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Dynamics of 'quantumness' measures in the decohering harmonic oscillator

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Abstract. We studied the behaviour under decoherence of four different measures of the distance between quantum states and classical states for the harmonic oscillator coupled to a linear Markovian bath. Three of these are relative measures, using different definitions of the distance between the given quantum states and the set of all classical states. The fourth measure is an absolute one, the negative volume of the Wigner function of the state. All four measures are found to agree, in general, with each other. When applied to the eigenstates $|n\rangle$, all four measures behave non-trivially as a function of time during dynamical decoherence. First, we find that the first set of classical states to which the set of eigenstate evolves is (by all measures used) closest to the initial set. That is, all the states decohere to classicality along the 'shortest path'. Finding this closest classical set of states helps improve the behaviour of all the relative distance measures. Second, at each point in time before becoming classical, all measures have a state n^* with maximal quantum-classical distance; the value n^* decreases as a function of time. Finally, we explore the dynamics of these non-classicality measures for more general states.

Keywords. Decoherence; non-classicality; quantum harmonic oscillator.

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1. Introduction

Contemporary investigations of the transition from quantum to classical are motivated by fundamental considerations as well as practical issues of control, engineering, and the study of decoherence [1]. The transition is being investigated in nanomechanical systems [2,3], mesoscopic systems such as Josephson junction devices [4], as well as cavity–QED (quantum electrodynamics) systems [5], among others. The decohering harmonic oscillator has in particular been studied extensively [6–13]. It is foundational for a fundamental understanding of quantum behaviour. It also has a more applied relevance in proposals to encode qubits and qudits in the harmonic oscillator using finite superpositions of eigenstates [14]. This decohering harmonic oscillator has also been experimentally studied; for example, Brune *et al* successfully excited Fock states in a cavity–QED system, and monitored their decay using tomography [13] and Wang *et al* [15] did an equivalent experiment in a superconducting quantum circuit.

An issue of deep interest in these studies is quantifying the non-classicality of quantum states. Defining non-classicality is complicated even in a static situation (i.e. closed systems, where the measures do not change with time). For example, often the purity of a quantum state is used, but the degree to which a state is mixed is not necessarily the same as how classical it is behaving. Similarly, a squeezed Gaussian state is highly non-classical in quantum optics but is part of the range of very classical states when looking at phase-space issues such as quantum-classical correspondence in dynamical systems. The approach taken by some researchers in this context is to define non-classicality in terms of a supremum over all possible classical states in terms of a sensible quantum state distance metric. Only recently has the time-dependence of non-classicality been studied for open systems. Marian and Marian [6] focussed attention on even coherent states ('cat states'), and used as diagnostics (a) the 2-entropy $= -\ln[\mathrm{Tr}\rho^2]$ of the state and (b) the high-order squeezing (if any) associated with the state. They found a transient increase of high-order squeezing as a function of time, as well as a peak in the behaviour of the 2-entropy in time and concluded that coupling to a heat-bath non-trivially affects non-classicality compared to a closed system. Paavola *et al* [7], in a recent detailed study of the behaviour of even coherent states, used a variety of measures including the negative volume of the Wigner function of the state, and found that the quantum-to-classical transition occurs at a finite time, but that initially more non-classical states do not necessarily decohere faster. Dodonov *et al* [8] have also studied squeezing and Wigner function negativity for cat states, albeit for a parametrically excited quadratic oscillator. These studies indicate that the non-classicality for open systems as a function of time, state parameters, Hamiltonian and environmental coupling is an interesting landscape with plenty to be explored yet.

In this paper, we are particularly motivated by two questions about this landscape: (1) Non-classicality measures have typically been studied formally and in detail using relative measures and for time-independent systems, i.e., with measures using different definitions of the distance between the given quantum states and the set of all classical states, and where there is no evolution, either pure Hamiltonian or open system evolution. On the other hand, the behaviour of non-classicality under decoherence has so far been studied using absolute measures. This motivates the following two-part question: how do relative measures behave in decohering systems? And how do absolute and relative measures compare to each other in static systems? (2) While studies so far have looked at the 'cat states', which are relatively complex, how does non-classicality behave in a decohering system for simpler states such as Fock states (which correspond to the well-studied experimental situation)?

Now, we do side-by-side comparisons of different measures of non-classicality during decoherence, specifically the relative measures termed as (1) the Hillery measure [16] denoted by η_{H}, (2) the Bures measure [17] denoted by η_{B} and (3) the Dodonov measure [18] denoted by η_{D}, as well as the absolute measure (4), the Wigner function negativity [19] denoted by η_{W}; all these measures are going to be defined fully. Further, since the Wigner function has now been directly measured [20–23], arguably η_{W} is an experimentally accessible measure of non-classicality. We focus mainly on the Fock states, i.e., the eigenstates $|n\rangle$ of the harmonic oscillator. Even for the static (closed system) case, considering the behaviour of decohering Fock states allows us to introduce a novel set of states that improve upon previous analyses of the well-studied relative measures, η_{H}, η_{B} and η_{D}. That is, the first two of these non-classicality measures are defined using the infimum (inf) and the last using the supremum (sup) of particular quantum distances over the set of all possible classical states. As such, any new set of states that is added to the set of classical states considered can only in principle improve these calculations. We use the observation that each Fock state decoheres to a classical state after a finite time (t^*) to define a new set of states. When this set is used, it outperforms all other sets previously considered in the calculation of these supremum and infimum distances i.e., this set is measurably closest to the Fock states – according to all three relative measures – of all classical sets considered even before any decoherence occurs. Arguably, therefore, Fock states decohere along the shortest path from non-classicality to classicality. The somewhat intuitive absolute measure, η_{W} [8,19] has essentially the same behaviour as the other three more formal relative measures for the closed case.

We then analyse the decohering harmonic oscillator, conducting to the best of our knowledge the first dynamical study of the relative measures η_{H}, η_{B} and η_{D}. We find that all the measures display non-trivial dynamics. Specifically, as the system evolves, different eigenstates such as $n1, n2$ have different rates of change for all measures of η. In particular, given any $n1 > n2$, we initially have $\eta^{n1}_{\mathrm{H,B,W}}(0) > \eta^{n2}_{\mathrm{H,B,W}}(0)$ for the Hillery, Bures and negativity measures and $\eta^{n1}_{\mathrm{D}}(0) < \eta^{n2}_{\mathrm{H}}(0)$ for the Dodonov measure. As the system decoheres, there exists a time t (different for each measure and for each pair of $n1, n2$), where this does not hold true any more. In full generality, a peak in non-classicality as a function of n emerges, with the location of the peak changing as a function of time. We discuss each of this in detail now.

We start our presentation in §2 with a quick review of how non-classicality has been defined and quantified, before introducing the four formal measures of non-classicality that we use as representatives of the different techniques described the literature. Section 3 contains our results, starting with the static case where we are able to generalize previous results with a novel classical basis; this basis proves critical for the open system results. We then consider Fock states coupled to the vacuum, where we find the time-dependent peak in non-classicality as a function of eigenstate n; we also find essentially the same results at finite temperature. We close with a short conclusion and discussion of our findings in §4.

2. Measures of non-classicality

To quantify how non-classical a given state is, it is necessary to first define what constitutes a classical state. One definition of a classical state is one that can be a possible solution for a classical dynamical system (even if the state also satisfies the laws of quantum mechanics). The formal answer to the question of non-classicality starts from this intuition, with the work of Klauder, Glauber and Sudarshan, for example in considering the coherent states in quantum optics [24–26]; it was later shown [27] that coherent states could be thought of as classical states. The formal criterion for a state to be classical is that its Glauber–Sudarshan P-representation is positive definite and no more singular than a delta function.

Another way of determining classicality comes from the intuition developed via the work on quantum-classical correspondence, particularly as it pertains to classically chaotic systems (see for example early definitive work by Berry, and later work represented by Wilkie [28–31]). Berry pioneered the understanding that quantum-classical correspondence was best analysed in the Wigner–Weyl representation, which he proved is the only phase-space formulation of quantum mechanics that approaches the classical limit of Liouville phase-space mechanics. In this case, the criteria for a state to be considered classical are that its Wigner representation be continuous, smooth, positive definite and normalized. Coherent states are Gaussians in the Wigner representation that minimize the uncertainty principle, and satisfy all the properties of a classical probability distribution. They are indistinguishable from a classical probability distribution with minimum uncertainty.

A physically-argued justification that coherent states are the quantum analogues of points in phase-space also comes from Zurek *et al* [32], who showed that coherent states are the most stable in a thermal bath. Specifically, they studied decohering pure states weakly coupled to a thermal bath, and showed that the coherent states produced the least entropy as they decayed. It has also been established elsewhere that the coherent states are the only pure states that satisfy the rules for classical distributions [33]. In phase-space, this argument takes the form of a theorem that only the positive-definite pure state Wigner functions correspond to Gaussian states.

Looking beyond pure states, thermal states – whether centred at the origin or displaced – fit all the above descriptions of classical states and hence are a standard addition to the set of classical states when studying relative measures (for example, see [17,18,34]). Thermal states centred at the origin are the steady-state solutions to eq. (10), and are defined as

$$\rho_{\text{th}} = \frac{1}{N+1}\sum_{n=0}^{\infty}\left(\frac{N}{N+1}\right)^{n} |n\rangle\langle n|, \tag{1}$$

and represent a thermal distribution of the number states, with N defined, as in eq. (10), to be the mean number of thermal photons in the bath. Displaced thermal states are defined using the displacement operator $\hat{D}(\alpha) = \exp(\hat{\alpha} a^{\dagger} - \alpha^{*}\hat{a})$:

$$\hat{D}(\alpha)\rho_{\text{th}}\hat{D}^{-1}(\alpha). \tag{2}$$

In §3, we present a novel addition to the basis of classical states that is an improvement over all previous analyses for the relative measure in the context of the harmonic oscillator; these are found by studying the dynamics of Fock states, and can be understood as thermally-broadened number states and are explicitly defined below.

We note here that, while the quantum optics and the phase-space perspectives agree on the classicality of thermal states, they disagree about squeezed coherent states. Specifically, the phase-space perspective considers all positive-definite normalized distrbutions to be classical as they satisfy the criteria that define a valid classical Liouville distribution (even though they also satisfy quantum equations). This includes squeezed coherent states, even though such states are considered non-classical in quantum optics as they exhibit sub-Poissonian photon statistics. We work in phase-space and do not consider squeezed states to be non-classical; a careful analysis of the dynamics of squeezing under decoherence, has been previously done by Marian and Marian [6] and Dodonov *et al* [8]. It is also important here to stress that we are only considering unipartite systems. If, for example two coherent states are entangled, the total multipartite system would certainly not be classical.

Having established a set of classical states, it is then possible to classify how non-classical another state is by considering how different it is from the set of classical states. We shall consider three standard ways of quantifying this distance between an unknown state and the set of classical states. These measures are relative in that they compare unknown states to those known to be classical. They are formally well-defined; however, as we see now, the non-classicality of a state is defined in terms of the infimum (or in the case of classicality measures, the supremum) of the distance from all classical states. Given that it is impossible to search through this infinite set of all classical states, there is a certain amount of ambiguity about these relative measures. Progress can be made using an incomplete basis, however, as has been previously achieved, and in particular in some cases upper and lower bounds on the non-classical distance can be established [16].

Other ways of quantifying non-classicality are absolute measures in that they depend only on inherent properties of the state under consideration, and avoid defining a set of classical states. Perhaps the simplest and the most well-known examples of this type of measure are the purity, $\mathrm{Tr}\rho^2$, or its close relative, the 2-entropy, $-\ln[\mathrm{Tr}\rho^2]$. For example, as mentioned, Zurek *et al* [32] studied decoherence dynamics by calculating the 2-entropy generated by decohering pure states, and offered further evidence that the coherent states are the most classical pure states. Marian and Marian [9] have also used the same measure to show that at low temperatures, as Fock states decay in the Markovian bath, they reach a point of maximal mixing, after which time the purity or 2-entropy increases again. This result is intriguing and troubling in that it is counterintuitive that the non-classicality of a simple object like an eigenstate should decay to a minimum, and then increase again under the effects of decoherence. Further, coherent states, which have been defined as the most classical states of all by many authors (see the discussion about relative measures), have unit (maximum) purity. Dodonov *et al* have used such results to argue against using purity as a measure of non-classicality [18]. But, as the 2-entropy has been previously well-studied, we do not consider it further in this paper.

Other absolute measures involve representing the uncharacterized state in phase-space and measuring some property of that representation. Paavola *et al*, Marian and Marian, and Dodonov *et al* have used several absolute measures to study the decoherence of cat states [6–8]. Both Paavola *et al* and Marian and Marian considered the time it takes for the Glauber–Sudarshan function to become fully positive. Paavola *et al* further studied the time at which the Wigner function became fully positive, the time it took for the interference fringe in the Wigner representation to disappear, the Vogel criterion (which measures properties of the characteristic function) and the Klyshko criterion (which measures properties of the P-representation). Paavola *et al* showed that, of these five measures, all but the interference fringe technique showed that the cat state became classical in a finite amount of time, and that those times are all on the same order of magnitude. As they have already shown that these four approaches give similar results, we shall consider only one of these measures in this paper. Dodonov *et al* also studied the negative volume of the Wigner function of cat states and presented analytical formulas when the Wigner functions of cat states become fully positive.

With this as the background, the four measures for which we present the results are as follows:

(1) The first, a measure discussed in detail by Hillery, is defined as the infimum of the trace norm of the difference between the quantum state and the set of all classical states ρ_{cl} [16]:

$$\eta_{\mathrm{H}}[\rho] = \inf_{\rho_{\mathrm{cl}}} \|\rho - \rho_{\mathrm{cl}}\|, \tag{3}$$

where $\|A\| = \mathrm{Tr}[\sqrt{AA^{\dagger}}]$ is the trace norm of a matrix. Hillery used this measure to prove that there will always be a finite distance between non-classical states and classical states. He also used it to study the harmonic oscillator eigenstates, and proved that η_{H} is bounded from above and below by

$$1 - \gamma_n \le \eta_{\mathrm{H}} \le 2\sqrt{1-\gamma_n}, \tag{4}$$

where $\gamma_n = \mathrm{e}^{-n} n^n/n! \approx (2\pi n)^{-1/2}$. This result simply rests on the existence of a classical basis, and does not require its construction. In §3, we use a partial classical basis to show that this non-classicality for harmonic oscillator eigenstates in fact increases monotonically and asymptotically to the maximum value predicted (which is 2). Though it is not immediately evident from eq. (3), the maximum value of η_{H} is in fact 2, meaning that number states of increasing n approach the maximal value of non-classicality.

(2) The second measure, the Bures distance, uses the Uhlmann fidelity, given by

$$B[\rho_1, \rho_2] = \mathrm{Tr}[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}],$$

which was shown to be the density operator generalization of $|\langle\psi_1|\psi_2\rangle|$ by Uhlmann [35]. The actual Bures distance is calculated using [17]

$$\eta_{\mathrm{B}} = \inf_{\rho_{\mathrm{cl}}} \sqrt{2(1 - |B[\rho, \rho_{\mathrm{cl}}]|)}. \tag{5}$$

This measure was used to analytically determine the non-classicality of squeezed, displaced thermal states [17], and later the non-classicality of two such states when entangled [34]. In §3, we show that this measure agrees well with the Hillery measure in the case of the number states. One trivial difference is that η_{B} increases asymptotically to $\sqrt{2}$ instead of 2, simply due to its construction. Equation (5) clearly shows that the maximum value η_{B} can attain is $\sqrt{2}$.

(3) The third measure (the Dodonov measure), is a measure of classicality, rather than non-classicality, as it computes how close, rather than far, a state is to the classical basis. It is presented in [18], and is a modified form of the overlap of density operators:

$$\eta_{\mathrm{D}}[\rho] = \sup_{\rho_{\mathrm{cl}}} \mathrm{Tr}[\rho' \rho'_{\mathrm{cl}}], \tag{6}$$

where $\rho' = \rho/\sqrt{\mathrm{Tr}[\rho^2]}$ is the density operator renormalized by its purity. The purpose of this renormalization is to make it easier to compare mixed states. Notice that for mixed states $\mathrm{Tr}[\rho'_1 \rho'_2] = 1$, if $\rho_1 = \rho_2$, whereas $\mathrm{Tr}[\rho_1 \rho_2] < 1$ in the same case. Thus, the renormalized overlap allows one to easily see how similar two density operators are, regardless of the degree of mixedness. Dodonov and Reno [18] used this measure to show that the classicality of the number states decreases monotonically and asymptotically to 0. Again, this means that increasing number states approach the maximal value of non-classicality. Specifically they showed that $\eta_{\mathrm{D}} = \gamma_n$, as defined above.

(4) The fourth measure we consider is from the family of absolute measures. It is the negative volume of the Wigner function, or negativity,

$$\eta_{\mathrm{W}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|W[x, p]| - W[x, p]) \mathrm{d}x \mathrm{d}p, \tag{7}$$

which is particularly appealing because it is easily calculated using numerical integration. It is also consistent with the intuition resulting from the study of quantum-classical correspondence in phase-space following Berry. Kenfack and Zyczkowski have already used η_{W} to study the harmonic oscillator eigenstates (for the closed, static problem). Their results agreed very well with those of Dodonov *et al* in that Kenfack and Zyczkowski found that η_{W} increases asymptotically like $\sqrt{n}$ [19].

We now proceed to quantify non-classicality using all these measures, starting with the previously well-studied (see e.g. [16,18]) base-line case, where there is no coupling with the environment (alternatively understood as the initial condition for decohering situation).

3. Results

3.1 *No environmental coupling*

As just discussed, the Hillery and Dodonov *et al* measures have been previously used to show that the non-classicality of Fock states increases with increasing n in the absence of any environmental coupling. Dodonov *et al* derived an analytical expression for this increase using the classical basis composed of the displaced thermal states. In studying the decohering problem we found a new basis (broadened microcanonical states, described below). When these states are included in the total classical basis, the value of η_{D} increases and the values of η_{H} and η_{B} decrease. Given that η_{D} seeks supremum over all classical states and $\eta_{\mathrm{H}}, \eta_{\mathrm{B}}$ seek the infimum over all classical states, calculations using the broadened microcanonical basis are found to be superior to calculations solely using bases of coherent states and the displaced thermal states.

We discovered this new basis through our own numerical experiments as well as from intuition drawn from previous results on the behaviour of initial number states [36]. What has been observed in these previous calculations is as follows: the Wigner functions for the number states start as Laguerre polynomials in phase-space, with multiple non-classical fringes. As the states evolve in the presence of the environment, they lose their initial fringes until they become somewhat broadened versions of the classical microcanonical states. These broadened microcanonical states are the 'first' permissible classical basis that arise during the decoherence of Fock states. These positive-definite distributions are sharply peaked in energy space on the appropriate classical energy, and are otherwise evenly distributed along the classical orbit. The classical microcanonical state at a given energy E is, of course, $\delta(E - H(p, q))$. These new states are not quite as singular as delta functions and have a slight width in energy. The classical microcanonical states violate the uncertainty principle of course, but these new states derived from quantum evolution naturally satisfy both the uncertainty principle as well as the requirements of a classical probability distribution. These 'thermally-broadened' microcanonical states are a natural choice as classical states which might be closest to the quantum eigenstates. In studying quantum-classical correspondence averaging over neighbouring states has been shown to be necessary [28–30], yielding states similar to these thermally-broadened microcanonical states. Irrespective of the intuition, as we show below, empirically these states work very well indeed.

We represent these states by

$$\rho^+ = \sum_{n=0}^{\infty} w_n \rho_n^+, \tag{8}$$

where $\sum_{n=0}^{\infty} w_n = 1$, $w_n \in [0, 1]$ and ρ_n^+ represents a number state that has evolved according to eq. (10) to a time $t_* = \gamma^{-1}\{\ln[(2N+2)/(2N+1)]\}$ [8], which guarantees that it is positive definite in the Wigner representation. This is an extremely large set and is impossible to explore completely. For our purposes, we found that the most useful subset of ρ^+ is

$$\rho_\nu^+ = (x+1-\nu)\rho_x^+ + (\nu - x)\rho_{x+1}^+, \tag{9}$$

where $x = \mathrm{Int}[\nu]$ is the truncated integer of ν, and ν varies continuously. For example, by this definition $\rho_{2.9}^+ = 0.1\rho_2^+ + 0.9\rho_3^+$.

In figure 1, we show the results of computing these measures of non-classicality for eigenstates of the harmonic oscillator using multiple measures, and using three different classical bases: the set of all coherent states ρ_α, the set of all thermal states ρ_{th} and the set ρ_ν^+. As ρ_ν^+ (the first positive definite set of states that arise from the decoherence) outperforms the other bases, we conclude that decoherence finds the closest classical states dynamically. Also, given that η_{W} determines the ρ_ν^+ states to be the first classical set of states as a result of decoherence, this gives us confidence in that absolute measure, which is enhanced by the general agreement of all four graphs in figure 1.

3.2 *Zero temperature*

The results of the previous section seem to violate the correspondence principle on the face of it, as they show higher n states as more non-classical in contradiction to the rule of thumb that higher number states should be less non-classical. This apparent paradox has in fact been well known for decades: that higher n states display more non-classicality – via rapidly oscillating fringes, for example – in violation of the correspondence principle intuition. It has been previously shown that the correct way to approach this issue is to average the Wigner function over a small energy spread or phase-space spread (usually evoked as a resolution limit). When such averaging is done for the Fock states, it appropriately smooths the fringes, or

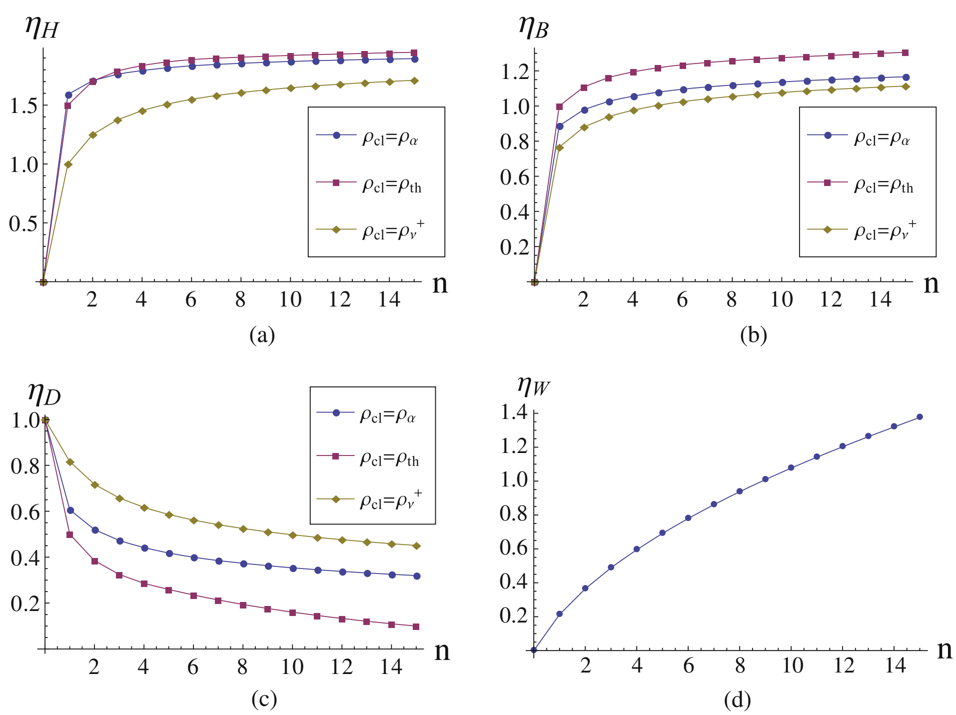

Figure 1. For the static (or closed/non-interacting) case, we show different measures of non-classicality for eigenstates of the harmonic oscillator. Specifically we show **(a)** Hillery distance, η_{H}, **(b)** Bures distance, η_{B} and **(c)** Dodonov overlap, η_{D}, each computed for three different classical bases and **(d)** negativity, η_{W}.

oscillations, of the number states [28,29]. The same smoothing effect occurs explicitly when dynamics are considered. As soon as environmental coupling is considered, the number states evolve over time into states resembling the microcanonical states (in this case the 'averaging' is due to the environment). Higher n states are much more sensitive to decoherence than lower n states, which means smoothing occurs much more rapidly for higher n states, and yields the appropriate correspondence principle intuition.

Another way of understanding this idea, that the correspondence principle requires averaging (either mathematical or environmental), starts from the notion that the behaviour of closed systems is singular and non-physical, in that, states in such systems evolve completely independent of their environment; they do not even interact with the vacuum. This behaviour does not survive when even the smallest amount of interaction with the environment is introduced.

For a harmonic oscillator coupled to a linear Markovian bath, assuming that the bath is composed of a continuum of oscillators, neglecting the back-action of the system on the bath, and making the Markovian approximation for the interaction, the time evolution of the density matrix ρ is governed in the interaction picture by the master equation [27]

$$\dot{\rho} = \frac{\gamma}{2}\left(NL[a^{\dagger}]\rho + (N+1)L[a]\rho\right). \tag{10}$$

Here, the dot represents the time derivative, the Lindblad superoperator is defined as $L[O]\rho \equiv 2O\rho O^{\dagger} - O^{\dagger}O\rho - \rho O^{\dagger}O$, $a^{\dagger}$ and a are the raising and lowering operators for the harmonic oscillator, respectively, γ represents the degree of coupling of the oscillator to its environment and N corresponds to the mean number of thermal photons in the bath. Solutions to this equation are well-known; we have used an analytical solution in the number basis, which is a variation of that from ref. [12], which allows for convenient and computationally fast calcuations of the non-classicality measures as a function of time.

A zero-temperature bath (that is, with $N = 0$) changes the non-classicality of the states markedly from that displayed in figure 1. We use the expanded classical basis outlined above, that is, adding the broadened microcanonical states ρ^{+} to the thermal states ρ_{th} and the coherent states ρ_{α}. With this as the set of classical states, each of the three relative measures yields non-trivial behaviour (see figure 2), as does the absolute measure η_{W}. That is, all the non-classicality curves (shown for different finite γt) show peaks that asymptotically decrease to 0 with increasing n (although this is harder to see for the Hillery and Bures measures). The thermally broadened microcanonical states basis ρ^{+} is critical in detecting this feature. Without it (i.e., using only the displaced thermal states and the coherent states), the various infima and suprema are incompletely found, and the relative measures for the

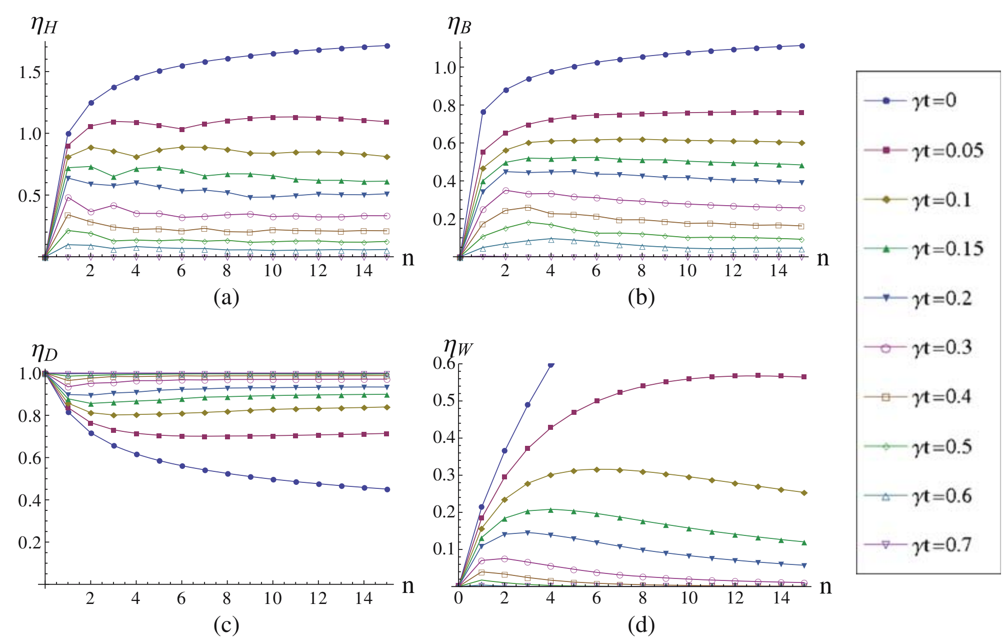

Figure 2. All four plots show the evolution of non-classicality for the zero-temperature case. Each curve represents a different snapshot in time of non-classicality plotted as a function of initial eigenstate n. We show this for **(a)** η_{H}, the Hillery distance, **(b)** η_{B}, the Bures distance, **(c)** η_{D}, the Dodonov overlap and **(d)** η_{W}, the negativity.

non-classicality of the decohering eigenstates show a mistaken monotonic increase as a function of n for all γt. This underscores the trouble (even danger) with using relative measures to study dynamics: without a sufficiently large basis, in particular without the thermally broadened microcanonical states in the mix, all three give erroneous results. For these three measures, reasonable results were obtained only through intuition gained by considering the problem in phase-space.

Each of the measures shows the same general result but differs somewhat on the details. According to (c) η_{D}, the Dodonov overlap and (d) η_{W}, the negativity, this non-monotonicity is relatively straightforward. Each state decays smoothly, and the curves for each state cross that of any other state only once, so that the non-classicality peak decreases in n monotonically. However, the (a) Hillery and (b) Bures distances tell a more complicated story. As can be seen in figure 3a, individual eigenstates do not decay smoothly as a function of time according to the Hillery distance; specifically the 4th and 7th excited states exhibit multiple corners, giving rise to multiple crossovers. This naturally leads to the more complex behaviour seen in figure 2a. The time curves in this case display not only a maximum value, but also local minima.

The Bures distance shows individual eigenstates decaying more smoothly than the Hillery distance, though figure 2b does show occasional local minima. The more unusual signature present in this measure is that many of the eigenstates never cross the decay curve of the 1st eigenstate, as seen in figure 3b. For this reason, the non-classicality peak never reaches $n = 1$. The lowest value it reaches is $n = 2$, after which it begins moving up again.

These unexpected features can be explained by the fact that we used a classical basis that is far from complete. The Hilbert space relative measures rely on searching over the entire set of classical states, which is practically impossible. Unfortunately, this inability to represent the complete classical basis can lead to incorrect results. Indeed, as we discussed above, we get the wrong monotonic behaviour when using the coherent or thermal states as reference states. It is reasonable to assume that the non-classicality peak would come to rest at $n = 1$, according to the Bures distance as well if a more complete basis were used. Similarly, it is arguable that the Hillery distance would decay smoothly for all harmonic oscillator eigenstates if a more complete basis was found. In particular, it is possible that if the basis described in eq. (8) were fully enumerated, i.e., if we used an infinite sum over the thermally-broadened microcanonical states, instead of the subset defined in eq. (9) – these artifacts might well disappear.

Independent of these speculative arguments about these unusual features, all four measures show qualitatively similar behaviour. That is, in all cases, an

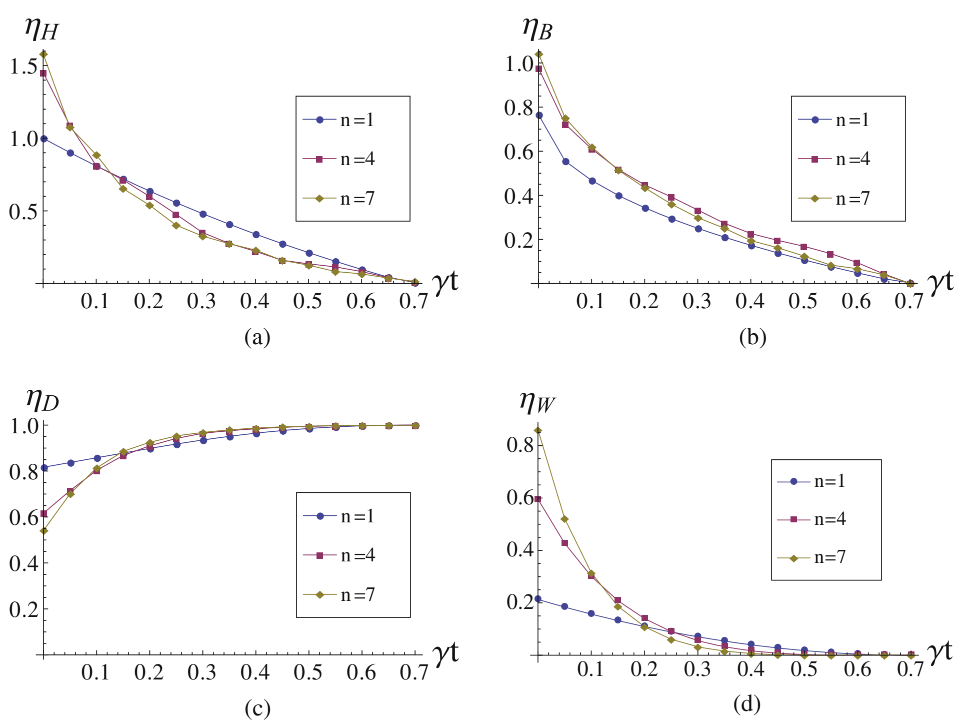

Figure 3. The decrease in non-classicality as a function of time for the 1st, 4th and 7th Fock states as measured by **(a)** η_{H}, the Hillery distance, **(b)** η_{B}, the Bures distance, **(c)** η_{D}, the Dodonov overlap and **(d)** η_{W}, the negativity.

initially ($\gamma t = 0$) monotonic dependence on quantum number transforms into a non-monotonic behaviour in the presence of decoherence.

Before we consider more complicated situations, we comment here on the relative computational difficulty of the different measures. We have just demonstrated that at zero temperature the absolute measure η_W shows good agreement with the three relative measures considered. However, calculating the relative measures is formidably challenging in general. Specifically, we have to search for the best classical basis for each temperature and for each initial condition, and having found this basis, minimize the measure at each time-step. Further, note that the zero temperature problem is relatively computationally easier, because it yields finite density matrices in the number representation. Non-zero temperature leads to infinite density matrices in number space, and makes calculating non-classicality more and more difficult as temperature increases (we need more terms in the basis expansion for a convergent calculation as the temperature increases). In contrast, the absolute measure η_W requires only one numerical integral to be calculated at every time step. Hence we chose to use solely the negativity η_W as our measure of non-classicality for the rest of our calculations. This choice is consistent with two previous studies of dynamics [7,8] of non-classicality measures which also solely used phase-space measures. We also reiterate that as the Wigner function has now been directly measured [20–23], arguably η_W is the truly experimentally accessible non-classicality measure, making this is a sensible restriction.

3.3 *Non-zero temperature*

The effect of a finite-temperature bath is incorporated by using a non-zero number of thermal photons ($N \neq 0$) in the evolution equations. In figure 4a, we show negativity vs. Fock state for different values of γt at finite temperature ($N = 0.06$). As for the zero-temperature case, at $\gamma t = 0$, the negativity increases monotonically in n, but that at any non-zero time there is a peak in negativity. This is reinforced by figure 4b, which shows the evolution of negativity of the lowest 16 harmonic oscillator eigenstates until the time $\gamma t = 0.15$, with each curve representing different bath temperatures. As we have already seen in figure 2d, even the $N = 0$ curve has a peak. Aside from this striking difference (at $\gamma t = 0$), temperature has a somewhat similar effect on non-classicality as scaled

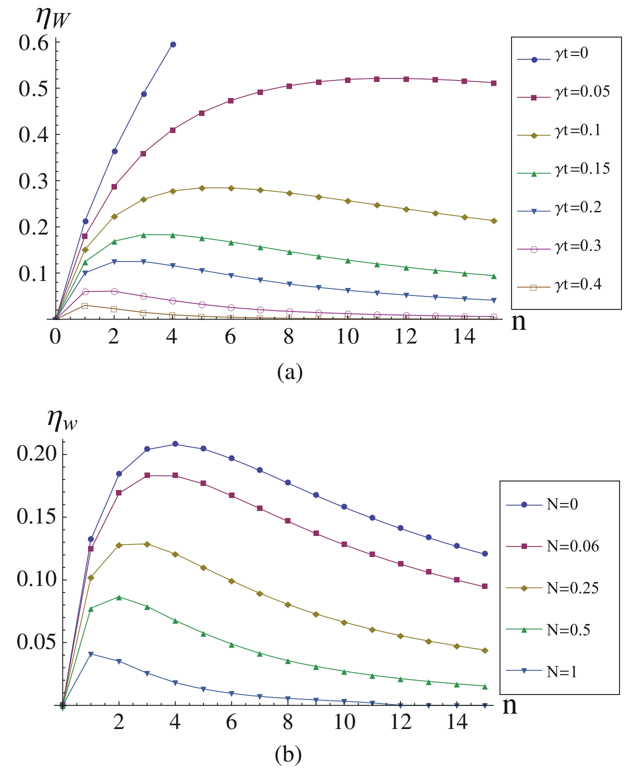

Figure 4. **(a)** Shows the time dependence (at constant bath temperature) of negativity of the Wigner function for Fock states evolving in a bath of mean photon number $N = 0.06$. Each curve is a snapshot at different values of time of negativity as a function of eigennumber n. **(b)** Shows the temperature dependence (at constant time) of negativity of the Wigner function as a function of n. Each curve is a snapshot at different values of bath temperature, of the negativity as a function of eigennumber n, all taken at the same unitless time $\gamma t = 0.15$.

time γt, in that, higher temperature corresponds to greater decoherence, just as greater values of γt correspond to greater decoherence.

4. Conclusions

In summary, we have studied the behaviour of non-classicality in the decohering harmonic oscillator using four different measures of non-classicality (three relative measures: η_H, η_B, η_D and one absolute measure η_W). We have used as initial conditions, the number (Fock) states.

A new result in the static ($t = 0$) case was the introduction of a new classical basis (a broadened microcanonical basis), which improves quantitatively on previous choices of bases for the three relative measures. This basis arises as the first positive-definite set of states into which the decohering Fock states evolve. That this set of states is measurably closer (using three different measures) to the undecohered Fock states than all other states considered previously, shows that decoherence arguably takes the shortest path from quantum to classical states, and also implicitly confirms the validity of using the negativity – η_W – as an absolute measure of non-classicality. We then studied the dynamics of Fock states in the zero-temperature case when the harmonic oscillator is coupled to a Markovian bath using all four measures for non-classicality. We believe that this is the first study of the behaviour of η_H, η_B and η_D in a decohering system. We find that this system displays a non-monotonic transition as a function of n. This is in keeping with the fact that higher n quantum states, while intrinsically more complicated than lower n states, are also far more sensitive to environmental effects and quickly become nearly classical even though all states transition to true classicality at the same time t^*. This is fundamentally a validation of the correspondence principle.

To return to our initial motivating questions, we find that (a) η_W, an absolute measure of non-classicality, agrees well with the relative measures we have studied in the zero-temperature case, where these relative measures are computationally tractable. Unlike the relative measures, η_W can be calculated in more complicated situations and can be used as an experimental measure of non-classicality. We have also used η_W to study the dynamics of number states decaying in a non-zero temperature bath where we found more general but substantially similar results and (b) the study of number states have a similar complicated landscape of non-classicality while decohering as shown by the cat states. Our work thus builds on and agrees with previous studies of the dynamics of non-classicality [6–9] and reinforces the perspective that there is much non-trivial behaviour to be uncovered via such analyses.

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