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Statistical model of stress corrosion cracking based on extended form of Dirichlet energy: Part 2

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Abstract. In the previous paper (*Pramana – J. Phys.* **81(6)**, 1009 (2013)), the mechanism of stress corrosion cracking (SCC) based on non-quadratic form of Dirichlet energy was proposed and its statistical features were discussed. Following those results, we discuss here how SCC propagates on pipe wall statistically. It reveals that SCC growth distribution is described with Cauchy problem of time-dependent first-order partial differential equation characterized by the convolution of the initial distribution of SCC over time. We also discuss the extension of the above results to the SCC in two-dimensional space and its statistical features with a simple example.

Keywords. Stress corrosion cracking; Dirichlet energy; variational principle; Cauchy problem; convolution.

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1. Introduction

As stress corrosion cracking (SCC) was recognized to bring serious structural failures, various mechanisms of generating SCC have been proposed. Phenomenally, the presence of SCC on the pipe surface is associated with corrosive environment and susceptible materials under substantial tensile stress [1]. The influence of environment on the materials are classified as the dissolution, removal of materials and the embrittlement caused by hydrogen diffusion, while I tried in the previous paper [2] to represent those relations with a single parameter β which is related to the deviation from linear stress–strain relation. Practically, β is used in the modified form of Dirichlet energy for describing the behaviour of SCC on the pipe surface. It gives the distribution of SCC in the early stage of corrosion. However, it does not state how that distribution changes over time. In this paper we discuss the time-dependent nature of SCC. Particularly, the growth of an individual crack consisting of an SCC colony is dealt with from stochastic viewpoint. Regarding stochastic approach for SCC growth, several models have been proposed. Mullins *et al* [3] proposed the kinetic model for the rate of surface recession including white noise which induces the fractal structure of corroded surface. In PRAISE [4], time to initiate SCC is represented by the product of semiempirically determined functions of loading, environment and material variables. Based on the methodology of PRAISE, Anoop *et al* [5] modelled the initiation and propagation stages of SCC using fuzzy set. Regarding probabilistic approach, Savoia [6] discussed the relation of probability measures with fuzzy set applying possibility theory. In this paper we focus on the transition of SCC distribution under some simple probabilistic premises, and investigate the stochastic properties of the time-dependent partial differential equation describing that transition. We also discuss the relation of their speed of growth with the parameter β in the quasi-Dirichlet energy mentioned above.

As in most cases the hoop stress is predominant on the pipe wall, the main focus of discussion in the previous paper was spatially one-dimensional, or axially-extended SCC. However, when the longitudinal stress is not negligible, spatially two-dimensional SCC must be considered. In this paper, the quasi-Dirichlet energy applied to one-dimensional case is extended for two-dimensional case, and the partial differential equation representing two-dimensional SCC is induced by applying variational principle. Also the time-dependent feature of the two-dimensional SCC distribution is discussed using a simple example.

2. Stochastic properties of SCC

In this section we focus on SCC growing one-dimensionally. We assume here that the each event of generating a single SCC is independent and identically distributed, and the growth of SCC, which is measured by its length, is always non-negative. Several SCCs may generate separately and eventually link each other after growing. However, in the following discussion we exclude such a case, i.e., we focus on a single SCC and its growth only. Then the probability that the SCC length is x at time $t_1 + t_2$ would be expressed with the convolution of probability of SCC length at times t_1 and t_2, viz. when the probability density function (PDF) of the SCC at time t and length x is denoted as $f(t, x)$, the PDF at time $t_1 + t_2$ is expressed with the convolution of the PDFs at time t_1 and t_2:

$$f(t_1 + t_2, x) = (f(t_1) * f(t_2))(x) = \int_0^x \mathrm{d}\xi f(t_1, x - \xi) f(t_2, \xi). \quad (1)$$

It is convenient to apply Laplace transform for analysing this equation because the convolution operator in the equation is transformed to ordinary multiplication. Applying Laplace transform to it, we restate the above equation as

$$F(t_1 + t_2, s) = F(t_1, s)F(t_2, s), \quad (2)$$

where $F(t, s)$ is the Laplace transform of $f(t, x)$. If $F(t, x)$ is sufficiently smooth, it is expressed as

$$F(t, s) = \mathrm{e}^{H(s)t}, \quad (3)$$

where $H(s)$ is the function determined by the initial condition of SCC distribution discussed later. Here $H(s)$ is supposed as $H(s) \to -\infty$ when $s \to \infty$ for the sake of applying inverse Laplace transform to $F(t, s)$. $F(t, s)$ is expanded with Taylor series by t as

$$F(t, s) = 1 + H(s)t + \frac{H(s)^2 t^2}{2!} + \frac{H(s)^3 t^3}{3!} + \cdots. \quad (4)$$

Applying inverse Laplace transform to this Taylor expansion formally, we get

$$f(t, x) = \delta(x) + h(x)t + \frac{(h * h)(x)}{2!} t^2 + \frac{(h * h * h)(x)}{3!} t^3 + \cdots. \quad (5)$$

This equation indicates that $f(0, x)$ is the delta function of x.

Equation (3) implies that $F(t, s)$ satisfies the following equation:

$$\frac{\partial F(t, s)}{\partial t} = H(s)F(t, s) \quad (6)$$

or applying inverse Laplace transform to it we get

$$\frac{\partial f(t, x)}{\partial t} = (h * f(t))(x). \quad (7)$$

We call this equation 'SCC growth distribution equation'. As $f(0, x) = \delta(x)$ as mentioned above, we obtain the following initial condition:

$$\left.\frac{\partial f(t, x)}{\partial t}\right|_{t=0} = (h * f(0))(x) = (h * \delta)(x) = h(x). \quad (8)$$

Although the general solution for SCC growth distribution equation is given by (5), here we review some special solutions for understanding the behaviour of SCC growth in a more tangible manner. Firstly we survey the case $H(s) = -c \cdot s$ where c is a constant. Then eq. (3) is restated as

$$F(t, s) = \mathrm{e}^{-cts} \quad (9)$$

and its inverse Laplace transform is

$$f(t, x) = \delta(x - ct). \quad (10)$$

This equation indicates that the SCC length is determined as ct at time t without probabilistic uncertainty, i.e., the SCC grows with the speed of c constantly. Equation (10) is thought to be the simplest solution of SCC growth distribution eq. (7) with the initial condition:

$$\left.\frac{\partial f(t, x)}{\partial t}\right|_{t=0} = h(x) = -c\delta'(x). \quad (11)$$

Next we deal with gamma distribution [7] as the solution of SCC growth distribution equation. As gamma distribution $\Gamma(k, \theta)$ satisfies the following relation:

$$\Gamma(k_1 + k_2, \theta) = \Gamma(k_1, \theta) * \Gamma(k_2, \theta) \quad (*\text{: convolution}) \quad (12)$$

it implies that gamma distribution satisfies SCC growth distribution equation when the parameter k is proportional to time. Hence we consider the gamma distribution in the following form:

$$\Gamma(\nu t, \theta) = \frac{x^{\nu t - 1} \mathrm{e}^{-x/\theta}}{\Gamma(\nu t)\theta^{\nu t}}, \quad (13)$$

where $\Gamma(\nu t)$ is the gamma function and ν is a constant having the dimension (/sec). The mean value of $\Gamma(\nu t, \theta)$ is $\nu t\theta$, and it implies that the mean growth speed of SCC represented with the above gamma distribution is $\nu\theta$. The Laplace transform of the gamma distribution (13) is

$$G(\nu t, \theta) = \frac{1}{(1+\theta s)^{\nu t}}. \tag{14}$$

It is restated as,

$$G(\nu t, \theta) = \mathrm{e}^{-\nu t \log(1+\theta s)}. \tag{15}$$

Thus,

$$H(s) = -\nu \log(1+\theta s). \tag{16}$$

Although the inverse Laplace transform of $H(s)$ does not exist in general meaning, here we discuss how it is expressed asymptotically. Firstly, we consider the following function:

$$L(s) = s^{\varepsilon}, \tag{17}$$

where ε is any small positive real number. Although its inverse Laplace transform $l(x)$ also does not exist, we calculate it using analytical continuation formally as

$$l(x) = \frac{x^{-\varepsilon-1}}{\Gamma(-\varepsilon)}. \tag{18}$$

Hence, the inverse Laplace transform of the function

$$F(s) = \frac{s^{\varepsilon}-1}{\varepsilon} \tag{19}$$

is expressed as

$$f(x) = \frac{x^{-\varepsilon-1}}{\Gamma(-\varepsilon)\varepsilon} - \frac{\delta(x)}{\varepsilon}. \tag{20}$$

When $\varepsilon \to 0$, $F(s)$ converges to $\log(s)$, while the above $f(x)$ is expressed asymptotically as

$$f(x) \sim -x^{-1} - \lim_{\varepsilon\to 0}\frac{\delta(x)}{\varepsilon} \tag{21}$$

because $\Gamma(-\varepsilon) \to -\infty$ when $\varepsilon \to 0$. Actually, $\Gamma(0)$ is the first-order pole which has the residue 1. Thus, the inverse Laplace transform of $H(s)$ in (16) is expressed with the asymptotic expansion as

$$h(x) \sim \nu\left\{\frac{\mathrm{e}^{-x/\theta}}{x} + \lim_{\varepsilon\to 0}\frac{\delta(x)}{\varepsilon}\right\}. \tag{22}$$

Therefore, the initial condition of SCC growth equation is expressed as

$$\left.\frac{\partial\Gamma(\nu t,\theta)}{\partial t}\right|_{t=0} \sim \nu\left\{\frac{\mathrm{e}^{-x/\theta}}{x} + \lim_{\varepsilon\to 0}\frac{\delta(x)}{\varepsilon}\right\}. \tag{23}$$

In general, if the function

$$F(\nu t, s) = \frac{1}{(1+a_1 s + a_2 s^2 + \cdots + a_n s^n)^{\nu t}} \tag{24}$$

is factorized as

$$F(\nu t, s) = \frac{1}{(1+\theta_1 s)^{\nu t}(1+\theta_2 s)^{\nu t}\cdots(1+\theta_n s)^{\nu t}}, \tag{25}$$

where all θ_i are positive real numbers, then the inverse Laplace transform of $F(\nu t, s)$ by s is expressed with the convolution of gamma distributions as

$$f(\nu t, s) = \Gamma(\nu t, \theta_1) * \Gamma(\nu t, \theta_2) * \cdots * \Gamma(\nu t, \theta_n). \tag{26}$$

$f(\nu t, s)$ is the solution of SCC growth distribution eq. (7) with the following $H(s)$:

$$H(s) = -\nu\{\log(1+\theta_1 s) + \log(1+\theta_2 s) + \cdots + \log(1+\theta_n s)\} \tag{27}$$

namely,

$$\left.\frac{\partial F(\nu t,\theta)}{\partial t}\right|_{t=0} = h(x) \sim \nu\left\{\frac{1}{x}\sum_{i=1}^{n}\mathrm{e}^{-x/\theta_i} + \lim_{\varepsilon\to 0}\frac{n\delta(x)}{\varepsilon}\right\}. \tag{28}$$

Now we estimate the relation between the speed of SCC growth and stress on the pipe surface based on a simplified model. Figure 1 shows the comparison of two states on the pipe surface degenerated one-dimensionally. State I consists of single SCC and the

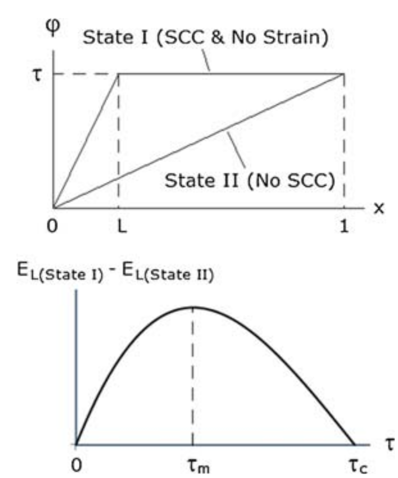

Figure 1. SCC and non-SCC states and difference of their quasi-Dirichlet energies.

remaining part that has no tensile stress and State II has no SCC and uniformly strained. The difference of energy between those states is calculated based on the quasi-Dirichlet energy E_L defined by the formula (9) in the previous paper [2] as

$$\begin{aligned} E_{L(\text{State I})} - E_{L(\text{State II})} &= \frac{\tau}{8\beta\tau_c} - \frac{1}{2}\frac{\tau^2}{1+\beta\tau^2} \\ &= \frac{\tau(\tau-\tau_c)(\tau-3\tau_c)}{8\beta\tau_c(1+\beta\tau^2)}, \end{aligned} \quad (29)$$

where $\tau_c = \sqrt{(1/3\beta)}$ and it is equal to the strain on SCC. Equation (29) indicates that the quasi-Dirichlet energy of State I under $0 < \tau < \tau_c$ is greater than that of State II, viz. to remove the tensile stress on the pipe surface with single SCC generation needs additional energy. Also, when τ_1 and τ_2 are greater than the maximum point (τ_m) of $E_{L(\text{State I})} - E_{L(\text{State II})}$ on $[0, \tau_c]$ and $\tau_1 < \tau_2$, the SCC at τ_1 is thought to grow slower than that of τ_2 because the energy required to cause SCC in the former is greater than in the latter according to (29), which is physically plausible. On the other hand, when τ_1 and τ_2 are smaller than the maximum point, the energy required to cause SCC in the former is less than in the latter. That is, the smaller the strain, or the stress is, the faster the SCC grows, although the smaller the strain (τ) is, the smaller the maximum SCC length (L) is, as they are proportional in this model. It implies SCC can grow even if the stress is very small.

3. Two-dimensional SCC

So far we have dealt with one-dimensional SCC only under the assumption that any stress other than the hoop stress on the pipe surface is negligible. However, in some cases the longitudinal tensile stress is practically not small enough to neglect. For those cases, we extend the results obtained from one-dimensional case to two-dimensional case.

The quasi-Dirichlet energy is defined by formula (9) in the previous paper:

$$E_L[\varphi] = \frac{1}{2}\int_0^D \mathrm{d}x \frac{\varphi_x(x)^2}{1+\beta\varphi_x(x)^2}, \quad (30)$$

where φ is the displacement on the pipe surface. It is naturally extended two-dimensionally as

$$E_{L\times L}[\varphi] = \frac{1}{2}\int_0^{D_1} \mathrm{d}x_1 \int_0^{D_2} \mathrm{d}x_2 \frac{\varphi_1^2+\varphi_2^2}{1+\beta(\varphi_1^2+\varphi_2^2)}, \quad (31)$$

where $\varphi_j = \partial\varphi/\partial x_j$. Applying the variational principle to the above quasi-Dirichlet energy, we obtain the following equation:

$$\begin{aligned} &-\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\frac{\partial E_{L\times L}[\varphi]}{\partial \varphi_j} \\ &= -\frac{\varphi_{11}(1-3\beta\varphi_1^2+\beta\varphi_2^2)+\varphi_{22}(1-3\beta\varphi_2^2+\beta\varphi_1^2)}{\{1+\beta(\varphi_1^2+\varphi_2^2)\}^3} \\ &= 0, \end{aligned} \quad (32)$$

where $\varphi_{jj} = \partial^2\varphi/\partial x_j^2$. When $\varphi(x_1, x_2) = \varphi(x_1)$, i.e. φ is independent of x_2, the above equation is reduced to one-dimensional case:

$$-\frac{\varphi_{11}(1-3\beta\varphi_1^2)}{(1+\beta\varphi_1^2)^3} = 0. \quad (33)$$

The conditions for satisfying the above equation are $\varphi_{11} = 0$, $1 - 3\beta\varphi_1^2 = 0$ or $\varphi_1 \to \pm\infty$, while in two-dimensional case the conditions for satisfying (32) are rather subtle. Here we deal with only the following particular condition: $\varphi_{22} = 0$ almost everywhere, i.e. the longitudinal strain φ_2 is assumed to be piecewise constant. As most of the SCC events on the pipe surface are deemed approximately occurring one-dimensionally, it is thought plausible if SCC generates only longitudinally and is perpendicular to the stress causing it. Then (32) is restated as

$$-\frac{\varphi_{11}(1-3\beta\varphi_1^2+\beta a^2)}{\{1+\beta(a^2+\varphi_1^2)\}^3} = 0, \quad (34)$$

where $\varphi_2 = a$. The condition corresponding to $1 - 3\beta\varphi_1^2 = 0$ in one-dimensional case is $1-3\beta\varphi_1^2+\beta a^2 = 0$ and solved as

$$\varphi_1 = \pm\sqrt{\frac{1+\beta a^2}{3\beta}} = \pm\sqrt{\frac{1}{3\beta'}} \quad (35)$$

where $\beta' = \beta/(1+\beta a^2)$. That is, the virtual β or β', is smaller than real β. As SCC length L_{SCC} is expressed as $L_{\text{SCC}} = S\lambda\sqrt{(1/3\beta)}$ from eqs (12) and (13) in the previous paper [2], the greater longitudinal tensile stress a has the effect to make the SCC longer. When a is sufficiently greater than β, the SCC length is approximately proportional to a, or longitudinal strain. As longitudinal strain is approximately proportional to longitudinal stress in a certain range, it is restated that the SCC length is approximately proportional to the longitudinal stress under the above-mentioned condition. In figure 2, the sample of two-dimensional solution is illustrated.

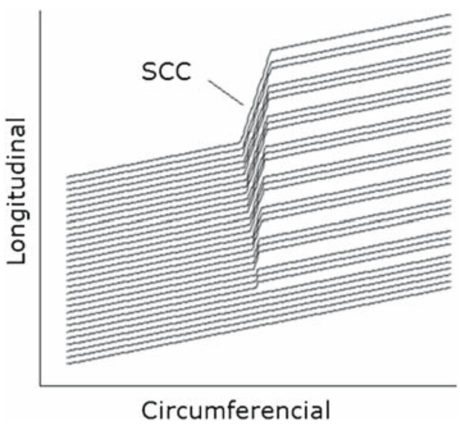

Figure 2. Two-dimensional displacement on the pipe surface.

Next we extend the one-dimensional SCC growth model mentioned in the previous section two-dimensionally. Firstly, relation (1) is naturally extended as

$$\begin{aligned} f(t_1+t_2,x_1,x_2) &= (f(t_1)*f(t_2))(x_1,x_2) \\ &= \int_0^{x_1}\mathrm{d}\xi_1\int_0^{x_2}\mathrm{d}\xi_2 \\ &\quad \times f(t_1,x_1-\xi_1,x_2-\xi_2)f(t_2,\xi_1,\xi_2). \end{aligned} \tag{36}$$

Applying Laplace transforms by x_1 and x_2,

$$F(t_1+t_2,s_1,s_2) = F(t_1,s_1,s_2)F(t_2,s_1,s_2) \tag{37}$$

and (3) is extended as

$$F(t,s_1,s_2) = \mathrm{e}^{H(s_1,s_2)t}. \tag{38}$$

$F(t,s_1,s_2)=1$ when $t=0$, and hence $f(0,x_1,x_2)=\delta(x_1)\delta(x_2)$. By applying inverse Laplace transform to $H(s_1,s_2)$, the one-dimensional SCC growth distribution eq. (7) is extended two-dimensionally as

$$\frac{\partial f(t,x_1,x_2)}{\partial t} = (h*f(t))(x_1,x_2) \tag{39}$$

with the following initial condition:

$$\left.\frac{\partial f(t,x_1,x_2)}{\partial t}\right|_{t=0} = h(x_1,x_2). \tag{40}$$

Equation (9) is extended two-dimensionally as

$$F(t,s_1,s_2) = \mathrm{e}^{-t(c_1s_1+c_2s_2)}. \tag{41}$$

As s_1 and s_2 are separable as

$$\mathrm{e}^{-t(c_1s_1+c_2s_2)} = \mathrm{e}^{-tc_1s_1}\mathrm{e}^{-tc_2s_2} \tag{42}$$

$F(t,s_1,s_2)$ is expressed with the product of $F_1(t,s_1)$ and $F_2(t,s_2)$. By applying inverse Laplace transform to them individually, the following solution is derived:

$$f(t,x_1,x_2) = f_1(t,x_1)f_2(t,x_2) = \delta(x_1-c_1t)\delta(x_2-c_2t). \tag{43}$$

$H(s_1,s_2)$ in this example is expressed as

$$H(s_1,s_2) = -c_1s_1 - c_2s_2 \tag{44}$$

and its inverse Laplace transform by s_1 and s_2 is

$$h(x_1,x_2) = -c_1\delta'(x_1)\delta(x_2) - c_2\delta(x_1)\delta'(x_2). \tag{45}$$

Hence the initial condition is

$$\left.\frac{\partial f(t,x_1,x_2)}{\partial t}\right|_{t=0} = -c_1\delta'(x_1)\delta(x_2) - c_2\delta(x_1)\delta'(x_2), \tag{46}$$

where c_1 and c_2 are actually the velocities of the SCC growth towards x and y directions respectively. It should be noted that the SCC grows one-dimensionally in this case, although the solution itself is extended to two-dimensional pipe surface.

4. Summary

The features of the SCC growth are discussed by applying the results from the previous paper. Here, each one-dimensional single SCC is assumed to be generated independently and identically distributed. Then the SCC distribution has the property expressed with the convolution regarding time as shown in (1). Based on that property, eq. (7) for describing SCC growth distribution is induced. That equation is a first-order time-dependent partial differential equation and is solved uniquely as Cauchy problem.

The speed of SCC growth depends on tensile stress. Their relation is estimated by comparing the SCC state and non-SCC state regarding their quasi-Dirichlet energy which was discussed in the previous paper. The result implies that even if the tensile stress is very small, SCC can still grow.

The above discussion is for one-dimensional SCC on the pipe surface where the hoop stress is predominant. However, when the longitudinal tensile stress is not negligibly small, we need to consider two-dimensional SCC. The partial differential equation for describing two-dimensional SCC is the natural extension of that of the one-dimensional SCC, while the general solution of the former is more complicated than the latter. Here, only a special solution for the two-dimensional case is dealt with, and it reveals the relation between the SCC length and longitudinal tensile stress, which implies they are approximately proportional if the latter is sufficiently greater than the factor β in quasi-Dirichlet energy. Also a simple example of the SCC growth on the two-dimensional space is demonstrated based on the solution of the above-mentioned partial differential equation.

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