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A note on analytical solutions of nonlinear fractional 2D heat equation with non-local integral terms

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Abstract. In this paper, we consider the (2+1) nonlinear fractional heat equation with non-local integral terms and investigate two different cases of such non-local integral terms. The first has to do with the time-dependent non-local integral term and the second is the space-dependent non-local integral term. Apart from the nonlinear nature of these formulations, the complexity due to the presence of the non-local integral terms impelled us to use a relatively new analytical technique called q-homotopy analysis method to obtain analytical solutions to both cases in the form of convergent series with easily computable components. Our numerical analysis enables us to show the effects of non-local terms and the fractional-order derivative on the solutions obtained by this method.

Keywords. Diffusion; non-local integral term, fractional derivative; q-homotopy analysis method.

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1. Introduction and preliminaries

The heat equation in one or m dimensions has been studied extensively and one can find the formulation and solutions of such problems in Carslaw and Jaeger [1]. In these problems, the thermal diffusivity of the medium is either assumed to be constant, or in the case of non-homogeneous medium, as a function of position or time. The homogeneous equation in such cases is given as

$$u_t - \underline{\nabla} \cdot (D\underline{\nabla} u) = 0, \tag{1}$$

where D is the thermal diffusivity. However, in many cases the thermal diffusivity is found to be dependent on temperature (see Ozisik [2] in particular) and the thermal properties may be proportional to the temperature. In such a case, the nonlinear heat equation is given as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(D(u)\frac{\partial u}{\partial y}\right) = 0. \tag{2}$$

Such problems have been studied from Lie symmetry point of view by Ahmad *et al* in [3]. An interesting case with a non-local term has been studied by Bokhari *et al* in [4,5].

The subject of this paper is a nonlinear fractional two-dimensional heat equation with two types of non-local terms. Though the non-integer (fractional) derivative is as old as the integer (classical) derivative, it is only recently that many researchers started seeing the importance of the former in real-life models. The enormous usefulness of this type of derivative in explaining some phenomena which cannot be modelled with integer derivative has given it recognition over the past few decades. Some of these applications have been explained in recent publications by many researchers such as Caputo [6]. He showed how he used the modified form of the Darcy's law to incorporate the memory term to model transport through porous media. Cooper and Cowan [7] illustrated on how being able to calculate any order of derivative allows the maximum order to be used that is commensurate with the noise levels. This then enabled them to achieve the optimum spatial resolution in geophysical data. Many other applications can be found in reactive flows and semiconductors, meteorology, ground water flow, cancer tumor with treatment profile and astrophysics [8–11].

Several researchers have worked to obtain analytical solutions of nonlinear differential equations including fractional type, due to the complexity involved and due to the fact that many such problems do not have exact solutions. The problems we are considering here are nonlinear but the presence of non-local integral terms makes them more interesting in situations where input due to non-local source is present. This paper uses one of the powerful analytical methods known as q-homotopy analysis method to obtain analytical solutions to nonlinear fractional two-dimensional heat equation, with two types of non-local terms; see [12–14] for details. We also gave some numerical illustrations to show the effect of non-local term and the fractional order on the series solution obtained. We present a lemma and associated definitions.

Lemma 1.1. *Let* $t \in (a, b]$. *Then*

$$\left[I_a^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \alpha \geqslant 0, \quad \beta > 0, \tag{3}$$

$$\left[D_a^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad \alpha \geqslant 0, \quad \beta > 0. \tag{4}$$

The operators I^{α} and D^{α} are defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad t > 0, \quad \alpha > 0,$$

$$\mathcal{D}^{\alpha} f(t) = I^{1-\alpha} D f(t), \qquad D = \frac{\mathrm{d}}{\mathrm{d}t},$$

where Γ is the gamma function, $0 < \alpha \leq 1$ and $I^0 f(t) = f(t)$. Also we denote $\mathcal{D}^{\alpha} = \partial^{\alpha}/\partial t^{\alpha}$.

2. q-Homotopy analysis method (q-HAM)

Here, we consider a relatively new but well-known method called q-homotopy analysis method. The main difference between this method and classical homotopy analysis method is briefly highlighted here. The reader is encouraged to see [12–14] for detail.

Differential equation of the form

$$N\left[\mathcal{D}_t^{\alpha} u(x,t)\right] - f(x,t) = 0 \tag{5}$$

is considered, where N is a non-linear operator, f is a known function and u is an unknown function. The zeroth-order deformation equation is constructed to generalize the original homotopy method as

$$(1-nq)L(\Phi(x,t;q) - u_0(x,t)) = qhH(x,t)\left(N[\mathcal{D}_t^{\alpha}\Phi(x,t;q)] - f(x,t)\right), \tag{6}$$

where $n \geqslant 1$, $q \in [0, 1/n]$ denotes the so-called embedded parameter, L ia an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter and $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = 1/n$, eq. (6) becomes

$$\Phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \Phi\left(x,t;\frac{1}{n}\right) = u(x,t) \tag{7}$$

respectively. So, as q increases from 0 to $1/n$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Therefore, we have the q-HAM series representation as

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{8}$$

Remark 2.1. The presence of the factor $(1/n)^m$ gives more chances for better convergence, faster than the solution obtained by the standard homotopy method. Of course, when $n = 1$, the method is called the standard homotopy method. The complete details of qHAM can be explored in references cited earlier.

3. Nonlinear fractional 2D heat equation with non-local integral terms

We consider nonlinear fractional two-dimensional heat equation with different non-local integral terms in this section. The aim is to obtain analytical solutions of these types of problems using q-homotopy analysis method in series form.

3.1 *Case* 1: *Time-dependent non-local integral term*

Consider

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right) - \lambda\int_0^t u(x,y,\tau)\mathrm{d}\tau \tag{9}$$

with initial condition

$$u(x,y,0) = \mathrm{e}^{a(x+y)}. \tag{10}$$

We choose the linear operator as

$$L[\phi(x,y,t;q)] = \mathcal{D}_t^{\alpha}\phi(x,y,t;q),$$

with a property that $L[k] = 0$, k is constant, where $\mathcal{D}_t^{\alpha} = \partial^{\alpha}/\partial t^{\alpha}$.

We use initial approximation $u_0(x, y, t) = \mathrm{e}^{a(x+y)}$ and define non-linear operator as

$$\begin{aligned}N[\phi(x, y, t; q)] &= \mathcal{D}_t^\alpha \phi(x, y, t; q) - \phi\phi_{xx}(x, y, t; q)\\ &\quad - (\phi_x(x, y, t; q))^2 - \phi\phi_{yy}(x, y, t; q)\\ &\quad - \left(\phi_y(x, y, t; q)\right)^2 + \lambda\int_0^t \phi(x, y, \tau; q)\mathrm{d}\tau.\end{aligned}$$

By the q-HAM method, using $H(x, y, t) = 1$, solution to eq. (9) for $m \geqslant 1$ becomes

$$u_m(x, y, t) = \chi_m^* u_{m-1} + hI_t^\alpha[\mathcal{R}_m(\vec{u}_{m-1})], \tag{11}$$

with initial condition for $m \geqslant 1$, $u_m(x, 0) = 0$, χ_m^* is as defined in eq. (16) of [15] and

$$\begin{aligned}&\mathcal{R}_m\left(\vec{u}_{m-1}\right)\\ &= \mathcal{D}_t^\alpha u_{(m-1)} - \sum_{k=0}^{m-1} u_k(u_{m-1-k})_{xx}\\ &\quad - \sum_{k=0}^{m-1}(u_k)_x(u_{m-1-k})_x - \sum_{k=0}^{m-1} u_k(u_{m-1-k})_{yy}\\ &\quad - \sum_{k=0}^{m-1}(u_k)_y(u_{m-1-k})_y + \lambda\int_0^t u_{m-1}(x, y, \tau)\mathrm{d}\tau.\end{aligned} \tag{12}$$

We therefore obtain components of the solution using q-HAM successively as follows:

$$\begin{aligned}&u_1(x, y, t)\\ &= \chi_1^* u_0 + hI_t^\alpha\left[\mathcal{D}_t^\alpha u_0 - u_0(u_0)_{xx} - (u_0)_x^2\right.\\ &\quad \left. - u_0(u_0)_{yy} - (u_0)_y^2 + \lambda\int_0^t u_0(x, y, \tau)\mathrm{d}\tau\right]\\ &= -4a^2 h\mathrm{e}^{2a(x+y)}\frac{t^\alpha}{\Gamma(1+\alpha)}\\ &\quad + \lambda h\mathrm{e}^{a(x+y)}\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}.\end{aligned} \tag{13}$$

$$\begin{aligned}&u_2(x, y, t)\\ &= \chi_2^* u_1 + hI_t^\alpha\left[\mathcal{D}_t^\alpha u_1 - u_0(u_1)_{xx} - u_1(u_0)_{xx}\right.\\ &\quad \left. - 2(u_0)_x(u_1)_x - u_0(u_1)_{yy}\right]\\ &\quad + hI_t^\alpha\left[- u_1(u_0)_{yy} - 2(u_0)_y(u_1)_y\right.\\ &\quad \left. + \lambda\int_0^t u_1(x, y, \tau)d\tau\right]\\ &= -4a^2(n+h)h\mathrm{e}^{2a(x+y)}\frac{t^\alpha}{\Gamma(1+\alpha)}\\ &\quad + \lambda(n+h)h\mathrm{e}^{a(x+y)}\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}\\ &\quad + 72a^4h^2\mathrm{e}^{3a(x+y)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &\quad - 12\lambda a^2h^2\mathrm{e}^{2a(x+y)}\frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}\\ &\quad + \lambda^2h^2\mathrm{e}^{a(x+y)}\frac{t^{2+\alpha}}{\Gamma(3+\alpha)}.\end{aligned} \tag{14}$$

In the same way, $u_m(x, t)$ for $m = 4, 5, 6, \ldots$ can be obtained using symbolic mathematics package/ symbolic computation software such as *Mathematica*. For example, to compute $u_3(x, t)$, the following formula is used:

$$\begin{aligned}u_3(x, y, t) &= \chi_3^* u_2 + hI_t^\alpha\left[\mathcal{D}_t^\alpha u_2 - u_0(u_2)_{xx} - u_1(u_1)_{xx}\right.\\ &\quad \left. - u_2(u_0)_{xx} - 2(u_0)_x(u_2)_x - (u_1)_x^2\right]\\ &\quad + hI_t^\alpha\left[- u_0(u_2)_{yy} - u_1(u_1)_{yy} - u_2(u_0)_{yy}\right.\\ &\quad \left. - 2(u_0)_y(u_2)_y - (u_1)_y^2\right]\\ &\quad + hI_t^\alpha\left[\lambda\int_0^t u_2(x, y, \tau)\mathrm{d}\tau\right].\end{aligned}$$

Then, the series solution expression by q-HAM can be written as

$$u(x, t; n; h) = \mathrm{e}^{kx} + \sum_{i=1}^{\infty} u_i(x, t; n; h)\left(\frac{1}{n}\right)^i, \tag{15}$$

which is an appropriate solution to the problem (9) in terms of convergence parameter h and n.

3.2 *Case* 2: *Space-dependent non-local integral term*

Consider

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)\\ &\quad - \sigma\int_0^x\int_0^y u(\eta, \gamma, t)\mathrm{d}\gamma\, \mathrm{d}\eta\end{aligned} \tag{16}$$

with initial condition

$$u(x, y, 0) = a\sin(xy). \tag{17}$$

Following similar procedure as in (9), using initial approximation $u_0(x, y, t) = a\sin(xy)$ and changing the non-local integral term appropriately as

$$\sigma\int_0^x\int_0^y \phi(\eta, \gamma, t; q)\mathrm{d}\gamma\mathrm{d}\eta,$$

we obtain the q-HAM series solution for this case as

$$\begin{aligned}
u_1(x, y, t) &= \chi_1^* u_0 + hI_t^\alpha\Big[\mathcal{D}_t^\alpha u_0 - u_0(u_0)_{xx} \\
&\quad -(u_0)_x^2 - u_0(u_0)_{yy} - (u_0)_y^2\Big] \\
&\quad + hI_t^\alpha\left[\sigma\int_0^x\int_0^y u_0(\eta, \gamma, t)\mathrm{d}\gamma\mathrm{d}\eta\right] \\
&= \sigma ahx\frac{t^\alpha}{\Gamma(1+\alpha)} - \sigma ah\sin(xy)\frac{t^\alpha}{\Gamma(1+\alpha)}. \quad (18)
\end{aligned}$$

$$\begin{aligned}
u_2(x, y, t) &= \chi_2^* u_1 + hI_t^\alpha\left[\mathcal{D}_t^\alpha u_1 - u_0(u_1)_{xx}\right. \\
&\quad \left. - u_1(u_0)_{xx} - 2(u_0)_x(u_1)_x - u_0(u_1)_{yy}\right] \\
&\quad + hI_t^\alpha\Big[-u_1(u_0)_{yy} - 2(u_0)_y(u_1)_y \\
&\quad + \sigma\int_0^x\int_0^y u_1(\eta, \gamma, t)\mathrm{d}\gamma\mathrm{d}\eta\Big] \\
&= a\sigma(n+h)hx\frac{t^\alpha}{\Gamma(1+\alpha)} \\
&\quad - a\sigma(n+h)h\sin(xy)\frac{t^\alpha}{\Gamma(1+\alpha)} \\
&\quad + 4a^2\sigma h^2\cos(2xy)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + 2a^2\sigma h^2 x\sin(xy)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + a\sigma^2 h^2\sin(xy)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad - 2a^2\sigma h^2\cos(xy)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + a\sigma^2 h^2 x^2 y\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad - a\sigma^2 h^2 x\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad (19) \\
&\ \ \vdots
\end{aligned}$$

Hence, the series solution expression by q-HAM can be written as

$$u(x, t; n; h) = a\sin(xy) + \sum_{i=1}^{\infty} u_i(x, t; n; h)\left(\frac{1}{n}\right)^i. \quad (20)$$

4. Numerical results and discussion

In this section, we present numerical analysis of the approximate solution obtained for both cases considered and observe the effect of non-local terms λ and σ and the fractional-order α on the solution given by U_2.

The behaviour of the solution given by U_2 for different values of α is displayed in figures 1 and 2 for Case1 and figures 3 and 4 for Case 2, with the values $a = 0.2$, $h = -0.8$ and $n = 1$ for a fixed time $t = 0.2$.

Figures 5 and 6 and figures 7 and 8 show the effect of different values of the non-local term λ and σ respectively. The plots here are done with $a = 0.2$, $h = -0.8$ and $n = 1$ for a fixed time $t = 0.2$.

Remark 4.1. Figures 5 and 6 show that the increasing or decreasing trend is affected by the choice of the sign of lambda as it determines whether the heat is given or lost. However, the choice of alpha or fractional order of diffusion term does not affect the trend .

Remark 4.2. We observe also for Case 2 that the sign of σ greatly changes the profile of the solution and this

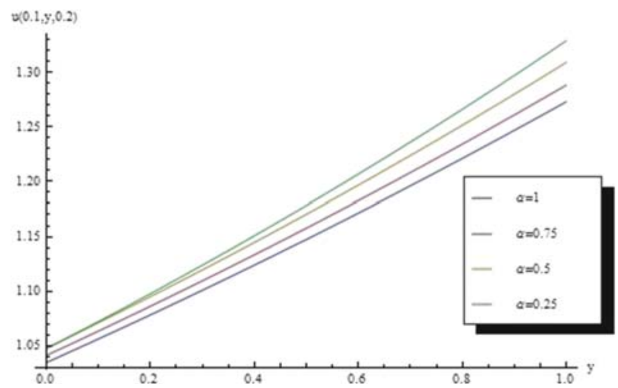

Figure 1. Plots for different values of α when x is fixed for Case 1.

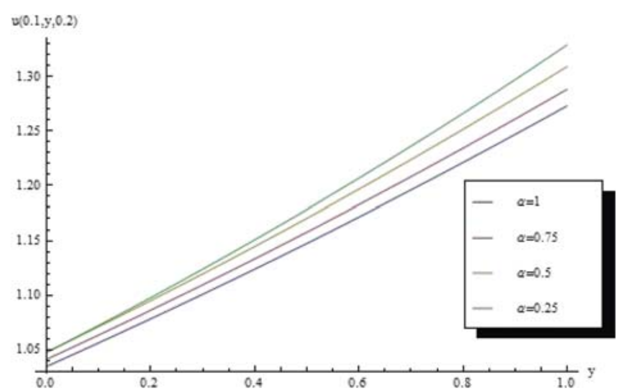

Figure 2. Plots for different values of α when y is fixed for Case 1.

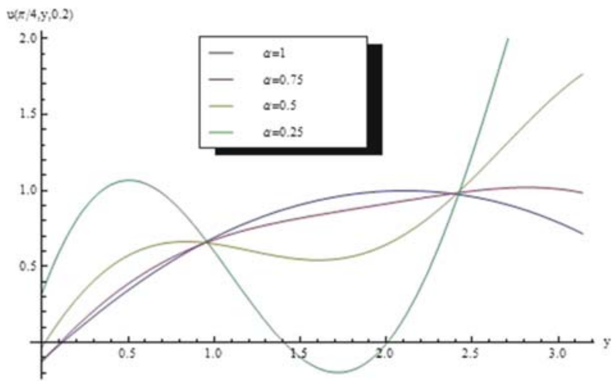

Figure 3. Plots for different values of α when x is fixed for Case 2.

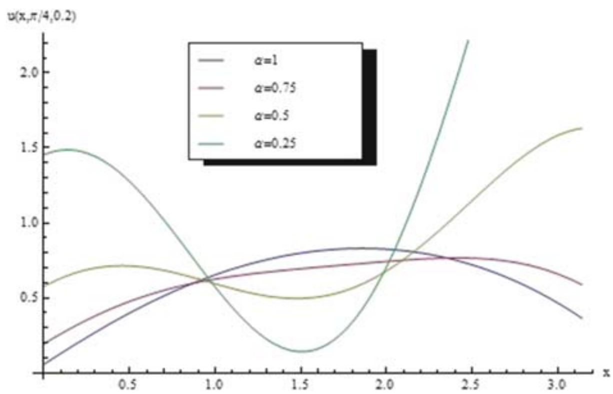

Figure 4. Plots for different values of α when y is fixed for Case 2.

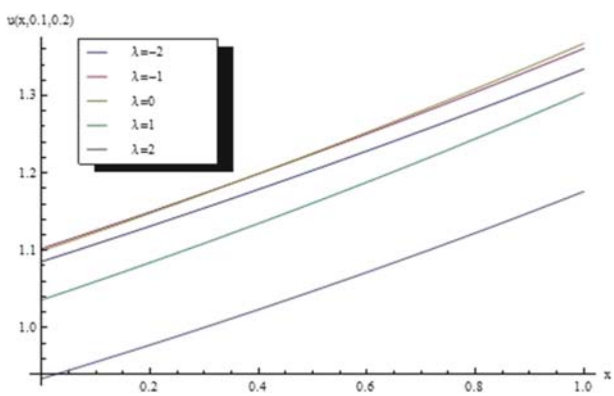

Figure 5. U_2 plot for different values of λ when x is fixed.

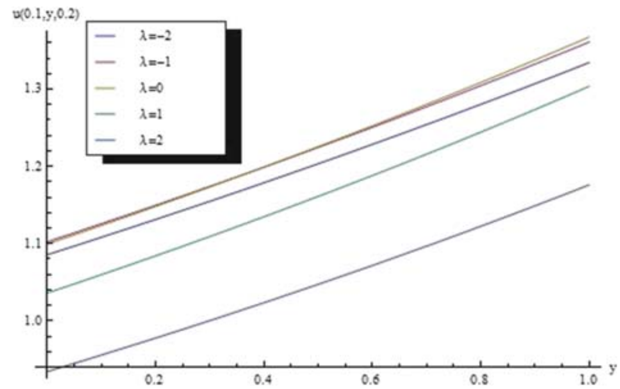

Figure 6. U_2 plot for different values of λ when y is fixed.

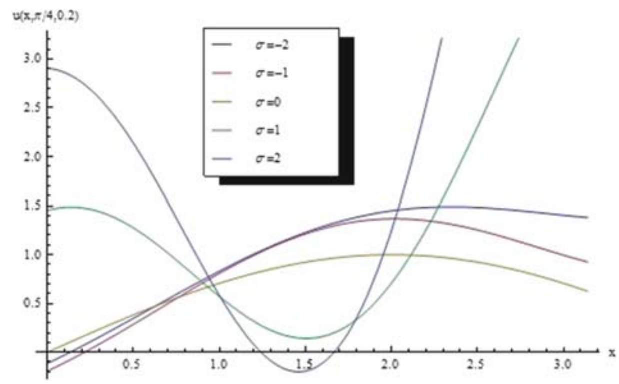

Figure 7. U_2 plot for different values of σ when x is fixed.

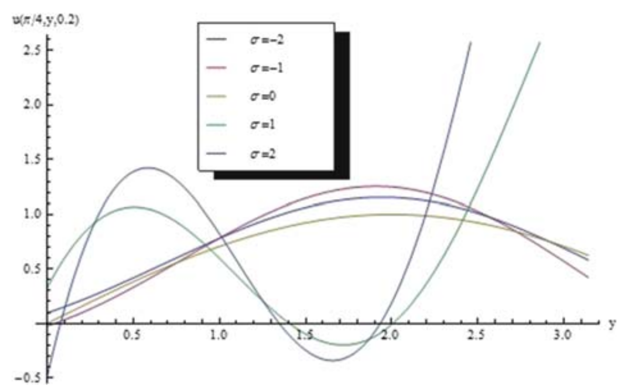

Figure 8. U_2 plot for different values of σ when y is fixed.

can be seen in figures 7 and 8. In this case, choosing space-dependent non-local term could be very interesting due to its effect on the solution.

Remark 4.3. We remark here that the detail about the way to choose appropriate h for fast convergence of the series solution could be obtained through h-curve (see [15] for details).

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