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The classification of the single travelling wave solutions to the variant Boussinesq equations

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Abstract. The discrimination system for the polynomial method is applied to variant Boussinesq equations to classify single travelling wave solutions. In particular, we construct corresponding solutions to the concrete parameters to show that each solution in the classification can be realized.

Keywords. Exact solution; discrimination system for the polynomial method; variant Boussinesq equations; travelling wave solutions.

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1. Introduction

The construction of exact solutions to nonlinear differential equations is an important and difficult task. It plays a significant role in mathematics and physics. For obtaining exact solutions, many methods such as direct method [1], canonical-like method [2], symmetrical method [3,4], etc. as proposed.

Recently, Liu proposed a powerful method called the complete discrimination system for polynomial method, to classify single travelling wave solutions to a series of nonlinear differential equations [5–10]. By his method, many difficult nonlinear partial differential equations have been solved. If we take the travelling wave transformation and then integrate it, the nonlinear differential equation can be reduced as

$$\pm(\xi - \xi_0) = \int \sqrt{P_n(u)}\mathrm{d}u, \tag{1}$$

where $P_n(u)$ is a rational function. Then we can classify all travelling wave solutions to eq. (1). For example, Fan used Liu's method [11,12] to invest the generalized equal width equation and Pochhammer–Chree equation, and she obtained all the possible travelling wave solutions including elliptic functions and hyperelliptic functions.

In this paper, we consider the variant Boussinesq equations [13]

$$H_t + HH_x + gu_x + \frac{1}{3}au_{ttx} = 0, \tag{2}$$

$$u_t + Hu + uH_x = 0, \tag{3}$$

where g and a ($a \neq 0$) are real constants, $u(x, t)$ is the velocity, $H(x, t)$ is the total depth and the subscripts denote the partial derivative. This model was originally introduced by Boussinesq to describe the propagation of long waves in shallow water [14]. Furthermore, the equations also arise in many other physical applications such as nonlinear lattice waves [15], ion sound waves in plasma [16] and vibrations in a nonlinear string [17], and they were also applied to study the percolation of water in porous subsurface strata [18]. So, finding exact solutions to the variant Boussinesq equations is a very important task. In ref. [18], Li and his colleagues obtained smooth, cusped solitary wave solutions of the variant Boussinesq equations and analysed their analytical and dynamical behaviour, but they have not given all the travelling wave solutions to eqs (2) and (3). In this paper, we use the complete discrimination system for polynomial method to study the equations and give all concrete solutions for the classification under the concrete parameters. This means that all solutions in the classification can be realized.

2. All possible exact travelling wave solutions to the variant Bossinesq equation

To find all possible exact travelling wave solutions to variant Boussinesq equations, we set

$$H(x,t)=\phi(\xi),\quad u(x,t)=\psi(\xi),\quad \xi=x-\lambda t, \tag{4}$$

where λ is a constant to be determined. Substituting eq. (4) into eqs (2) and (3) yields

$$-\lambda\phi'+\phi\phi'+g\psi'+\frac{1}{3}a\lambda^2\psi'''=0, \tag{5}$$

$$-\lambda\psi'+\phi\psi'+\psi\phi'=0. \tag{6}$$

Integrating them with respect to ξ, we have

$$-\lambda\phi+\frac{1}{2}\phi^2+g\psi+\frac{1}{3}a\lambda^2\psi''=c_1, \tag{7}$$

$$-\lambda\psi+\phi\psi=c_2, \tag{8}$$

where c_1 and c_2 are constants. Substituting eq. (8) into eq. (7) yields

$$-\frac{\lambda^2}{2}+\frac{c_2^2}{2\psi^2}+g\psi+\frac{1}{3}a\lambda^2\psi''=c_1. \tag{9}$$

Furthermore, we get

$$(\psi')^2=\frac{-3g\psi^3+6(c_1+(\lambda^2/2))\psi^2+6c_3\psi+3c_2^2}{a\lambda^2\psi}, \tag{10}$$

where c_3 is also a constant.

Case 1. $b_3\neq 0$. In this case, eq. (10) becomes

$$\xi-\xi_0=\int\sqrt{\frac{\psi}{b_3\psi^3+b_2\psi^2+b_1\psi+b_0}}\mathrm{d}\psi, \tag{11}$$

where

$$b_3=\frac{-3g}{a\lambda^2},\quad b_2=\frac{6(c_1+(\lambda^2/2))}{a\lambda^2},$$

$$b_1=\frac{6c_3}{a\lambda^2},\quad b_0=\frac{3c_2^2}{a\lambda^2}.$$

According to eq. (11), we shall give single travelling wave solutions to Boussinesq equations.

Case 1.1. $b_0=0$, that is $c_2=0$. We denote

$$F(\psi)=\psi^2+\alpha\psi+\beta, \tag{12}$$

where $\alpha=b_2/b_3$ and $\beta=b_1/b_3$. Then we write its complete discrimination system as follows:

$$\Delta=\alpha^2-4\beta, \tag{13}$$

and eq. (11) becomes

$$\sqrt{b_3}(\xi-\xi_0)=\int\frac{\mathrm{d}\psi}{\sqrt{F(\psi)}}, \tag{14}$$

where ξ_0 is an integral constant, $\sqrt{b_3}$ is a real number when $b_3\geq 0$, and it becomes a imaginary number when $b_3<0$. According to the complete discrimination system, we give the corresponding solutions to eq. (14).

Case 1.1.1. $\Delta<0$. $F(\psi)$ has a pair of conjugate complex roots, namely

$$F(\psi)=(\psi+s)^2+l^2, \tag{15}$$

where $l>0$. By using eq. (14), we have

$$\sqrt{b_3}(\xi-\xi_0)=\int\frac{\mathrm{d}\psi}{\sqrt{(\psi+s)^2+l^2}}, \tag{16}$$

and integrating the right side of eq. (16), we get

$$\sqrt{b_3}(\xi-\xi_0)=\ln\left(\frac{\psi+s}{l}+\sqrt{\left(\frac{\psi+s}{l}\right)^2+1}\right). \tag{17}$$

So the solution of eq. (14) is

$$\psi=\frac{l[\mathrm{e}^{2\sqrt{b_3}(\xi-\xi_0)}-1]}{\mathrm{e}^{\sqrt{b_3}(\xi-\xi_0)}}-s. \tag{18}$$

For example, when $a=\lambda=1, g=-1/3, c_1=-1/2, c_3=1/6$, we have $s=0, l=1$ and $b_3=1$. Then we get

$$\psi=\frac{\mathrm{e}^{2(\xi-\xi_0)}-1}{\mathrm{e}^{\xi-\xi_0}}-1. \tag{19}$$

Case 1.1.2. $\Delta=0$. $F(\psi)$ has a real root with multiplicity two

$$F(\psi)=(\psi+s)^2. \tag{20}$$

By using eq. (14), we have

$$\sqrt{b_3}(\xi-\xi_0)=\int\frac{\mathrm{d}\psi}{\psi+s}=\ln|\psi+s|, \tag{21}$$

then we derive the solution of eq. (14) as

$$\psi=\mathrm{e}^{\pm\sqrt{b_3}(\xi-\xi_0)}-s. \tag{22}$$

For example, when $a = \lambda = 1, g = -1/3, c_1 = -1/6, c_3 = 1/6$, we have $s = 0$, that is

$$\psi = \mathrm{e}^{\pm(\xi-\xi_0)} - 1. \tag{23}$$

Case 1.1.3. $\Delta > 0$. $F(\psi)$ has two distinct real roots,

$$F(\psi) = (\psi + s)^2 - l^2, \tag{24}$$

where $l > 0$. From eq. (14), we have

$$\sqrt{b_3}(\xi - \xi_0) = \int \frac{\mathrm{d}\psi}{\sqrt{(\psi+s)^2 - l^2}} = \ln\left|\frac{\psi+s}{l} + \sqrt{\left(\frac{\psi+s}{l}\right)^2 - 1}\right|. \tag{25}$$

By using eq. (25), we have

$$\psi = \frac{l[\mathrm{e}^{\pm 2\sqrt{b_3}(\xi-\xi_0)} + 1]}{\mathrm{e}^{\pm\sqrt{b_3}(\xi-\xi_0)}} - s, \tag{26}$$

the two $\pm$ signs are independent. For example, when $a = \lambda = 1, g = -1/3, c_1 = -1/6, c_3 = 0$. So $b_3 = 1, b_2 = 2, b_1 = 0$, and we have $s = 1, l = 1$. That is

$$\psi = \frac{\mathrm{e}^{\pm 2(\xi-\xi_0)} + 1}{\mathrm{e}^{\pm(\xi-\xi_0)}} - 1. \tag{27}$$

Case 1.2. $b_0 \neq 0$. We denote

$$G(\psi) = \psi^3 + \alpha\psi^2 + \beta\psi + \gamma, \tag{28}$$

where $\alpha = b_2/b_3$, $\beta = b_1/b_3$ and $\gamma = b_0/b_3$. Then we write its complete discrimination system as follows:

$$A = \alpha^2 - 3\beta, \quad B = \alpha\beta - 9\gamma, \quad C = \beta^2 - 3\alpha\gamma, \quad \Delta = B^2 - 4AC, \tag{29}$$

and the integration becomes

$$\sqrt{b_3}(\xi - \xi_0) = \int \sqrt{\frac{\psi}{G(\psi)}}\mathrm{d}\psi. \tag{30}$$

Case 1.2.1. $A = B = 0$, $F(\psi)$ has a real root with multiplicity three, i.e.

$$G(\psi) = (\psi - s)^3. \tag{31}$$

Then eq. (30) becomes

$$\sqrt{b_3}(\xi - \xi_0) = \int \frac{1}{\psi - s}\sqrt{\frac{\psi}{\psi - s}}\mathrm{d}\psi = -2\sqrt{\frac{\psi}{\psi - s}} + \ln\left|\frac{\sqrt{\psi/(\psi-s)}+1}{\sqrt{\psi/(\psi-s)}-1}\right|. \tag{32}$$

For example, when $a = -1, \lambda = 1, g = 1/3, c_1 = 0, c_2 = \sqrt{3}/3, c_3 = -1/2$, we have $s = 1$, that is

$$\xi - \xi_0 = -2\sqrt{\frac{\psi}{\psi - 1}} + \ln\left|\frac{\sqrt{\psi/(\psi-1)}+1}{\sqrt{\psi/(\psi-1)}-1}\right|. \tag{33}$$

Case 1.2.2. $\Delta > 0$, $F(\psi)$ has a real root and a pair of conjugate complex roots, that is

$$G(\psi) = (\psi - s)(\psi^2 + l^2). \tag{34}$$

By using eq. (30), we have

$$\sqrt{b_3}(\xi - \xi_0) = \int \sqrt{\frac{\psi}{(\psi - s)(\psi^2 + l^2)}}\mathrm{d}\psi. \tag{35}$$

If $y^2 = \psi$, we get

$$\sqrt{b_3}(\xi - \xi_0) = \int \frac{2y^2}{\sqrt{(y^2 - s)(y^4 + l^2)}}\mathrm{d}y. \tag{36}$$

We can see that the result is a hyperelliptic function. For example, when $a = -1, \lambda = 1, g = 1/3, c_1 = -1/3, c_2 = \sqrt{3}/3, c_3 = -1/6$, we have $s = 1, l = 1$, that is

$$\sqrt{b_3}(\xi - \xi_0) = \int \frac{2y^2}{\sqrt{(y^2 - 1)(y^4 + 1)}}\mathrm{d}y. \tag{37}$$

Case 1.2.3. $\Delta = 0$. $G(\psi)$ has a real root with multiplicity one and a real root with multiplicity two, i.e.

$$G(\psi) = (\psi - s)^2(\psi - l). \tag{38}$$

By eq. (30), we have

$$\sqrt{b_3}(\xi - \xi_0) = \int \frac{1}{\psi - s}\sqrt{\frac{\psi}{\psi - l}}\mathrm{d}\psi, \tag{39}$$

where $s \neq l$ and $s, l \neq 0$. If $s/(l-s) < 0$, eq. (38) becomes

$$\sqrt{b_3}(\xi - \xi_0) = \ln\left|\frac{\sqrt{\psi/(\psi-l)}+1}{\sqrt{\psi/(\psi-l)}-1}\right| + \sqrt{\frac{s}{s-l}} \times \ln\left|\frac{\sqrt{\psi/(\psi-l)}-\sqrt{s/(s-l)}}{\sqrt{\psi/(\psi-l)}+\sqrt{s/(s-l)}}\right|, \tag{40}$$

and if $s/(l-s) > 0$, we can get

$$\sqrt{b_3}(\xi - \xi_0) = \ln\left|\frac{\sqrt{\psi/(\psi-l)}+1}{\sqrt{\psi/(\psi-l)}-1}\right| - 2\sqrt{\frac{s}{l-s}}\arctan\sqrt{\frac{(l-s)\psi}{s(\psi-l)}}. \tag{41}$$

For example, when $a = \lambda = 1, g = -1/3, c_1 = 1/6, c_2 = \sqrt{6}/3, c_3 = -5/6$, we have $s = 1, l = 2$, that is

$$\xi - \xi_0 = \ln\left|\frac{\sqrt{\psi/(\psi-2)}+1}{\sqrt{\psi/(\psi-2)}-1}\right| - 2\arctan\sqrt{\frac{\psi}{(\psi-2)}}. \tag{42}$$

Case 1.2.4. $\Delta < 0$, $G(\psi)$ has three distinct real roots, namely

$$G(\psi) = (\psi - s)(\psi - l)(\psi - m). \tag{43}$$

From eq. (30), we get

$$\sqrt{b_3}(\xi - \xi_0) = \int\sqrt{\frac{\psi}{(\psi - s)(\psi - l)(\psi - m)}}\mathrm{d}\psi. \tag{44}$$

Setting $y^2 = \psi$, we have

$$\sqrt{b_3}(\xi - \xi_0) = \int\frac{2y^2}{\sqrt{(y^2 - s)(y^2 - l^2)(y^2 - m^2)}}\mathrm{d}y. \tag{45}$$

We can see that the result is a hyperelliptic function. For example, when $a = -1, \lambda = 1, g = 1/3, c_1 = 1/2, c_2 = \sqrt{2}, c_3 = -11/6$, we have $s = 1, l = \sqrt{2}, m = \sqrt{3}$. So (45) becomes

$$\sqrt{b_3}(\xi - \xi_0) = \int\frac{2y^2}{\sqrt{(y^2 - 1)(y^2 - 2)(y^2 - 3)}}\mathrm{d}y. \tag{46}$$

Case 2. $b_3 = 0$, that is $g = 0$. In this case, we have

$$I(\psi) = b_2\psi^2 + b_1\psi + b_0. \tag{47}$$

Then, eq. (10) becomes

$$\xi - \xi_0 = \int\sqrt{\frac{\psi}{I(\psi)}}\mathrm{d}\psi. \tag{48}$$

According to the complete discrimination system, we also give the corresponding solutions to eq. (47).

Case 2.1. $b_0 = 0$, that is $c_2 = 0$, we can easily see that if $b_2 \neq 0$

$$\xi - \xi_0 = \int\frac{\mathrm{d}\psi}{\sqrt{b_2\psi + b_1}} = \frac{2\sqrt{b_2\psi + b_1}}{b_2}. \tag{49}$$

Then we get

$$\psi = \frac{b_2^2(\xi - \xi_0)^2 - 4b_1}{4b_2}. \tag{50}$$

If $b_2 = 0$, eq. (48) becomes

$$\xi - \xi_0 = \int\frac{\mathrm{d}\psi}{\sqrt{b_1}} = \frac{\psi}{\sqrt{b_1}}, \tag{51}$$

that is

$$\psi = \sqrt{b_1}(\xi - \xi_0). \tag{52}$$

For example, when $a = 1, \lambda = 1, c_1 = -1/3, c_3 = 1/6$, we have $b_1 = b_2 = 1$. So (52) becomes

$$\psi = \frac{(\xi - \xi_0)^2 - 4}{4}. \tag{53}$$

Case 2.2. $b_0 \neq 0$. In this case, if $b_2 = 0$, the solutions of eq. (48) are

$$\xi - \xi_0 = \sqrt{b_1\psi^2 + b_0\psi} - \frac{b_0}{\sqrt{b_1^3}} \times \ln|\sqrt{b_1\psi} + \sqrt{b_1\psi + b_0}|(b_1 \neq 0) \tag{54}$$

and

$$\psi = \left(\frac{3\sqrt{b_0}(\xi - \xi_0)}{2}\right)^{2/3} \ (b_1 = 0). \tag{55}$$

If $b_2 \neq 0$, we denote

$$\Delta = \alpha^2 - 4\beta. \tag{56}$$

Case 2.2.1. $\Delta < 0$. In this case, $I(\psi)$ has a pair of conjugate complex roots, i.e.

$$I(\psi) = (\psi + s)^2 + l^2, \tag{57}$$

where $l > 0$. By using eq. (48), we have

$$\sqrt{b_2}(\xi - \xi_0) = \int\sqrt{\frac{\psi}{(\psi + s)^2 + l^2}}\mathrm{d}\psi. \tag{58}$$

The result is an elliptic function of the third kind. For example when $a = 1, \lambda = 1, c_1 = -1/3, c_2 = 1, c_3 = 1/3$, we have $s = l = 1$. So (58) becomes

$$(\xi - \xi_0) = \int\sqrt{\frac{\psi}{(\psi + 1)^2 + 1}}\mathrm{d}\psi. \tag{59}$$

Case 2.2.2. $\Delta = 0$. In this case, $I(\psi)$ has a real root with multiplicity two, that is

$$I(\psi) = (\psi + s)^2. \tag{60}$$

If $s < 0$, eq. (48) becomes

$$\sqrt{b_2}(\xi - \xi_0) = \sqrt{-s}\ln\left|\frac{\sqrt{\psi} - \sqrt{-s}}{\sqrt{\psi} + \sqrt{-s}}\right| + 2\sqrt{\psi} \tag{61}$$

and if $s > 0$, we have

$$\sqrt{b_2}(\xi - \xi_0) = 2\sqrt{\psi} - 2\sqrt{s}\arctan\sqrt{\frac{\psi}{s}}. \tag{62}$$

For example when $a = 1, \lambda = 1, c_1 = -1/3, c_2 = \sqrt{3}/3, c_3 = 1/3$, we have $s = 1$. So (62) becomes

$$\sqrt{b_2}(\xi - \xi_0) = 2\sqrt{\psi} - 2\arctan\sqrt{\psi}. \tag{63}$$

Case 2.2.3. $\Delta > 0$, $I(\psi)$ has two real distinct roots such that

$$I(\psi) = (\psi + s)^2 - l^2, \tag{64}$$

where $l > 0$. By using eq. (48), we have

$$\sqrt{b_2}(\xi - \xi_0) = \int\sqrt{\frac{\psi}{(\psi + s)^2 - l^2}}\mathrm{d}\psi. \tag{65}$$

The result is an elliptic function of the third kind. For example, when $a = -1, \lambda = 1, c_1 = -2/3, c_2 = \sqrt{3}/3, c_3 = -1/3$, we have $s = 1, l = \sqrt{2}$. So (65) becomes

$$(\xi - \xi_0) = \int\sqrt{\frac{\psi}{(\psi + 1)^2 - 2}}\mathrm{d}\psi. \tag{66}$$

3. Conclusion

In this paper, all possible travelling wave solutions for the variant Boussinesq equations have been given. In particular, we construct all solutions for concrete parameters to show that each solution can be realized.

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