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Periodic solutions of Wick-type stochastic Korteweg–de Vries equations

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Abstract. Nonlinear stochastic partial differential equations have a wide range of applications in science and engineering. Finding exact solutions of the Wick-type stochastic equation will be helpful in the theories and numerical studies of such equations. In this paper, Kudrayshov method together with Hermite transform is implemented to obtain exact solutions of Wick-type stochastic Korteweg–de Vries equation. Further, graphical illustrations in two- and three-dimensional plots of the obtained solutions depending on time and space are also given with white noise functionals.

Keywords. Wick-type stochastic Korteweg–de Vries equation; Hermite transform; Kudrayshov method; white noise functionals.

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1. Introduction

Nonlinear partial differential equations (NPDEs) have a wide range of applications in physics, chemistry, biology and economics from various points of view [1]. More precisely, in order to describe the realistic physical phenomena accurately, it is necessary and important to study NPDEs in random surroundings [2–4]. The Korteweg–de Vries equation models the propagation of weakly nonlinear dispersive waves in various areas: plasma physics, surface waves on the top of an incompressible irrotational inviscid fluid, beam propagation [5], etc. From a mathematical point of view, the Korteweg–de Vries equation with external noise was first discussed by Wadati [6]. He also obtained the large-time behaviour of one-soliton solutions under Gaussian noise. Song and Zhang [7] obtained a series of stochastic wave solutions for (2 + 1)-dimensional stochastic dispersive long wave system by means of Jacobi elliptic function rational expansion method.

In particular, many researchers were interested in two-dimensional surface waves generated by a localized pressure distribution and the Korteweg–de Vries (KdV) equation is used to model them in subcritical flows [8–12]. In this paper, we shall obtain periodic solutions of Wick-type stochastic KdV equations in the following form [14]:

$$U_t + F(t)\diamond U\diamond U_x + G(t)\diamond U_{xxx} = 0, \tag{1}$$

where '$\diamond$' is the Wick product on the Hida distribution space $(S(\mathbb{R}^d))^*$, $F(t)$ and $G(t)$ are the white noise functionals which are given in [13,14]; $F(t) = f(t) + K_1 W_t$ and $G(t) = g(t) + K_2 W_t$, $f(t)$ and $g(t)$ are functions of t, W_t is the Gaussian white noise that satisfies $W_t = \dot{B}_t$, B_t is a Brownian motion, K_1 and K_2 are arbitrary constants. Equation (1) is the perturbation of the coefficients $f(t)$ and $g(t)$ of the variable coefficient KdV equation

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{2}$$

where the coefficients $f(t)$ and $g(t)$ are integrable functions on $\mathbb{R}_+$. Using the Hermite transform, homogeneous balance and white noise analysis method, Xie [14] obtained positonic solutions for Wick-type stochastic KdV equation. By means of the Hermite transformation together with rational expansion method, a series of stochastic non-travelling wave solutions for articles stochastic mKdV equation was obtained in [15].

Recently, several analytic methods have been successfully developed and applied for constructing exact solutions to nonlinear stochastic partial differential equations such as extended Jacobi elliptic function rational expansion method [7], (G'/G)-expansion method [16], homotopy perturbation method [17], F-expansion method [18], exp-function method [19] and so on. However, there is no unified method that can be used to solve all types of nonlinear stochastic evolution equations. Recently, Kudryashov [20] proposed a novel, powerful and effective approach called the simplest equation method for finding exact solutions of nonlinear differential equations and successfully used it for finding exact solutions of nonlinear evolution equations arising in mathematical physics [21]. A detailed description of the simplest equation method is presented in [20,21]. Now, let us present the algorithm of the simplest equation method for finding exact solutions of nonlinear partial differential equations. Consider the nonlinear PDE in the following polynomial form:

$$E(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \tag{3}$$

By taking travelling wave solutions $u(x,t) = y(\eta)$, $\eta = kx - wt$, eq. (3) can be reduced to the nonlinear ordinary differential equation (ODE)

$$E_1(y, -wy_\eta, ky_\eta, w^2 y_{\eta\eta}, k^2 y_{\eta\eta}, \ldots) = 0. \tag{4}$$

To find the dominant terms, we substitute $y(\eta) = \eta^{-p}$, $p > 0$ into all terms of eq. (4). In particular, we look for exact solution of (4) in the form

$$y(\eta) = a_0 + a_1 Q(\eta) + a_2 Q(\eta)^2 + \cdots + a_N Q(\eta)^N, \tag{5}$$

where $a_i, i = 1, 2, \ldots, N$ are unknown constants, and $Q(\eta)$ takes the form

$$Q(\eta) = \frac{1}{1 + \exp\{\eta\}}. \tag{6}$$

Then, we compare the degrees of all terms in eq. (4) and comparing the two or more terms with the smallest powers we find the value for N. It should be noted that this method can be applied when N is an integer. If N is a non-integer, we have to use the transformation of the solution $y(\eta)$. It should be noted that the function Q is the solution of equation

$$Q_\eta = Q^2 - Q \tag{7}$$

which allows us to find derivatives of y_η, $y_{\eta\eta}$ and so on. For example, we consider the general case when N is arbitrary. Differentiating eq. (5) with respect to η and considering eq. (7), we get

$$y_\eta = \sum_{i=1}^{N} a_i i (Q - 1) Q^i,$$
$$y_{\eta\eta} = \sum_{i=1}^{N} a_i i ((i+1)Q^2 - (2i+1)Q + i) Q^i. \tag{8}$$

Further, substitute expressions (5), (6) and (7) in eq. (3) and collect all terms with the same powers of function $Q(\eta)$ and equate the resulting expression to zero. Finally, we obtain a system of algebraic equations and on solving it, we obtain values of coefficients $a_0, a_1, \ldots, a_N$ and relations for the parameters of eq. (4). As a result, we can obtain exact solutions of eq. (4) in the form of eq. (5).

In this paper, the simplest equation method will be employed to obtain new exact solutions for the Wick-type stochastic KdV equation. More precisely, Wick products in Wick-type stochastic KdV equation will be transformed into an ordinary one by using the Hermite transform. Further, the solution of the ordinary differential equation can be obtained. In order to obtain the solution of the Wick-type stochastic KdV equation, one can take inverse Hermite transform of the resulting solution.

2. Preliminaries

Let $(S(\mathbb{R}^d))$ and $(S(\mathbb{R}^d))^*$ be the Hida test function space and the Hida distribution space on $\mathbb{R}^d$, respectively. And let $h_n(x)$ be the d-order Hermite polynomials. Put $\xi_n(x) = \mathrm{e}^{-\frac{1}{2}x^2} h_n(\sqrt{2}x)/(\pi(n-1)!)^{1/2}, n \geq 1$. Then, the collection $\{\xi_n\}_{n\geq 1}$ constitutes an orthogonal basis for $\mathcal{L}^2(\mathbb{R})$.

If we assume $\alpha = (\alpha_1, \ldots \alpha_d)$ as d-dimensional multi-indices with $\alpha_1, \ldots \alpha_d \in \mathbb{N}$, we get a family of tensor products $\xi_\alpha = \xi_{(\alpha_1, \cdots \alpha_d)} = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d} (\alpha \in \mathbb{N}^d)$ forms an orthogonal basis for $\mathcal{L}^2(\mathbb{R}^d)$. Let $\alpha^{(i)} = (\alpha_1^{(i)}, \ldots \alpha_d^{(i)})$ be the ith multi-index number in some fixed ordering of all d-dimensional multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. We can, and will, assume that this ordering has the property

$$i < j \Rightarrow \alpha_1^{(i)} + \cdots + \alpha_d^{(i)} \leq \alpha_1^{(j)} + \cdots + \alpha_d^{(j)}, \tag{9}$$

i.e., $\{\alpha^{(j)}\}_{j=1}^\infty$ occurs in an increasing order. Now define

$$\eta_i = \xi_{\alpha^{(i)}} = \xi_{\alpha_1^{(i)}} \otimes \cdots \otimes \xi_{\alpha_d^{(i)}}, \qquad i \geq 1. \tag{10}$$

We denote multi-indices as elements of the space $(\mathbb{N}_0^d)_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. For $\alpha \in (\mathbb{N}_0^{\mathbb{N}})_c$, let us define

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \qquad \omega \in (S(\mathbb{R}^d))^*. \quad (11)$$

For a fixed $n \in \mathbb{N}$, let $(S)_1^n$ consist of those $f(\omega)x = \sum_\alpha c_\alpha H_\alpha(\omega) \in \oplus_{k=1}^n \mathfrak{L}^2(\nu)$ with $c_\alpha \in \mathbb{R}^n$ such that $\|f(\omega)\|_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty, \forall k \in \mathbb{N}$ with $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, \ldots, c_\alpha^{(n)}) \in \mathbb{R}^n$, where ν is the white noise measure on $(S^*(\mathbb{R}), B(S^*(\mathbb{R})))$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$ and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha j}$ for $\alpha = (\alpha_1, \alpha_2, \ldots) \in J$.

The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$ with $b_\alpha \in \mathbb{R}^n$ such that $\|F\|_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $\|f\|_{1,k}, k \in \mathbb{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the relation

$$\langle F, f \rangle = \sum_\alpha (b_\alpha, c_\alpha)\alpha! \quad (12)$$

and (b_α, c_α) is the usual inner product in $\mathbb{R}^n$.

The Wick product $f \diamond F$ of two elements $f = \sum_\alpha a_\alpha H_\alpha$, $F = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ with $a_\alpha, b_\beta \in \mathbb{R}^n$, is defined by

$$f \diamond F = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta}. \quad (13)$$

We can prove that the spaces $(S(\mathbb{R}^d))$, $(S(\mathbb{R}^d))^*$, $(S)_1$ and $(S)_{-1}$ are closed under Wick products.

For $F = \sum_\alpha b_\alpha H_\alpha \in (S)_{-1}^N$, with $b_\alpha \in \mathbb{R}^n$, the Hermite transformation of F, denoted by $\mathcal{H}(F)$ or $\tilde{F}$ is defined by

$$\mathcal{H}(F) = \tilde{F}(z) = \sum b_\alpha z^\alpha \in \mathbb{C}^{\mathbb{N}}, \quad \text{when convergent}, \quad (14)$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and $z^\alpha = (z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots)$ for $\alpha = (\alpha_1, \alpha_2, \ldots) \in J$, where $z_j^0 = 1$.

For $F, G \in (S)_{-1}^N$, by this definition we have

$$\widetilde{F \diamond G}(z) = \tilde{F}(z) \dot{\tilde{G}}(z), \quad (15)$$

for all z such that $\tilde{F}(z)$ and $\tilde{G}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^{\mathbb{N}}$ defined by $(z_1^1, \ldots, z_n^1)\dot{}(z_1^2, \ldots, z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$, where $z_k^i \in \mathbb{C}$.

Let $X = \sum_\alpha a_\alpha H_\alpha \in (S)_{-1}^{\mathbb{N}}$. Then the vector $c_0 = \tilde{X}(0) \in \mathbb{R}^N$ is called the generalized expectation of X and is denoted by $E(X)$. Suppose that $g : U \to \mathbb{C}^M$ is an analytic function, where U is a neighbourhood of $\xi_0 := E(X)$. Assume that the Taylor series of g around ξ_0 has coefficients in $\mathbb{R}^M$. Then, the Wick version $g^\diamond(X) = \mathcal{H}^{-1}(g \circ \tilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z) = \sum a_\alpha (z - \xi_0)^\alpha$ with $a_\alpha \in \mathbb{R}^M$, then $g \diamond (X) = \sum a_\alpha (X - \xi_0)^{\diamond\alpha} \in (S)_{-1}^M$.

We next outline the main steps of the compatibility method for Wick-type SPDEs.

Suppose that modelling considerations lead us to consider an SPDE expressed formally as

$$A(t, x, \partial_t, \nabla_x, U, \omega) = 0, \quad (16)$$

where A is a given function, $U = U(t, x, \omega)$ is the unknown (generalized) stochastic process, and where the operators $\partial_t = \partial/\partial t, \nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ when $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Step 1. With the aid of the Hermite transformation, we transform the Wick-type equation

$$A^\diamond(t, x, \partial_t, \nabla_x, U, \omega) = 0 \quad (17)$$

into an ordinary products equation (variable coefficient PDE)

$$\tilde{A}(t, x, \partial_t, \nabla_x, \tilde{U}, z_1, z_2, \ldots) = 0, \quad (18)$$

where $\tilde{U} = \mathcal{H}(\mathcal{U})$ is the Hermite transform of U and $z_1, z_2, \ldots$ are complex numbers.

Step 2. Suppose that we can find a solution $u = \tilde{U}(t, x, z)$ of eq. (18) for each $z \in \mathbb{K}_q(r)$, where $\mathbb{K}_q(r) = \{z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ and $\sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < r^2\}$ for some q, r.

Step 3. Substituting Kudrayshov's solutions (6) into eq. (18) yields the highest-order derivative term of u with respect to x_i and t. Substituting (6) in eq. (18), we get the highest-order derivative term of u with respect to x_i and t. Then, setting the coefficients of u and their differential terms to zero, we get a set of overdetermined PDEs for some undetermined coefficients in the Kudrayshov method.

Step 4. Solving the system of overdetermined PDEs derived in Step 3, we would end up with the explicit expressions for some undetermined coefficients. Then we can obtain exact solutions of eq. (18) by substituting them into $u = \tilde{U}(t, x, z)$.

Step 5. Under certain conditions, we can take the inverse Hermite transform $U = \mathcal{H}^{-1}u \in (S)_{-1}$, thereby obtain a solution U of the original Wick-type stochastic equation (17). We have the following theorem, which was proved by Holden *et al* in [3].

Theorem 2.1. *Suppose $u(t, x, z)$ is a solution (in the usual strong, pointwise sense) of* eq. (18) *for (t, x) in some bounded open set $\mathbf{G} \subset \mathbb{R} \times \mathbb{R}^d$, and for all $x \in \mathbb{K}_q(r)$, for some q, r. Moreover, suppose that $u(t, x, z)$ and all its partial derivatives, which are involved in* (18), *are bounded for $(t, x, z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t, x) \in \mathbf{G}$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$, for all $(t, x) \in \mathbf{G}$. Then there exists $U(t, x) \in (S)_{-1}$ such that $u(t, x, z) = (\tilde{U}(t, x))(z)$ for all $(t, x, z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and $U(t, x)$ solves* (*in the strong sense in $(S)_{-1}$*) eq. (17) *in $(S)_{-1}$.*

3. Solution of stochastic KdV equation

Based on the idea of [12], by taking the Hermite transform of eq. (1), we can obtain the following equation:

$$\tilde{U}_t(t, x, z) + \tilde{F}(t, z)\diamond\tilde{U}(t, x, z)\diamond\tilde{U}_x(t, x, z) + \tilde{G}(t, z)\diamond\tilde{U}_{xxx}(t, x, z) = 0, \quad (19)$$

where $z = (z_1, z_2, \ldots) \in (\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter. Further, for simplicity, if we take $u(t, x, z) = \tilde{U}(t, x, z)$, $F(t, z) = \tilde{F}(t, z)$ and $G(t, z) = \tilde{G}(t, z)$, then we can have the following equation:

$$u_t + Fuu_x + Gu_{xxx} = 0. \quad (20)$$

In order to obtain the solution of stochastic KdV equation (19), we consider the following transformation:

$$u(t, x, z) = u(\eta), \qquad \eta = xq(t, z) + r(t, z), \quad (21)$$

where $q(t, z)$ and $r(t, z)$ are non-zero functions to be determined later.

Moreover, by substituting transformation (21) in eq. (20), it can be converted to the following ordinary differential equation:

$$(q_t x + r_t)u' + Fquu' + Gq^3u''' = 0. \quad (22)$$

Balancing uu' and u''' in eq. (22), we obtain $2n+1 = n + 3$ which gives $n = 2$. Suppose that the solution of eq. (22) can be expressed by a polynomial in $Q(\eta)$ as follows:

$$u(\eta) = a_2(t, z)Q(\eta)^2 + a_1(t, z)Q(\eta) + a_0(t, z), \quad a_2(t, z) \neq 0, \quad (23)$$

where $Q(\eta)$ satisfies the first-order ordinary differential equation

$$Q_\eta = Q^2 - Q. \quad (24)$$

By substituting eq. (23) in eq. (22) and collecting all terms with the same power of Q together, the left-hand side of eq. (22) is converted into polynomial in Q. Equating each coefficient of this polynomial to zero, we obtain the following set of algebraic equations with respect to the unknowns $a_2(t, z)$, $a_1(t, z)$, $a_0(t, z)$ and $r(t, z)$:

$$\begin{cases} 2Fqa_2^2 + 24Gq^3a_2 = 0, \\ -(\mathrm{d}r/\mathrm{d}t)a_1 - x(\mathrm{d}q/\mathrm{d}t)a_1 - Gq^3a_1 - Fqa_0a_1 = 0, \\ 3Fqa_1a_2 + 6Gq^3a_1 - 54Gq^3a_2 - 2Fqa_2^2 = 0, \\ 2Fqa_0a_2 + 2(\mathrm{d}r/\mathrm{d}t)a_2 + 2x(\mathrm{d}q/\mathrm{d}t)a_2 + Fqa_1^2 \\ \quad -12Gq^3a_1 - 3Fqa_1a_2 + 38Gq^3a_2 = 0, \\ -2(\mathrm{d}r/\mathrm{d}t)a_2 + Fqa_0a_1 - 2x(\mathrm{d}q/\mathrm{d}t)a_2 - 8Gq^3a_2 \\ \quad -2Fqa_0a_2 + 7Gq^3a_1 - Fqa_1^2 + (\mathrm{d}r/\mathrm{d}t)a_1 \\ \quad +x(\mathrm{d}q/\mathrm{d}t)a_1 = 0, \end{cases} \quad (25)$$

where $F = F(t, z)$, $G = G(t, x)$, $q = q(t, z)$ and $r = r(t, z)$.

Solving the system of algebraic equations using *Maple*, we obtain the following sets of nontrivial solutions:

$$\begin{cases} a_2(t, z) = -\dfrac{12G(t, z)q^2(t, z)}{F(t, z)}, \\ a_1(t, z) = \dfrac{12G(t, z)q^2(t, z)}{F(t, z)}, \quad a_0(t, z) = a_0(t, z) \\ r(t, z) = -\displaystyle\int \left(\frac{\mathrm{d}q(t, z)}{\mathrm{d}t}x + G(t, z)q^3(t, z) \right. \\ \qquad \left. + F(t, z)q(t, z)a_0\right)\mathrm{d}t, \end{cases} \quad (26)$$

where $F(t, z)$, $G(t, z)$ and $q(t, z)$ are arbitrary functions.

Substituting eq. (26) in eq. (23), we obtain the solution of eq. (22) as follows:

$$u(t, x, z) = -\frac{12G(t, z)q^2(t, z)}{F(t, z)}\frac{1}{(1 + \exp\{\eta\})^2} + \frac{12G(t, z)q^2(t, z)}{F(t, z)}\frac{1}{1 + \exp\{\eta\}} + a_0(t, z), \quad (27)$$

where

$$\eta = \eta(t,x,z) = xq(t,z) - \int \left(\frac{\mathrm{d}q(t,z)}{\mathrm{d}t}x + G(t,z)q^3(t,z) + F(t,z)q(t,z)a_0(t,z)\right)\mathrm{d}t.$$

From solution (27) and the definition of $\tilde{W}(t,z)$, it is proved that there exists a bounded open set $\mathbf{G} \subset \mathbb{R}_+ \times \mathbb{R}$, $q > 0$ and $r > 0$ such that $u(t,x,z)$, $u_t(t,x,z)$, $u_x(t,x,z)$ and $u_{xxx}(t,x,z)$ are uniformly bounded for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t,x) \in \mathbf{G}$ and analytic with respect to $(t,x) \in \mathbf{G}$. Theorem 2.1 implies that there exists $U(t,x) \in (S)_{-1}$ such that $u(t,x,z) = (\mathcal{H}U(t,x))(z)$ for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and that $U(t,x)$ solves eq. (1). Here, $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, solution (27) can give the stochastic positonic solution of eq. (1) as [14]

$$U(t,x) = -\frac{12G(t)q^2(t)}{F(t)}\frac{1}{(1+\exp\{\eta\})^2} + \frac{12G(t)q^2(t)}{F(t)}\frac{1}{1+\exp\{\eta\}} + a_0(t), \tag{28}$$

where

$$\eta = \eta(t,x) = xq(t) - \int\left(\frac{\mathrm{d}q(t)}{\mathrm{d}t}x + G(t)q^3(t) + F(t)q(t)a_0(t)\right)\mathrm{d}t.$$

Further, if we take $G(t,z) = \alpha F(t,z)$, from solution (27), we obtain the following solution of eq. (19):

$$\tilde{U}(t,x,z) = -12\alpha q^2(t,z)\frac{1}{(1+\exp\{\eta\})^2} + 12\alpha q^2(t,z)\frac{1}{1+\exp\{\eta\}} + a_0(t,z), \tag{29}$$

where

$$\eta = \eta(t,x,z) = xq(t,z) - \int_0^t \left(\frac{\mathrm{d}q(s,z)}{\mathrm{d}s}x + \alpha F(s,z)q^3(s,z) + F(s,z)q(s,z)a_0(s,z)\right)\mathrm{d}s.$$

Example 3.1. It should be mentioned that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$ [12]. If $F(t) \neq 0$, then from solution (29), the exact solution of eq. (1) can be obtained as

$$U(t,x) = -12\alpha q^2(t)\frac{1}{\left(1+\exp^{\diamond}\{\eta\}\right)^2} + 12\alpha q^2(t)\frac{1}{1+\exp^{\diamond}\{\eta\}} + a_0(t), \tag{30}$$

where

$$\eta = \eta(t,x) = xq(t) - \int_0^t \left(\frac{\mathrm{d}q(s)}{\mathrm{d}s}x + \alpha F(s)q^3(s) + F(s)q(s)a_0(s)\right)\mathrm{d}s.$$

Further, it should be pointed out that for different values of $F(t)$, we can obtain different solutions for the stochastic KdV equation (1).

Example 3.2. Assume $f(t)$ to be bounded or integrable function on $\mathbb{R}_+$ and put $F(t) = f(t) + W_t$, where W_t is the Gaussian white noise, i.e., $W_t = \dot{B}_t$, B_t is a Brownian motion. Further, we have the Hermite transformation: $F(t,z) = f(t) + \tilde{W}(t,z)$, where $\tilde{W}(t,z) = \sum_{k=1}^{\infty}\int_0^t \eta_k(s)\mathrm{d}s z_k$, here $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ is a parameter vector and $\eta_k(s)$ is defined in [14]. In this case, the solution of (19) can be obtained as

$$\tilde{U}(t,x,z) = -12\alpha q^2(t,z)\frac{1}{(1+\exp\{\eta\})^2} + 12\alpha q^2(t,z) \times \frac{1}{1+\exp\{\eta\}} + a_0(t,z), \tag{31}$$

where

$$\eta = \eta(t,x,z) = xq(t,z) - \int_0^t \left(\frac{\mathrm{d}q(s,z)}{\mathrm{d}s}x + \alpha(f(s) + \tilde{W}(s,z))q^3(s,z) + (f(s) + \tilde{W}(s,z))q(s,z)a_0(s,z)\right)\mathrm{d}s.$$

Example 3.3. Also, by the definition of $\tilde{W}(t,z)$, eq. (31) yields the exact solution of eq. (1) as follows:

$$U(t,x) = -12\alpha q^2(t)\frac{1}{(1+\exp^{\diamond}\{\eta\})^2} + 12\alpha q^2(t)\frac{1}{1+\exp^{\diamond}\{\eta\}} + a_0(t), \tag{32}$$

where

$$\eta = \eta(t,x) = xq(t) - \int_0^t \left(\frac{\mathrm{d}q(s)}{\mathrm{d}s}x + \alpha(f(s)+\tilde{W}(s))q^3(s) + (f(s)+\tilde{W}(s))q(s)a_0(s)\right)\mathrm{d}s.$$

Since $\exp^{\diamond}\{X\} = \exp\{X\}$ for nonrandom X, $\exp^{\diamond}\{B_t\} = \exp\{B_t - \frac{1}{2}t^2\}$. Further, if we assume $q(t) = A$ and $a_0(t) = A_0$ are arbitrary constants, from the above equation we obtain the following solution:

$$U(t,x) = -12\alpha A^2 \frac{1}{(1+\exp\{\eta\})^2} + 12\alpha A^2 \frac{1}{1+\exp\{\eta\}} + A_0, \qquad (33)$$

where

$$\eta = \eta(t,x) = Ax - (\alpha A^3 + AA_0) \times \left(\int_0^t f(s)\mathrm{d}s + t\left(B_t - \frac{1}{2}t^2\right)\right).$$

Remark 3.4. The behaviours of the obtained solution (32) are shown graphically as various types of expressions in figure 1. Figure 1a represents the evolutional behaviour of solution (32) without stochastic forcing term when $W(t) = 0$ and in figure 1b, it is concluded that the stochastic forcing term leads to the uncertainty of the wave amplitude with noise effect when $W(t) = t$ under $q(t) = \sin t, a_0(t) = 0, f(t) = 1, \alpha = 1$.

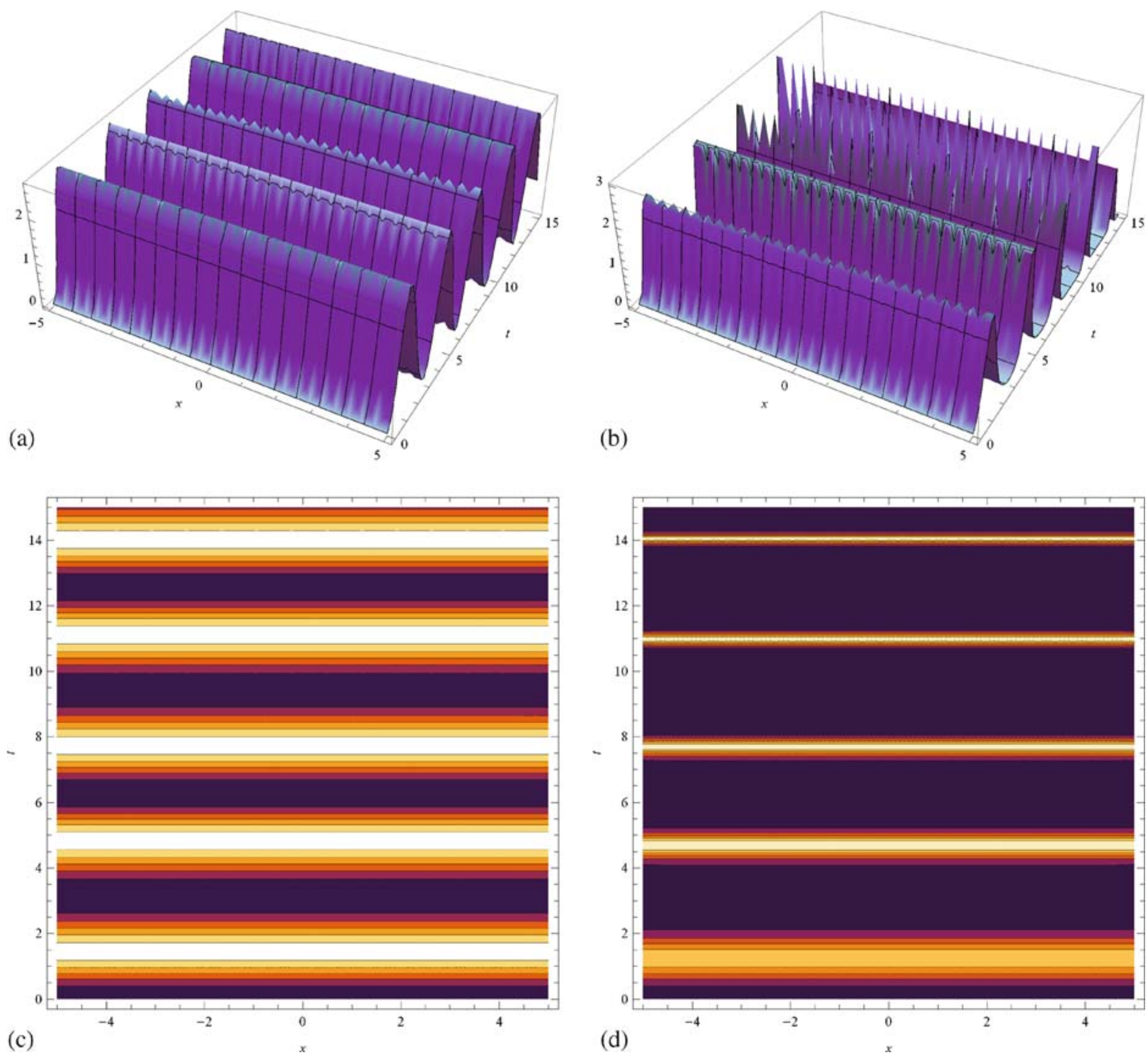

Figure 1. Plots of periodic solutions (32) (**a**) without stochastic forcing term when $W(t) = 0$, (**b**) with noise effect when $W(t) = t$. (**c**) and (**d**) are contours corresponding to (**a**) and (**b**), respectively, when $q(t) = \sin t$, $a_0(t) = 0$, $f(t) = 1$, $\alpha = 1$.

Figures 1c and 1d are contours according to space x and time t. Figure 2 represents a travelling wave solution (32) in time when $x = 2$ (see figures 3–5 for the different coefficient functions of special solutions (32) and (33) of eq. (1)).

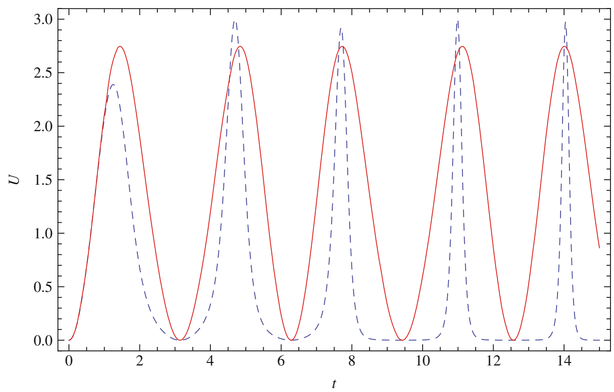

Figure 2. Plots of a travelling wave solution (32) without stochastic forcing term (solid line) and with noise effect (dashed line).

Remark 3.5. It should be noted that when Wick product $\diamond$ is an ordinary product in eq. (1), then we obtain the generalized KdV equation with variable coefficients as follows:

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{34}$$

where the coefficients $f(t)$ and $g(t)$ are integrable functions on $\mathbb{R}_+$ and eq. (34) can be regarded as the perturbation of eq. (1). Equation (34) was studied in [14], in which positonic solution was obtained by using the homogeneous balance principle and Hermite transform.

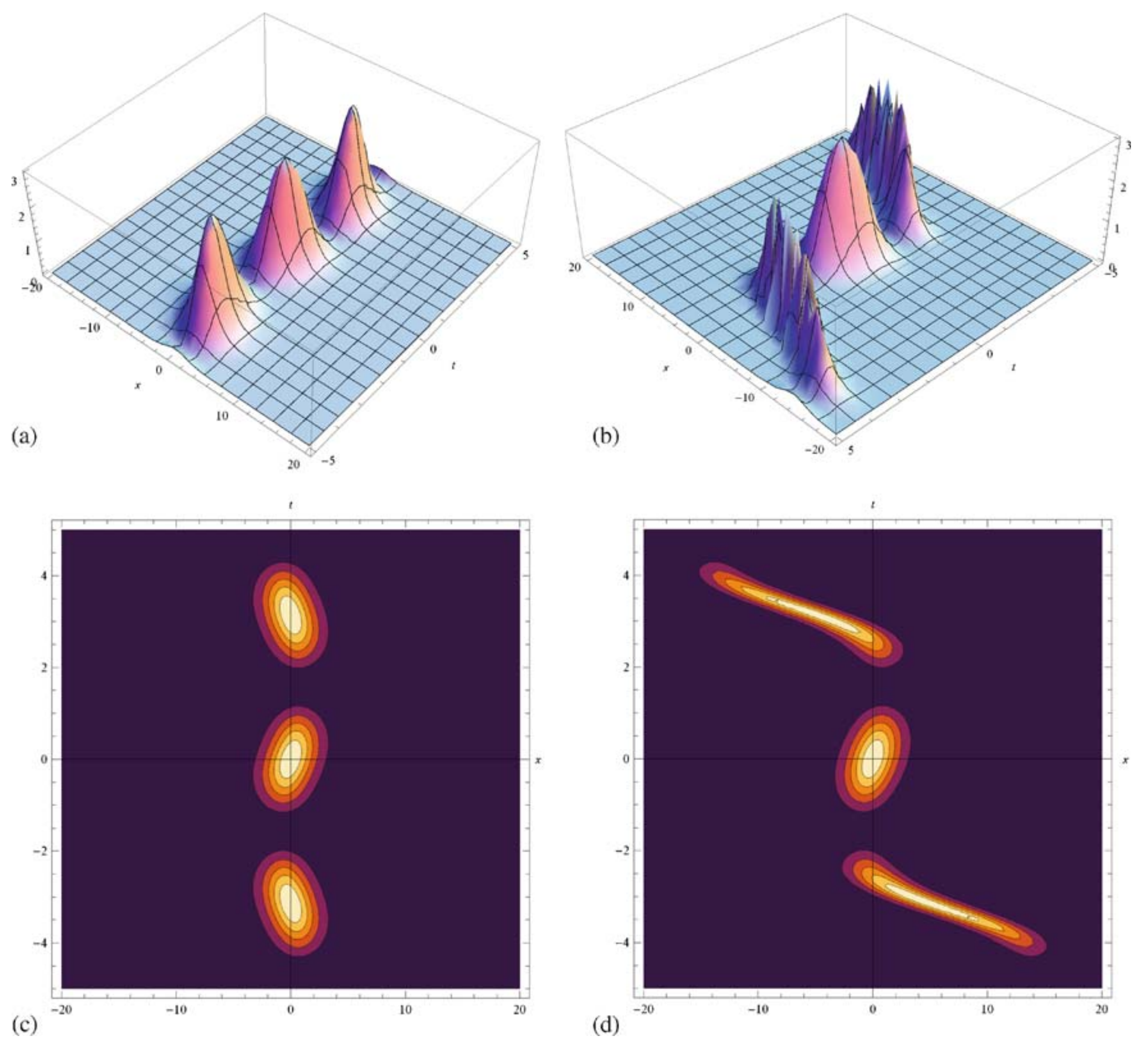

Figure 3. Plots of solution (32) with three peaks (**a**) without stochastic forcing term when $W(t) = 0$, (**b**) with noise effect when $W(t) = t$. (**c**) and (**d**) are contours corresponding to (**a**) and (**b**), respectively, when $q(t) = \cos t$, $a_0(t) = 0$, $f(t) = 1$, $\alpha = 1$.

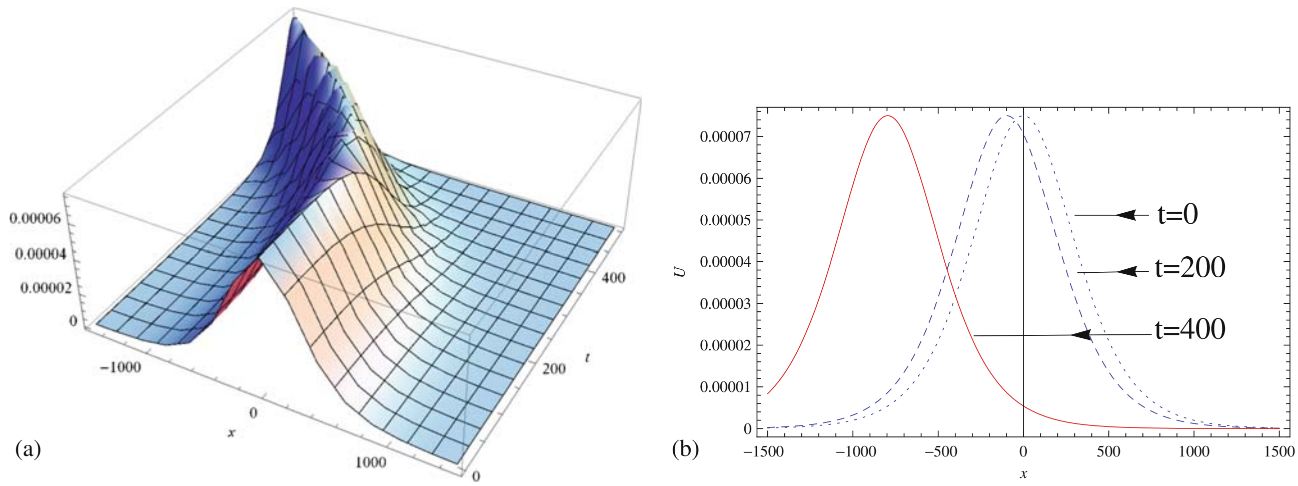

Figure 4. Plots of a single skewed soliton solution (33): (**a**) 3-D motion and (**b**) 2-D motions, which is moving to the left in the time when $A = 0.005$, $A_0 = 0$, $f(t) = \sin(2t) + \cos(2t)$, $B_t = t, \alpha = 1$.

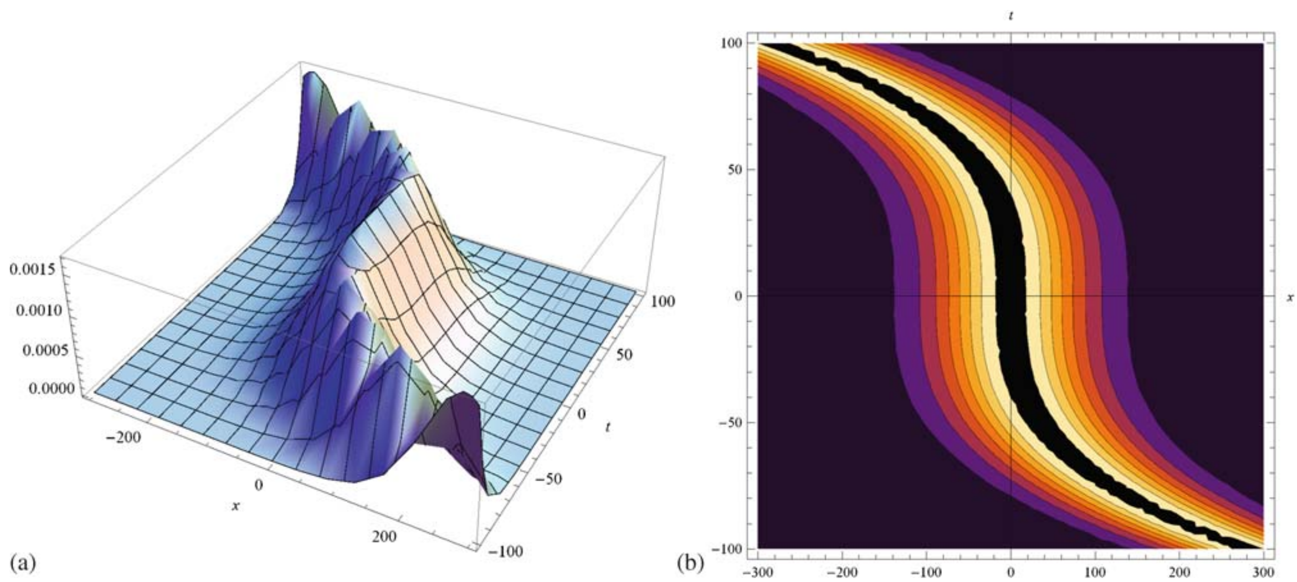

Figure 5. Plots of a curved soliton solution (33): (**a**) 3-D motion and (**b**) 2-D contour, when $A = 0.025$, $A_0 = 0$, $f(t) = 0.2$, $B_t = 0.5, \alpha = 0.9$.

4. Conclusion

In this paper, explicit solutions for Wick-type stochastic equations are found using the Kudrayshov method together with Hermite transformation and white noise theory. By using these solutions, one may get better insight into the physical aspects of the considered stochastic equations. It is concluded that the Kudrayshov method is a powerful technique for investigating exact solutions of Wick-type stochastic equations, particularly to seek different kinds of exact solutions for stochastic partial differential equations. Explaining various physical analyses and interpretation of travelling wave solutions of the Wick-type SPDE are our future endeavours.

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