*Pramana – J. Phys.* (2016) 87: 54
DOI 10.1007/s12043-016-1261-3

© Indian Academy of Sciences

# Calculation of energy spectrum of $^{12}$C isotope with modified Yukawa potential by cluster models

MOHAMMAD REZA SHOJAEI* and NAFISEH ROSHAN BAKHT

Department of Physics, Shahrood University of Technology, P.O. Box 36155-316, Shahrood, Iran
*Corresponding author. E-mail: nroshanbakht@yahoo.com

MS received 21 May 2015; revised 30 October 2015; accepted 16 December 2015; published online 12 September 2016

**Abstract.** In this paper, we have calculated the energy spectrum of $^{12}$C isotope in two-cluster models, $3\alpha$ cluster model and $^{8}$Be + $\alpha$ cluster model. We use the modified Yukawa potential for interaction between the clusters and solve the Schrödinger equation using Nikiforov–Uvarov method to calculate the energy spectrum. Then, we increase the accuracy by adding spin-orbit coupling and tensor force and solve them by perturbation theory in both models. Finally, the calculated results for both models are compared with each other and with the experimental data. The results show that the isotope $^{12}$C should be considered as a three-$\alpha$ cluster and the modified Yukawa potential is adaptable for cluster interactions.

**Keywords.** Cluster model; $^{12}$C isotope; modified Yukawa potential; energy levels.

**PACS No. 21.60.Gx**

## 1. Introduction

The energy levels of nuclei provide an idea of their nuclear structure. Therefore, all the nuclear models, which have evolved from 1930s to now, are expected to reproduce the energy levels. One of the fundamental models of nuclear structure is the cluster model which has a long history [1]. The cluster interpretation is suitable to describe nuclear states and has been successful in reproducing the energy spectra and other nuclear properties such as electromagnetic properties, $\alpha$-emission widths and $\alpha$-particle elastic scattering data in nuclei near the double closed shell. In 1936, when Bethe predicted that nuclei are made of $\alpha$-particles and also proposed a geometrical arrangement of $\alpha$-particles inside the nuclei, the cluster model was introduced [2]. In 1937 [7], Wheeler [3] extended this work, and similar models were suggested concurrently by Wefelmeier [4], Weizsacker [5], and Fano [6]. Freer and Merchant in 1997 [7], studied the role of clustering and cluster models in nuclear reactions and examined the evidence for $\alpha$-cluster chain configurations in the light even–even nuclei from $^{8}$Be to $^{28}$Si. Recently, several systems of even–even nucleus were studied with a cluster model and their results were fairly compared to the experimental spectra for ground state and some excited states [8–11].

The main contribution of this paper is to calculate the ground state of $^{12}$C and some excited states in two cluster models by using the modified Yukawa potential between clusters. The results verify the efficacy of the proposed models and show that the Yukawa potential can be used in cluster model studies.

## 2. Determination of energy spectrum of $^{12}$C isotope in the cluster models

The structure of the $^{12}$C nucleus has been one of the most important subjects in both experimental and theoretical studies. Many successful results have been obtained in various cluster structures [12–17]. Here, we discuss $^{12}$C structure in two cluster models. In the first model, we consider $^{12}$C isotope to be composed of $^{8}$Be core and $\alpha$ cluster, and in the other model the $^{12}$C isotope is considered to consist of three $\alpha$ clusters. In both the models, the internal structure of the $\alpha$ and effects of the Pauli principle between the nucleons in the $\alpha$ clusters are negligible, but modelling the effective interaction among clusters or between the core and the cluster is very important.

The cluster–core interaction leads to the identification of clustering in the nuclear matter and the description of clustering phenomenon in various nuclei. In the 1960s, Ali and Bodmer [18] used the experimental data on $\alpha$–$\alpha$ scattering and obtained potentials which were fitted to the scattering phase shifts. All their potentials had a repulsive part with strength $V_{0l}$ which was dependent on the angular momentum $l$ and an attractive part with a constant strength $V_l$ [1,18], so that

$$V(r) = V_{0l}\exp(-r^2/a^2) - V_l\exp(-r^2/a^2). \quad (1)$$

At a more microscopic level, the core–cluster interaction may be constructed from a nucleon–nucleon interaction. During the last decade, the modified phenomenological Saxon–Woods plus cubic Saxon–Woods cluster potential has successfully described various phenomena related to $\alpha$-clustering in light as well as in even–even heavy nuclei. The form of the potential is given by

$$V(r) = V_0\left[\frac{x}{1+\exp(\frac{r-R}{a})} + \frac{1-x}{1+\exp(\frac{r-R}{a})^3}\right]. \quad (2)$$

This potential is parametrized in terms of the potential depth $V_0$, nuclear radius $R$, diffuseness $a$, and $x$ is the mixing parameter [19]. Prior to the development of the Saxon–Wood plus Saxon–Wood cubic potential form, such a microscopic interaction had been employed in various forms to describe cluster-bound states in light nuclei [20] and the exotic decays in heavy nuclei [21]. Despite its success, the modified Saxon–Wood potential model tells us very little about the microscopic nature of clustering in closed shell nuclei. The Yukawa interaction, in theoretical nuclear physics, was considered as the phenomenological central potential between nucleons and was widely used in modern physics and became an important physical model for theoretical studies [22–27]. So, it is an ideal potential which may be used in the cluster models.

In this paper, for the interaction between the core and the clusters in both the models, we select modified Yukawa potential plus Coulomb repulsion as the central potential $V_c(r)$. Yukawa potential is short-range, and so is an ideal potential for the interaction between the clusters. Then, we add a spin-orbit force $V_{L\cdot S}$ and a tensor force $V_T$ as perturbation potentials and improve the accuracy. Therefore, the total potential becomes

$$V(r) = V_c(r) + V_{L\cdot S}(r)\vec{L}\cdot\vec{S} + V_T(r)\hat{S}_{12}, \quad (3)$$

where

$$V_c(r) = -V_0\frac{\exp(-\alpha r)}{r} + V_1\frac{\exp(-\alpha r)}{r^2} + \frac{k}{r}. \quad (4)$$

This potential has an attractive part with constant $V_0$, and a repulsive part with constant $V_1$ (because the nuclear forces saturate at very small distances) and $k$ is the Coulomb repulsion potential coefficient between the clusters.

In the following sections, we first examine each model separately and then will compare the two models with the experimental data and with each other.

### 2.1 *The $^8Be + \alpha$ cluster model*

$^{12}$C is synthesized in the triple-$\alpha$ process, whereby the two $\alpha$-particles are briefly mixed to form $^8$Be, and at sufficient densities there is a finite probability of capturing a third $\alpha$-particle to form $^{12}$C [28]. If the structure of the $^{12}$C ground state is influenced by clustering $^8\mathrm{Be}(0^+)+\alpha$, the system can be constructed from two particles. Solving the Schrödinger equation for a two-particle system is easier than for a many-body system. Our proposed potential used in this model has been introduced by eqs (3) and (4). The central potential between $^8\mathrm{Be}(0^+)$ core and $\alpha$ cluster is the modified Yukawa potential plus Coulomb repulsion. We solve the radial Schrödinger equation with our central potential between the core and the cluster by Nikiforov–Uvarov method and obtain the energy spectrum function and eigenfunctions as follows [29,30]:

$$(2n+1)\sqrt{\xi_1} - \xi_2 + 2\sqrt{\xi_1(0.25+\xi_3)} = 0 \quad (5)$$

and

$$\psi(r) = r^{-0.5+\sqrt{0.25+\xi_3}}\exp(-\sqrt{\xi_1}r)L_n^{2\sqrt{0.25+\xi_3}}(2\sqrt{\xi_1}r), \quad (6)$$

where

$$\xi_1 = \frac{-2\mu}{\hbar^2}(E_{nl} - \alpha V_0), \quad (7)$$

$$\xi_2 = \frac{-2\mu}{\hbar^2}(V_0 - \alpha V_1 + k), \quad (8)$$

$$\xi_3 = \frac{-2\mu}{\hbar^2}V_1 + l(l+1). \quad (9)$$

We have obtained the potential parameters by fitting the ground state and the first excited energy of the $^{12}$C isotope. In this way, the chosen parameters are $V_0 = -39.32$ MeV, $V_1 = 2.31$ MeV, $k = 5.75$ MeV and $\alpha = 0.01$ fm$^{-1}$. Then we have calculated other energy levels using eq. (5). The binding energy obtained with this configuration is $-92.19$ MeV, which is very close to the experimental value. The measured binding energy of $^{12}$C is equal to $-92.161$ MeV [31,32]. For the first excited state, $2^+$, the measured result is $-87.72$ MeV and we obtain $-87.75$ MeV. The difference between them is too

small. Our model can also successfully approximate the next excited states $0_2^+, 3_1^-, 2_1^-$ and we obtain $-87.16$ MeV, $-83.72$ MeV and $-76.96$ MeV, respectively. The calculated results for some levels are shown in figure 1. It can be observed that there is a gap between levels $3_1^-$ and $1_1^-$, because of neglecting the interaction among two $\alpha$ clusters in $^8$Be. Although the nuclear central force has short range and is strongly attractive at that range (which explains nuclear binding), the nuclear force has a spin-dependence too. A tensor force suggests a force that depends on the orientation of the spins of the nucleons with respect to the vector joining the two nucleons. Further, in heavier nuclei, a strong force between the spin of the nucleon and its orbital motion as a spin-orbit force can be explained.

Although tensor force and spin-orbit coupling are very small compared to the central potential, they are very important and finding exact solution for them is not possible. For improving the accuracy of the calculations, the shift of the energy levels due to them was calculated using the perturbation theory. Perturbation theory is an extremely important method to see how a quantum system will be affected by a small change in the potential and revolves around expressing the potential as some of the many separate potentials. It allows us to get good approximations for the systems where the eigenstates cannot be easily found. The first-order correction to energy is simply the expectation value of the perturbation Hamiltonian while the system is in the unperturbed state. The perturbation causes the average energy in the quantum state $|n\rangle$ to increase in our model by

$$\begin{aligned} E_n^{(1)} &= \langle n|V_{L\cdot S}(r)\vec{L}\cdot\vec{S} + V_T(r)\hat{S}_{12}|n\rangle \\ &= \int \psi_n^{(0)*}(r)V_{L\cdot S}(r)\vec{L}\cdot\vec{S} \\ &\quad + V_T(r)\hat{S}_{12}\psi_n^{(0)}(r)r^2\mathrm{d}r. \end{aligned} \tag{10}$$

$\psi_n^{(0)}(r)$ is the unperturbed wave function and calculated by eq. (6), $V_{L\cdot S}(r)$ is the spin-orbit term and $V_T(r)$ is the tensor term. Using eq. (10), the energy levels for some excited states after the first-order correction are calculated, and the results are shown in figure 1. Energy levels with zero spin will not be shifted, but other levels such as $2_1^-$ and $1_1^+$ will shift. We have calculated the energy levels $2_1^-$ and $1_1^+$ equal to $-77.44$ MeV and $-76.98$ MeV. It can be observed that the results can provide fairly good estimates of the energy spectrum in comparison with the experimental data, but still there is a relatively smaller gap between levels $3_1^-$ and $1_1^-$. This gap may be because of neglecting of interaction among two $\alpha$ clusters in $^8$Be. So, in the next section, three-$\alpha$ cluster model for $^{12}$C will be discussed.

### 2.2 *The three-α cluster model*

The structure of the $^{12}$C ground state is influenced by clustering or the symmetries thereof. So, the system can be constructed from a variety of geometric arrangements of three-$\alpha$ particles. It might be expected that the compact equilateral triangle arrangement is the lowest energy configuration [28]. For our purposes, three identical body forces of the internal particle motion are

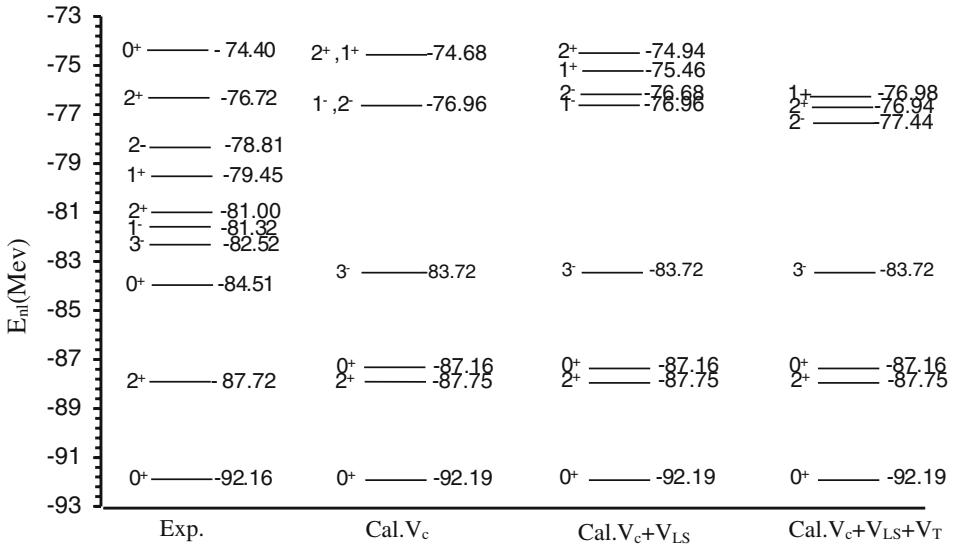


**Figure 1.** Calculated energy spectrum compared with experimental spectrum of $^{12}$C in Be + $\alpha$ model.

described in terms of the Jacobi relative coordinate $\rho$, $\lambda$ and $R$ is the centre of mass. Now, we can introduce the hyper-radius quantity $x$ and the hyper-angle $\xi$ as follows [33–36]:

$$x = \sqrt{\rho^2 + \lambda^2}, \quad \xi = \tan\left(\frac{\rho}{\lambda}\right), \tag{11}$$

where

$$\rho = \frac{\vec{r_1} - \vec{r_2}}{\sqrt{2}}$$
$$\lambda = \frac{\vec{r_1} + \vec{r_2} - 2\vec{r_3}}{\sqrt{6}},$$
$$R = \frac{\vec{r_1} + \vec{r_2} + \vec{r_3}}{\sqrt{3}} \tag{12}$$

and $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ are relative positions of the particles. Now, our central potential in the new coordinate is

$$V_c(x) = -V_0 \frac{\exp(-\alpha x)}{x} + V_1 \frac{\exp(-\alpha x)}{x^2} + \frac{k}{x}. \tag{13}$$

By placing the central potential into the Schrödinger equation in the Jacobi coordinate and making use of the NU method, the energy spectrum and eigenfunctions become

$$(2n+1)\sqrt{\xi_1} - \xi_2 + 2\sqrt{\xi_1(4+\xi_3)} = 0, \tag{14}$$

$$\psi(x) = x^{-2+\sqrt{4+\xi_3}} \exp(-\sqrt{\xi_1}x) L_n^{2\sqrt{4+\xi_3}}(2\sqrt{\xi_1}x), \tag{15}$$

where

$$\xi_1 = \frac{-2\mu}{\hbar^2}(E_{nl} - \alpha V_0), \tag{16}$$

$$\xi_2 = \frac{-2\mu}{\hbar^2}(V_0 - \alpha V_1 + k), \tag{17}$$

$$\xi_3 = \frac{-2\mu}{\hbar^2} V_1 + l(l+4). \tag{18}$$

In this model, similar to the previous model, the binding energy and the first excited state are fitted to the experimental data. The chosen parameters are $V_0 = -94.93$ MeV, $V_1 = 17.17$ MeV, $k = 6.18$ MeV and $\alpha = 0.01$ fm$^{-1}$. Typical value for the parameter in the Yukawa potential is approximately in the range of 30 MeV to 50 MeV but in our calculation, the constant coefficient $V_0$ is relatively more because we have two repulsive centres. The results for some excited states $0_2^+, 3_1^-, 2_2^+$ and $2_1^-$ are $-81.66$ MeV, $-82.52$ MeV, $-77.94$ MeV and $-80.06$ MeV respectively. By comparing the calculated results with the experimental data, we find only a small difference between them. Furthermore, the results in figure 2 show that the three-$\alpha$ cluster model is more appropriate than the $^8\mathrm{Be} + \alpha$ model and gap shown in figure 1 between the $3_1^-$ state and the $2_1^-$ state has been eliminated. For increasing the accuracy, we estimate the first-order correction to energy incurred by the spin-orbit coupling and tensor potential. All the results of the energy levels for the central potential and perturbation potentials are shown in figure 2. The result for the energy levels of the $2_2^+$ Hoyle state in $^{12}$C is $-80.32$ MeV. It is an important input in many recent studies [37–39], and our result is very close to the experimental data ($-81.00$ MeV). The calculated energy level of the $1_1^+$ state is $-78.85$ MeV, and it also is very near to the experimental data.

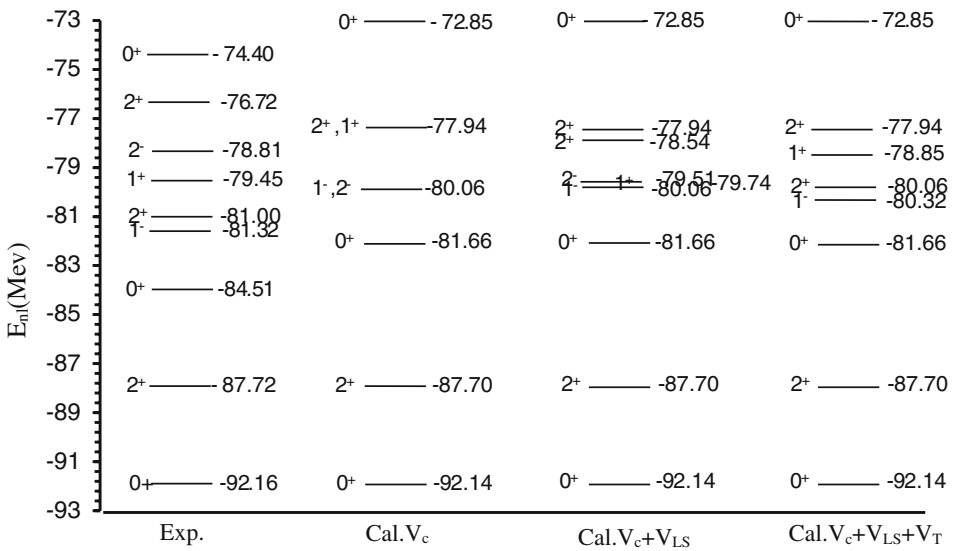


**Figure 2.** Calculated energy spectrum compared with the experimental spectrum of $^{12}$C in 3$\alpha$ cluster model.

## 3. Conclusion

In this paper, through the calculation of the energy levels of $^{12}$C isotopes in two cluster models, it was found that the modified Yukawa potential is quite appropriate to express the central potential between the clusters. Also, by comparing the two proposed models, it was shown that although the energy of the ground state and the first excited state can be fitted well to the experimental data in both the models, for the other excited states, the results of the three-$\alpha$ cluster model are closer to the experimental data than the $^8\mathrm{Be} + \alpha$ core–cluster model. More specifically, for the energy level of the $2_2^+$ Hoyle state in $^{12}$C, we obtained $-80.32$ MeV in three-$\alpha$ cluster model. This value is very close to the experimental value ($-81.00$ MeV) and points to the accuracy of our model. To conclude, the successful results of both the models show that the Yukawa potential may be used successfully in the cluster model studies.

## References


[1] D M Brink, *J. Phys. Conf. Ser.* **111**, 012001 (2008)
[2] M Freer, *Scholarpedia* **5**, 9652 (2010)
[3] J A Wheeler, *Phys. Rev.* **52**, 1083 (1937)
[4] W Z Wefelmeier, *Physik* **107**, 332 (1937)
[5] C F Von Weizsacker, *Naturwiss.* **26**, 209, 225 (1938)
[6] U Fano, *Naturwiss.* **25**, 602 (1937)
[7] M Freer and A C Merchant, *J. Phys. G* **23**, 206 (1997)
[8] H Yépez-Martínez *et al*, *Phys. Rev. C* **86**, 034309 (2012)
[9] H Yépez-Martínez *et al*, *Phys. Rev. C* **85**, 014316 (2012)
[10] P R Fraser *et al*, *Phys. Rev. C* **85**, 014317 (2012)
[11] Y Kanada-En'yo, *Phys. Rev. C* **89**, 024302 (2014)
[12] H Horiuchi, *Prog. Theor. Phys.* **51**, 1266 (1974)
[13] H Horiuchi, *Prog. Theor. Phys.* **53**, 447 (1975)
[14] A Tohsaki *et al*, *Phys. Rev. Lett.* **81**, 192501 (2001)
[15] Y Funaki *et al*, *Phys. Rev. C* **67**, 051306 (2003)
[16] T Yamada *et al*, *LNP* **848**, 229 (2012)
[17] Y Kanada-En'yo, *Prog. Theor. Phys.* **117**, 655 (2007)
[18] S Ali and A R Bodmer, *Nucl. Phys.* **80**, 99 (1966)
[19] S M Wyngaardt *et al*, *J. Phys. Conf. Ser.* **205**, 012013 (2010)
[20] B Buck *et al*, *Phys. Rev. C* **11**, 1803 (1975)
[21] B Buck *et al*, *Phys. Rev. C* **39**, 2097 (1989)
[22] R Machleidt, *Scholarpedia* **9**, 30710 (2014)
[23] E Epelbaum *et al*, *Rev. Mod. Phys.* **81**, 1773 (2009)
[24] J S Hernandez *et al*, *Phys. Rev. D* **85**, 071301 (2012)
[25] E Z Liverts *et al*, *Ann. Phys.* **324**, 388 (2009)
[26] K V Tretiakov *et al*, *Phys. Status Solidi* **251**, 385 (2014)
[27] J C Long *et al*, *Nature* **421**, 922 (2003)
[28] M Freer *et al*, *Phys. Rev. C* **76**, 034320 (2007)
[29] M G Miranda *et al*, *Int. J. Mod. Phys. E* **19**, 123 (2010)
[30] A F Nikiforov *et al*, *Special functions of mathematical physics* (Basel, Birkhauser, 1988)
[31] S Cherry *et al*, *Physics in nuclear medicine* (Expert Consult, 2012)
[32] M Wang *et al*, *Chin. Phys. C* **36**, 1287 (2012)
[33] M R Shojaei *et al*, *Mod. Phys. Lett. A* **23**, 267 (2008); 385 (2014)
[34] M R Shojaei *et al*, *Int. J. Mod. Phys. E* **17**, 6 (2006)
[35] M M Giannini *et al*, *Nucl. Phys. A* **699**, 308 (2002)
[36] A A Rjabi, *Few-Body Syst.* **37**, 267 (2005)
[37] M Itoh *et al*, *Nucl. Phys. A* **738**, 268 (2004)
[38] M Freer *et al*, *Phys. Rev. C* **83**, 034314 (2011)
[39] M Itoh *et al*, *Phys. Rev. C* **84**, 054308 (2011)