This suggests that the manifold in this case is flat. Also, the Noether symmetries lead to an additional six conservation laws.

In general, if a or b is not constant, only two symmetries (multipliers) are obtained, viz., ∂_s and ∂_t.

Another related metric from (33) is given by

$$\begin{aligned}
ds_2^2 = (2p+a+b)&\left[\frac{4f'g'}{(1+fg)^2}dudv + \frac{1}{(p+a)(p+b)}dp^2\right]\\
&-\frac{(p+a)(p+b)}{(2p+a+b)}\left[dt + \left(\frac{2fg'}{1+fg} - \frac{b'}{p+b} - \frac{g''}{g'}\right)dv\right.\\
&\left.-\left(\frac{2gf'}{1+fg} - \frac{a'}{p+a} - \frac{f''}{f'}\right)du\right]^2. \qquad (36)
\end{aligned}$$

A series of calculations as above leads to equivalent results. That is, constants a and b lead to 16 multipliers corresponding to 16 variational symmetries.

4. Conclusion

Even though the two particular cases discussed in §3.2 lead to a 10-dimensional algebra of Killing vectors suggesting that the manifolds are flat, it is not conclusive whether they are equivalent to either of the Minkowsi or de Sitter metrics; the algebra of Noether symmetries in the latter two cases are 17- and 12-dimensional, respectively, whilst the two classes in §3.2 are 16-dimensional. Finally, it is clear that the multiplier approach to construct the variational symmetries is more efficient and inclusive than the other approaches and one can easily identify the isometries, too, of the underlying manifold.

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