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# On phase-space representations of quantum mechanics using Glauber coherent states

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**Abstract.** A phase-space formulation of quantum mechanics is proposed by constructing two representations (identified as *pq* and *qp*) in terms of the Glauber coherent states, in which phase-space wave functions (probability amplitudes) play the central role, and position $q$ and momentum $p$ are treated on equal footing. After finding some basic properties of the *pq* and *qp* wave functions, the quantum operators in phase-space are represented by differential operators, and the Schrödinger equation is formulated in both pictures. Afterwards, the method is generalized to work with the density operator by converting the quantum Liouville equation into *pq* and *qp* equations of motion for two-point functions in phase-space. A coordinate transformation between those points allows one to construct a cell in phase-space, whose central point can be treated as a parameter. In this way, one gets equations of motion describing the evolution of one-point functions in phase-space. Finally, it is shown that some quantities obtained in this paper are related in a natural way with cross-Wigner functions, which are constructed with either the position or the momentum wave functions.

**Keywords.** Phase-space quantum mechanics; coherent states; Husimi function; Wigner function.

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## 1. Introduction

Quantum mechanics is a fundamental theory of physics that supports different formulations [1]: Schrödinger, Heisenberg, Feynman, .... Each formulation has its own strengths and weaknesses in the sense that it facilitates the understanding of some aspects of the theory or the application in specific fields. In this article, the focus is on the phase-space formulation of quantum mechanics, which started with the early contributions of Weyl (1927), Wigner (1932), Groenewold (1946) and Moyal (1949). As position and momentum variables, $q$ and $p$, are placed on equivalent footing, similarly to what happens in Hamilton mechanics, the phase-space formulation is a natural bridge for comparing the quantum and classical descriptions, improving the knowledge about quantum systems and understanding the corresponding classical limit.

In this paper, a quantum system with $f$ degrees of freedom is considered. The position and momentum operators in the Hilbert space, denoted by $\hat{q}=\{\hat{q}_1, \hat{q}_2, \dots, \hat{q}_f\}$ and $\hat{p}=\{\hat{p}_1, \hat{p}_2, \dots, \hat{p}_f\}$, satisfy the canonical commutation relations $[\hat{q}_n\, , \hat{p}_m] = i\hbar\, \hat{1}\, \delta_{nm}$, where $\hat{1}$ is the identity operator, $\hbar$ is the reduced Planck constant, $\delta_{nm}$ is the delta of Kronecker, and $n, m$=1, 2, ..., $f$. A multi-index notation will also be used: $n := (n_1, n_2, \dots, n_f)$ is an ordered set of $f$ non-negative integers such that $|n| = n_1+n_2+\dots+n_f$, $n! := n_1!n_2!\dots n_f!$, $z := (z_1, z_2, \dots, n_f)$ is a set of $f$ complex numbers and $z^n := z_1^{n_1} z_2^{n_2} \dots z_f^{n_f}$. The dimensionless symbols $\hat{a}=\kappa_0\hat{q}+i\chi_0\hat{p}$ and $\hat{a}^{+}=\kappa_0\hat{q}-i\chi_0\hat{p}$ denote annihilation and creation operators, respectively, where $\kappa_0 := 1/(q_0\sqrt{2})$ and $\chi_0 := 1/(p_0\sqrt{2})$ are related with the units of length ($q_0$) and momentum ($p_0$), which are restricted by the condition $q_0 p_0 = \hbar$. Note that there is only one independent parameter, either $q_0$ or $p_0$.

The non-Hermitian operator $\hat{a}$ satisfies the eigenvalue equation $\hat{a}\,|z\rangle = z\,|z\rangle$, for each complex number $z=\kappa_0 q + i\chi_0 p$, and the normalized eigenket $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$ is the so-called Glauber coherent state [2]. In terms of these states, a phase-space $\mathbb{C}^f = \{z =$

$\kappa_0 q + i\chi_0 p, i = \sqrt{-1}\} \cong \mathbb{R}^{2f} = \{(q, p)\}$ is associated with the quantum system, the resolution of the identity operator takes the form $\int \mathrm{d}\mu(z)\,|z\rangle\langle z| = \hat{1}$, with a measure $\mathrm{d}\mu(z) = \pi^{-f}\,\mathrm{d}^{2f}z = (2\pi\hbar)^{-f}\mathrm{d}q\mathrm{d}p := \mathrm{d}\Gamma_{qp}$, where $\mathrm{d}q = \mathrm{d}q_1\mathrm{d}q_2\ldots\mathrm{d}q_f$ and $\mathrm{d}p = \mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_f$, and for each one-dimensional variable the integration is over the reals, i.e., between $-\infty$ and $\infty$.

Regarding phase-space formulations of quantum mechanics, the literature may be classified into two general areas, which are briefly described in the following subsections.

### 1.1 *Phase-space formulation à la Wigner*

This research area is guided by the idea of getting a quantum formulation in close analogy with the structure of classical statistical mechanics in which (see for example [3–8] and references therein) (i) to the quantum mechanical state at time $t$, described either by the ket$|\Psi(t)\rangle$ or the density operator $\hat{\rho}(t)$, corresponds a phase-space function of the real-valued position and momentum variables, $q = (q_1, q_2, \ldots, q_f)$ and $p = (p_1, p_2, \ldots, p_f)$, (ii) observables are represented by functions in the phase-space, and (iii) equations of the Liouville-type give account of the quantum dynamics. The prototype of this approach is the Wigner function, $\rho_{\mathrm{W}}(q, p, t)$, which can be obtained from the momentum and position wave functions, $\Psi(q, t)$ and $\tilde{\Psi}(p, t)$, by the transformations

$$\begin{aligned}\rho_{\mathrm{W}}(q', p', t) &= \int \exp\left(\frac{i}{\hbar}q'p\right)\tilde{\Psi}^{\star}\left(p' - \frac{1}{2}p, t\right) \\ &\quad\times \tilde{\Psi}\left(p' + \frac{1}{2}q, t\right)\mathrm{d}p \\ &= \int \exp\left(-\frac{i}{\hbar}qp'\right)\Psi^{\star}\left(q' - \frac{1}{2}q, t\right) \\ &\quad\times \Psi\left(q' + \frac{1}{2}q, t\right)\mathrm{d}q, \end{aligned} \tag{1}$$

where $\star$ means complex conjugate. In general, the Wigner function is always real, but fails to act as a probability distribution because it may take negative values, although it accurately reproduces the marginal distributions of position and momentum:

$$\begin{aligned}&\int \rho_{\mathrm{W}}(q, p, t)\mathrm{d}p = |\Psi(q, t)|^2, \\ &\int \rho_{\mathrm{W}}(q, p, t)\mathrm{d}q = |\tilde{\Psi}(p, t)|^2. \end{aligned} \tag{2}$$

An alternative to $\rho_{\mathrm{W}}$ is the Husimi function [9] or coherent state representation, which is equal to a Gaussian smoothed Wigner function, e.g. [10],

$$\begin{aligned}&\rho_{\mathrm{H}}(q, p, t) := |\langle z\,|\Psi(t)\rangle|^2 \\ &= (2\pi\hbar)^f \int \rho_{\mathrm{W}}(q', p', t)[M(q'-q)\tilde{M}(p'-p)]^2\mathrm{d}q'\mathrm{d}p'. \end{aligned} \tag{3}$$

For getting (3), one needs the relation $\exp(-zz^{\star}) = (\pi\hbar)^{f/2}\,\tilde{M}(p)M(q)$, where

$$\begin{aligned}M(q) &:= (2\pi\hbar)^{-f/2}\int w(q, p)\tilde{M}(p)\mathrm{d}p \\ &= \left(q_0\sqrt{\pi}\right)^{-f/2}\exp(-(\kappa_0 q)^2) \end{aligned} \tag{4}$$

and

$$\begin{aligned}\tilde{M}(p) &:= (2\pi\hbar)^{-f/2}\int w^{\star}(q, p)M(q)\mathrm{d}q \\ &= \left(p_0\sqrt{\pi}\right)^{-f/2}\exp(-(\chi_0 p)^2) \end{aligned} \tag{5}$$

are Gaussian functions ([11], eqs (54a) and (54b)), which are related to each other by a Fourier transform pair on $\mathbb{R}^f$. In (3), as the Wigner function may display large oscillations and adopt negative values in some regions of phase-space, the Gaussian smoothing function $[\cdots]^2$ plays the role of a 'coarse-grained' description in which the finest details of the Wigner function are either smoothed over or averaged out, resulting in a non-negative probability distribution $\rho_{\mathrm{H}}\left(q', p', t\right)$ (see e.g. [10]).

### 1.2 *Phase-space formulation à la Bargmann*

A more recent line of development for the formulation of quantum mechanics on the phase-space has emerged in the last decades, starting with the work of Bargmann [12] in 1961. This research is guided by the idea of representing the quantum states $|\Psi(t)\rangle$ by phase-space wave functions, in analogy with the Schrödinger's approach used in constructing the position and momentum representations, see e.g. [11,13–22]. It is postulated that there are phase-space representations in which: (i) the state of the system $|\Psi(t)\rangle$ is described by a complex phase-space amplitude or wave function $\Psi(q, p, t)$, (ii) the position and momentum operators $\hat{q}$ and $\hat{p}$ are described by differential operators $\breve{Q}$ and $\breve{P}$ of the phase-space coordinates, $q$ and $p$, (iii) these operators preserve the Heisenberg commutation relations, and (iv) the Schrödinger or the quantum Liouville equations can be expressed in terms of phase-space variables. As a matter of notation, the circumflex ($\breve{}$) denotes operators acting on functions defined in the phase-space.

Instead of starting with the formulation based on the Wigner function $\rho_W(q, p, t)$, the approach presented in this paper constructs phase-space wave functions using the Glauber coherent states [2,23–25] and the Husimi amplitude, $\langle z | \Psi(t) \rangle$. Thus, in order to put into context the present paper, some previous literature in this area of research (*à la* Bargmann) will now be briefly reviewed.

(1) Bargmann [12] introduces the analytical representation of quantum states, in which wave functions $\Psi(q, t)$ are transformed from the position Hilbert space $\mathbb{L}^2\left(\mathbb{R}^f\right)$ onto a Hilbert subspace $F$ (the so-called Bargmann or Fock–Bargmann space) of the entire holomorphic functions on $\mathbb{C}^f$ that are square integrable with respect to the Gaussian measure $\mathrm{d}\mu_g(z) = \pi^{-f}\exp(-zz^\star)\mathrm{d}^{2f}z$. A quantum state $\Psi(z^\star) = \langle z | \Psi \rangle = \exp(-\frac{1}{2}zz^\star)$ $g(z^\star)$ is represented in $F$ by an analytical entire function $g(z^\star)$ of the complex variable $z^\star = \kappa_0 q - i\chi_0 p$, and creation and annihilation operators play the role of multiplication and differentiation with respect to $z^\star$, respectively (see e.g. [26]): $\langle z|\hat{a}^+|\Psi\rangle = z^\star\,\Psi(z^\star, t)$ and $\langle z|\hat{a}|\Psi\rangle = (\partial/\partial z^\star)\,\Psi(z^\star, t)$.

(2) Torres-Vega and Frederick [13,14] postulate the existence of a Hermitian phase-space operator $\hat{\Gamma}$ of unknown identity such that $\hat{\Gamma}\,|\Gamma\rangle = \Gamma\,|\Gamma\rangle$, and choose the orthonormal eigenvectors $|\Gamma\rangle = |p, q\rangle$ as the basis of the representation. This approach is based on a complex-valued wave function $\Psi(\Gamma) := \langle\Gamma|\Psi(t)\rangle$, and the operators of position and momentum [14] are chosen in the form $\breve{Q}\Psi(q, p) = (\alpha q + i\hbar\beta\partial/\partial p)\Psi(q, p)$ and $\breve{P}\Psi(q, p) = (\gamma p + i\hbar\delta\partial/\partial q)\Psi(q, p)$, where $\alpha, \beta, \gamma$ and $\delta$ are some real parameters such that $\beta\gamma - \alpha\delta = 1$. The application of a certain criterion leads to the canonical values $\alpha = \gamma = 1/2$ and $\beta = -\delta = 1$.

(3) Harriman [15] connects with the works of Torres-Vega and Frederick [13,14] and studies a linear transformation between the phase-space and the position (momentum) space that is implemented via integral transforms, in which the scalar product $\langle\phi\,|\Psi(t)\rangle$ between the state $|\Psi(t)\rangle$ and a Gaussian wave packet (or coherent state) $|\phi(t)\rangle$ is used. He finds that the integral kernels for the maps from the position and momentum representations to the phase-space representation must be a constant $C = 1/(\beta\sqrt{2\pi})$ times a normalized Gaussian wave packet. The parameter $\beta$ defines an overall scaling of the phase-space variables $q$ and $p$, and for $\beta = 1/\sqrt{2}$, a phase-space representation equivalent to Bargmann's formulation is obtained. In 1997, Møller *et al* [16] argued that the state-vector representation formulated in [14] and analysed in [15] comprises the class of all coherent state representations for the Heisenberg–Weyl group.

(4) Wlodarz [17] introduces a wave function $\Psi(p, q, t)$ by decomposing the Wigner distribution function in the form $\rho_W(p, q, t) = (\Psi \star \Psi)(p, q, t)$, where $\Psi(p, q, t)$ belongs to the space $\mathbb{L}^2\left(\mathbb{R}^2\right)$ of square-integrable phase-space functions, and $(a \star b)(p, q)$ is the star product of phase-space functions. He finds that the phase-space Schrödinger equation is equivalent to the Liouville equation for the Wigner distribution, and that $\Psi(p, q, t)$ describes the state of a quantum system, while at the same time $|\Psi(p, q, t)|^2$ seems to describe its classical limit ($\hbar \to 0$). Wlodarz's approach coincides with [14], when neglecting operator ordering problems.

(5) The relative-state formulation proposed by Ban [19] is obtained by enlarging the Hilbert space ($\mathcal{H}$) of a quantum system with the Hilbert space ($\mathcal{H}_r$) of an auxiliary reference quantum system, i.e., $\tilde{\mathcal{H}} = \mathcal{H}\otimes\mathcal{H}_r$. He follows [14] and concludes that under certain conditions, the relative-state formulation is equivalent to those obtained by Torres-Vega and Frederick.

(6) Because the ordinary coherent states in Hilbert space cannot satisfy the two requirements of the phase-space representations proposed in [14] and [15], Smith [21] follows Ban's formalism and in the augmented space $\tilde{\mathcal{H}}$ builds a complete (but not overcomplete) orthonormal set of state vectors $|\omega(r, k; s)\rangle$, which is overcomplete in $\mathcal{H}$; here $(r, k)$ are points in phase-space and $s$ is a real parameter. In this formalism, $\Psi_\sigma(p, q; t)$ is the Weyl symbol of the projector $|\Psi\rangle\langle\sigma|$, where $|\sigma\rangle$ is any fixed state and $\breve{Q}_\pm = q \pm i(\hbar/2)\partial/\partial p$ and $\breve{P}_\pm = \pm p - i(\hbar/2)\partial/\partial q$ represent the phase-space position and momentum operators ([21], see eqs (41) and (42)). These pairs of operators $(\breve{Q}_+, \breve{P}_+)$ and $(\breve{Q}_-, \breve{P}_-)$ act on functions in the phase-spaces associated with $\mathcal{H}$ and $\mathcal{H}_r$, respectively.

(7) Bracken and Watson [22] address several questions concerning the implications of the projector $|\Psi\rangle\langle\varphi_0|$ introduced by Smith [21] to extend the phase-space formalism. They use the Weyl–Wigner transform $\mathcal{W}(\cdots)$ to define phase-space

wave functions $\Psi(q,p,t) := (2\pi)^{-1/2}\,\mathcal{W}(|\Psi\rangle\langle\varphi_0|)(q,p,t)$, where the arbitrary normalizable state $\langle\varphi_0|$ plays the role of a 'window vector'. In the coordinate representation, these functions can be written as

$$\Psi(q,p,t) = \frac{1}{\sqrt{2\pi}}\int \varphi_0^{\star}\left(q+\frac{1}{2}q'\right) \times \Psi\left(q-\frac{1}{2}q',t\right)\mathrm{d}q', \qquad (6)$$

a result that is closely related to Gabor's 'windowed Fourier transform' of the function $\Psi(q,t)$ with respect to the window $\varphi_0(q)$. When a Gaussian function $\varphi_0(x) = \left(\beta^2/\pi\right)^{1/4}\exp\left(\beta^2\lambda(\lambda-\bar{\lambda})/4\right)\exp\left(-\beta^2(x-\lambda)^2/2\right)$ is used as a window, a generalized Bargmann representation is obtained in which, after defining $z = \sqrt{2}\,(\beta q - ip/\beta)$, the position and momentum operators are described by $\breve{Q} = (z+\partial/\partial z)/(\beta\sqrt{2})$ and $\breve{P} = i(\beta/\sqrt{2})(z-\partial/\partial z) - i\beta^2\,\bar{\lambda}$.

### 1.3 *Purpose of this work*

As noted previously, there are important contributions related to the formulation of quantum mechanics using phase-space wave functions. In this work, as the vector $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$ is a continuous function of the label $z$, or equivalently of $q$ and $p$, one can associate to the state $|\Psi(t)\rangle$, the joint position–momentum wave functions

$$\Psi(\theta|q,p,t) := w\left((2\theta-1)\frac{1}{2}q,p\right)\langle\kappa_0 q + i\chi_0 p|\Psi(t)\rangle = \begin{cases}\Psi_-(q,p,t) & \text{if } \theta=0\\ \Psi_+(q,p,t) & \text{if } \theta=1,\end{cases} \qquad (7)$$

defined on the phase-space $\Omega = \{(q,p)\} = \mathbb{R}^{2f}$, and using the auxiliary quantity

$$w(q,p) := \exp\left(\frac{i}{\hbar}qp\right). \qquad (8)$$

With this choice: (i) Given a coherent state $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$, the introduction of a phase factor $w(\pm q/2,p)$ provides a means for handling the position and momentum operators $\hat{q}$ and $\hat{p}$ on equal footing. (ii) Two phase-space wave functions can be associated with the state $|\Psi(t)\rangle$, which are linked to each other by the relation $\Psi_+(q,p,t) = w(q,p)\Psi_-(q,p,t)$, and when the squared moduli of the functions $\Psi_\pm(q,p,t)$ are calculated, one gets the probability of finding the system in a particular Glauber coherent state $|z\rangle$, i.e., the Husimi function [9]. (iii) As summarized in table 1, the phase-space representation of the position and momentum operators, $\hat{q}$ and $\hat{p}$, does not involve the use of parameters, unlike the approaches given in [14,15,22].

**Table 1.** Representatives of the state $|\Psi(t)\rangle$, and the position and momentum operators ($\hat{q}$ and $\hat{p}$) in the *pq*- and *qp*-representations.

| Representation | Operator $\hat{q}$ | Operator $\hat{p}$ | State $\lvert\Psi(t)\rangle$ | $\theta$ |
|---|---|---|---|---|
| *pq* | $i\hbar\,\partial/\partial p$ | $p - i\hbar\,\partial/\partial q$ | $\Psi_-(q,p,t)$ | 0 |
| *qp* | $q + i\hbar\,\partial/\partial p$ | $-i\hbar\,\partial/\partial q$ | $\Psi_+(q,p,t)$ | 1 |

This work is arranged as follows: In §2, the basis of the proposed method is introduced, which includes definitions of coherent states *pq* and *qp*, and the imposition of a symmetry condition between the variables $q$ and $p$. In §3, it is shown that phase-space wave functions $\Psi_\pm(q,p,t)$, defined in (7), offer a suitable framework for the formulation of quantum mechanics, in particular, because they allow the calculation of position and momentum wave functions along with their derivatives, $\Psi^{(\eta)}(q,t)$ and $\tilde{\Psi}^{(\eta)}(p,t)$. In §4, functions of the quantum position and momentum operators are converted to the language of the phase-space, and in §5, the Schrödinger and Liouville equations are reformulated in that picture. In §6, the relation between the treatment used in this paper and the method of Wigner for the formulation of quantum mechanics in phase-space is considered. Finally, in §7, summary and conclusions are given.

## 2. Basis of the proposed method

This section deals with the phase-space wave functions $\Psi_\pm(q,p,t)$ defined by (7), whose properties are given in §2 and 3. The wave functions $\Psi_-(q,p,t)$ and $\Psi_+(q,p,t) = w(q,p)\Psi_-(q,p,t)$ only differ in the phase factor $w(q,p)$, a fact that seems to contradict the belief that the overall phase of the wave function plays no role in the probabilistic predictions of quantum mechanics. Naturally, the phase factors $w(\pm\frac{1}{2}q,p)$ do not change the expectation values of Hermitian operators or the Husimi function (3), but introducing them is an original approach to ensure an overall consistency between the phase-space, position and momentum representations of quantum mechanics.

With regard to relations (7), the motivations to incorporate the phase factors $w(\pm\frac{1}{2}q,p)$ are explained in the following two subsections.

### 2.1 *Definitions of pq and qp coherent states*

According to Glauber [2,23], each complex number $z = \kappa_0 q + i\chi_0 p$ corresponds a normalized coherent state $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$ given by

$$|z\rangle = \hat{D}(q,p)\,|0\rangle = \left[(\pi\hbar)^{f/2}\,\tilde{M}(p)M(q)\right]^{1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle\,, \qquad (9)$$

where $|n\rangle$ are the eigenkets of the number operator $\hat{N} = \hat{a}^+\hat{a}$, and

$$\hat{D}(q,p) = \hat{D}(z) := \exp\left(\frac{i}{\hbar}\left[p\hat{q} - q\hat{p}\right]\right) = w^\star\left(\frac{1}{2}q,p\right)\hat{D}(0,p)\hat{D}(q,0) = w\left(\frac{1}{2}q,p\right)\hat{D}(q,0)\hat{D}(0,p) \qquad (10)$$

is the Weyl operator [27]. The notation emphasizes that $\hat{D}(q,p)$ depends parametrically on the points $(q,p)$ in a $2f$-dimensional real phase-space ($\Omega = \mathbb{R}^{2f}$) whose elements are given by all the possible position and momentum expectation values:

$$q := \langle z|\hat{q}|z\rangle = \frac{q_0}{\sqrt{2}}\left(z + z^\star\right),$$
$$p := \langle z|\hat{p}|z\rangle = -i\frac{p_0}{\sqrt{2}}\left(z - z^\star\right). \qquad (11)$$

Now, in quantum mechanics, a well-behaved function $F(\hat{a},\hat{a}^+)$ of the operators $\hat{a}$ and $\hat{a}^+$ can be written in many equivalent ways, for example, if $F(\hat{a},\hat{a}^+)$ admits a convergent power-series expansion, the function is normal (antinormal) ordered if all the operators $\hat{a}^+$ stand always at the left (right) of the operators $\hat{a}$. However, due to the properties of the Weyl operator, one may also use the standard (antistandard) ordering for functions of the $\hat{q}$ and $\hat{p}$ operators [28–30]: all powers of the operator $\hat{q}$ ($\hat{p}$) precede all powers of the operator $\hat{p}$ ($\hat{q}$). Thus, one can define *pq* and *qp* coherent states by the relations

$$|\theta,q,\ p\rangle = \begin{cases} |p,q\rangle := \hat{D}(0,p)\hat{D}(q,0)\,|0\rangle\,, & \text{if } \theta = 0 \\ |q,p\rangle = \hat{D}(q,0)\hat{D}(0,p)\,|0\rangle\,, & \text{if } \theta = 1. \end{cases} \qquad (12)$$

These states differ from the Glauber coherent state $|z\rangle$, and each other, by a phase factor, in particular $|p,q\rangle = w\,(q,p)\,|q,p\rangle$. From (12), one also concludes that the phase factors $w(\pm\frac{1}{2}q,p)$ are a consequence of the properties of the Weyl operator. In fact, starting with the Glauber coherent state $|z\rangle = \hat{D}(q,p)\,|0\rangle$ then, as the $\hat{q}$ and $\hat{p}$ operators do not commute, moving the operator $\exp\bigl(-iq\hat{p}/\hbar\bigr)$ to the right (+) or to the left (−) of the operator $\exp\bigl(ip\hat{q}/\hbar\bigr)$ requires a compensation by means of the phase factor $w(\pm\frac{1}{2}q,p)$.

### 2.2 *Implementation of the $q \leftrightarrow p$ symmetry condition*

From a purely conceptual point of view, one should treat the phase-space position and momentum $q$ and $p$ on equal footing (briefly, $q \leftrightarrow p$ symmetry condition), except for the implications due to the commutation relations $[\hat{q}_n,\hat{p}_m] = i\hbar\,\hat{1}\,\delta_{nm}$, i.e., $i \to -i$, because $[\hat{p}_m,\hat{q}_n] = -i\hbar\,\hat{1}\,\delta_{nm}$. To account for this fact, the coordinate ($q'$) and momentum ($p'$) representations of the Glauber coherent state $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$ are taken as

$$\varphi_z(q') := \langle q'|z\rangle = M(q'-q)w(q',p)\exp(i\mu/\hbar),$$
$$\tilde{\varphi}_z(p') := \langle p'|z\rangle = \tilde{M}(p'-p)w^\star(q,p'-p) \times \exp(i\mu/\hbar), \qquad (13)$$

where the arbitrary phase factor $\exp(i\mu/\hbar)$ is chosen so that $\exp(i\mu/\hbar) = w^\star(q,p/2)$, i.e.,

$$w(q',p)\exp(i\mu/\hbar) = w\bigl(q'-q/2,p\bigr)$$

and

$$w^\star(q,p'-p)\exp(i\mu/\hbar) = w^\star\bigl(q,p'-p/2\bigr)\,.$$

Thus, after exchanging the roles of $(q,p) \leftrightarrow (q',p')$ (dummy variables), the scalar products linking coordinate, momentum and coherent-state representations are given by

$$\langle q|p\rangle = \langle p|\,q\rangle^\star = (2\pi\hbar)^{-f/2}\,w(q,p),$$
$$\langle q|p',q'\rangle = w(q,p')M(q-q'),$$
$$\langle q|q',p'\rangle = w^\star(q',p')\,\langle q|p',q'\rangle = w(q-q',p')M(q-q'),$$
$$\langle p|p',q'\rangle = w^\star(q',p-p')\tilde{M}(p-p'),$$
$$\langle p|q',p'\rangle = w^\star(q',p')\,\langle p|p',q'\rangle = w^\star(q',p)\tilde{M}(p-p'). \qquad (14)$$

### 2.3 *Relation with other phase-space representations*

2.3.1 *Bargmann's approach.* Based on the expansion (9) for the Glauber coherent state $|z\rangle$, any quantum state $|\Psi(t)\,\rangle$ can be represented by the wave function

$$\Psi(z^\star,t) = \langle z\,|\Psi(t)\,\rangle = \exp\left(-\frac{1}{2}zz^\star\right)g(z^\star,t),$$

where the function

$$g(z^\star,t) = \sum_{n=0}^{\infty}\frac{(z^\star)^n}{\sqrt{n!}}\langle n|\Psi(t)\,\rangle, \qquad (15)$$

is an element of the Bargmann space of the entire functions [12]. By using the transformation $(q, p) \leftrightarrow (z, z^\star)$ given after (10), the functions $\Psi_\pm(q, p, t)$ of the real variables $q$ and $p$ used in this work can be written as

$$\Psi_\pm(q, p, t)|_{(q,p)\to(z,z^\star)} = \exp\left(\pm \frac{1}{4}[z^2 - (z^\star)^2]\right) \times \Psi(z^\star, t). \quad (16)$$

Then, one concludes that $\Psi_\pm(q, p, t)$ are not complex analytic (or holomorphic) functions in $z$ or $z^\star$ [31], because the normalization factor $\exp(-\frac{1}{2}zz^\star)$ and the phase factor $\exp(\pm\frac{1}{4}[z^2 - (z^\star)^2])$ depend on both complex variables, $z$ and $z^\star$.

2.3.2 *Husimi function.* The expectation value of the projector $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ with respect to the Glauber coherent state $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle$ is given by

$$\rho_{\mathrm{H}}(q, p, t) = \langle z|\hat{\rho}(t)|z\rangle = |\Psi_\pm(q, p, t)|^2. \quad (17)$$

As commented in (3), the quantity $\langle z|\hat{\rho}(t)|z\rangle$ is known as Husimi function, a name given in recognition of the work of Husimi published in 1940 [9], two decades before the work of Glauber [2]. The non-holomorphic phase factors $w(\pm\frac{1}{2}q, p)$ included in $\Psi_\pm(q, p, t)$ do not alter the Husimi function but instead of the marginal property of the Wigner function given by (2), they will provide relationships between the phase-space wave functions $\Psi_\pm(q, p, t)$ and the position and momentum Schrödinger wave functions.

2.3.3 $\Gamma$-*basis representation.* In this work, instead of considering the $\Gamma$-basis generated by the eigenvalue equation $\hat{\Gamma}\,|p, q\rangle = \Gamma\,|p, q\rangle$ of an unknown Hermitian operator $\hat{\Gamma}$ proposed in [14], the wave functions $\Psi_-(q, p, t)$ and $\Psi_+(q, p, t)$ given in (7) are constructed using the eigenkets $|z\rangle$ of the non-Hermitian annihilation operator, which are duly modified by phase factors arising out of the physical and mathematical considerations. Roughly, the present treatment is linked to the work of Møller *et al* [16] in which the Weyl operator $\hat{D}(q, p)$ is applied to a normalized fiducial vector $|\chi\rangle$ and the resulting set of coherent states is used as a basis in a state-vector representation of quantum mechanics. They use the operator mapping $\breve{Q} = \frac{1}{2}q + i\hbar\partial/\partial p$ and $\breve{P} = \frac{1}{2}p - i\hbar\partial/\partial q$, discuss its origin and the implications on the possibilities of doing wave mechanics in phase-space, whereas in the present work the resulting mappings (see table 1) are a consequence of introducing the wave functions $\Psi_\pm(q, p, t)$ defined by (7).

## 3. Properties of the phase-space wave functions, $\Psi_\pm(q, p, t)$

In this section, some properties of the phase-space wave functions $\Psi_\pm(q, p, t)$ are deduced: (i) As shown in eqs (20) and (23) below, the functions $\Psi_\pm(q, p, t)$ are separable in the sense that they can be expressed as the sum of an infinite number of terms, where each one is the product of a time-dependent function of a variable $u$ (either $q$ or $p$) only and a time-independent function of the complementary variable $v$ (either $p$ or $q$) only. (ii) If $H_\eta(\gamma_0 v_\pm)$ is the Hermite polynomial of order $\eta$, and $\gamma_0 v_\pm$ is a dimensionless variable, where $\gamma_0$ is either $\kappa_0$ or $\chi_0$, integration of $\Psi_\pm(q, p, t)$ $H_\eta(\gamma_0 v_\pm)$ over $v_\pm$ results in the $\eta$th derivative of the position or momentum wave function at the point $u_\pm$. (iii) In particular, when $\eta = 0$, the position and momentum probability amplitudes can be obtained from the phase-space wave functions $\Psi_\pm(q, p, t)$. In §3.1 and 3.2, results of the preliminary investigations of the author a few years ago have been discussed [32].

### 3.1 *Wave function $\Psi_+(q, p, t)$, $\theta = 1$*

The completeness relation for the position operator $\hat{q}$, the expression $\langle q'|q, p\rangle$ given by (14) and the position–space wave function $\Psi(q, t) := \langle q|\Psi(t)\rangle$ lead to

$$\Psi_+(q, p, t) := \langle q, p|\Psi(t)\rangle = \int w^\star\left(q'' - q, p\right) M\left(q'' - q\right) \Psi(q'', t)\mathrm{d}q''. \quad (18)$$

Comparison of this equation with the relation $\Psi_+(q, p, t) = w(q, p)\Psi_-(q, p, t)$ gives

$$\Psi_-(q, p, t) := \langle p, q|\Psi(t)\rangle = \int w^\star\left(q', p\right) M\left(q' - q\right)\Psi(q', t)\mathrm{d}q'. \quad (19)$$

The transformations (18) and (19) pass from the position wave function $\Psi(q, t)$ to the phase-space wave functions $\Psi_+(q, p, t) = w(q, p)\Psi_-(q, p, t)$ and $\Psi_-(q, p, t)$. All these functions constitute a subspace of the Hilbert space $\mathbb{L}^2\left(\mathbb{R}^f\right)$ formed by pairs of functions, which differ from each other by a phase factor $w(\pm q, p)$ and they represent the state $|\Psi(t)\rangle$ in the basis of the rephased coherent states defined by (12).

In (18), changing the integration variable to $q' := q'' - q$, and expanding the position wave function $\Psi(q' + q, t) = \exp\left(q'\partial/\partial q\right)\Psi(q, t)$ in powers of $q'$, one finds

$$\Psi_+(q, p, t) = (2\pi\hbar)^{f/2} \sum_{m=0}^{\infty} \frac{1}{m!}\Psi^{(m)}(q, t)\tilde{J}_m(p). \quad (20)$$

Here $m = (m_1, m_2, \ldots, m_f)$ is a multi-index, $\Psi^{(m)}(q, t) := \partial^m \Psi(q, t)/\partial q^m$ is the $m$th derivative of the space wave function at point $q$,

$$\tilde{J}_m(p) = (2\pi\hbar)^{-f/2} \int w^\star(q', p) M(q')(q')^m \mathrm{d}q' = \left(-i\frac{q_0}{\sqrt{2}}\right)^m H_m(\chi_0 p)\tilde{M}(p) \qquad (21)$$

are time-independent coefficients and $H_m(\chi_0\ p) = H_{m_1}(\chi_0 p_1) H_{m_2}(\chi_0 p_2) \ldots H_{m_f}(\chi_0 p_f)$ is a product of Hermite polynomials. Now, as the one-dimensional Hermite polynomials are orthogonal in the range $(-\infty, \infty)$, with respect to the weighting function $\exp(-y^2)$, (20) can be inverted, and the $\eta$th derivative of the position wave function ($\eta = 0, 1, 2, \ldots$) is given by

$$\Psi^{(\eta)}(q, t) = \left(2p_0\sqrt{\pi}\right)^{-f/2} (-iq_0\sqrt{2})^{-\eta} (2\pi\hbar)^{-f/2} \times \int H_\eta(\chi_0 p) \Psi_+(q, p, t) \mathrm{d}p. \qquad (22)$$

In particular, for $\eta = 0$, $H_\eta(\chi_0\ p) = 1$, thus when one integrates $\Psi_+(q, p, t)$ over the momentum variable $p$, the position–space wave function $\Psi(q, t) := \Psi^{(0)}(q, t)$ is obtained.

### 3.2 *Wave function $\Psi_-(q, p, t)$, $\theta = 0$*

A procedure similar to that of §3.1, using now the completeness relation for the momentum operator $\hat{p}$ and the wave function $\Psi_-(q, p, t) := \langle p, q|\Psi(t)\rangle$, gives

$$\Psi_-(q, p, t) = (2\pi\hbar)^{f/2} \sum_{m=0}^{\infty} \frac{1}{m!} \tilde{\Psi}^{(m)}(p, t) J_m(q), \qquad (23)$$

with coefficients

$$J_m(q) = (2\pi\hbar)^{-f/2} \int \mathrm{d}p'\, w(q, p')\tilde{M}(p')(p')^m = \left(+i\frac{p_0}{\sqrt{2}}\right)^m H_m(\kappa_0 q) M(q). \qquad (24)$$

Similarly, the $\eta$th derivative of the momentum wave function is given by ($\eta = 0, 1, 2, \ldots$)

$$\tilde{\Psi}^{(\eta)}(p, t) = \left(2q_0\sqrt{\pi}\right)^{-f/2} (+ip_0\sqrt{2})^{-\eta} \times (2\pi\hbar)^{-f/2} \int H_\eta(\kappa_0 q) \Psi_-(q, p, t) \mathrm{d}q. \qquad (25)$$

Again, as a particular case, one obtains that the phase-space function $\Psi_-(q, p, t)$ integrated over $q$ gives the momentum–space wave function $\tilde{\psi}(p, t) := \tilde{\psi}^{(0)}(p, t)$.

In general, from (22) and (25), integration of $\Psi_+(q, p, t)$ and $\Psi_-(q, p, t)$ over the variables $p$ and $q$, respectively, leads to the position and momentum wave functions, $\Psi(q, t)$ and $\tilde{\Psi}(q, t)$. Therefore, when the squared moduli of these functions are calculated, one gets the proper marginal density in the position and momentum representations, $|\Psi(q, t)|^2$ and $|\tilde{\Psi}(p, t)|^2$, which is equivalent to the property of the Wigner function given by (2). The formulations [14,17] do not lead to proper marginal density in the position and momentum representations, except if an indirect link with the Wigner function is used (comment annotated in [17]). An additional feature of $\Psi_\pm(q, p, t)$ is their ability to generate derivatives of the wave functions $\Psi(q, t)$ and $\tilde{\Psi}(p, t)$ to any order $\eta$.

After noting the similarity between (22) and (25), one concludes that the $q \leftrightarrow p$ symmetry condition introduced in the paragraph before eq. (13) is equivalent to a transformation between symbols: in fact, the equations only differ from each other in the sign of $\pm i$ and in renaming the variables. Based on the foregoing text, one concludes that $\Psi_-(q, p, t)$ and $\Psi_+(q, p, t)$ exhibit greater affinity or closeness with either the momentum or the position wave function, respectively.

### 3.3 *Relation with the eigenfunctions of the harmonic oscillator*

Using the multi-index notation $n = (n_1, n_2, \ldots, n_f)$, the position and momentum eigenfunctions of an $f$-dimensional harmonic oscillator can be written as

$$\varphi_n(q) = \frac{1}{\sqrt{2^n\, n!}} H_n(q/q_0) M(q),$$

$$\tilde{\varphi}_n(p) = \frac{(-i)^n}{\sqrt{2^n\, n!}} H_n(p/p_0)\tilde{M}(p). \qquad (26)$$

Therefore, recalling that $\kappa_0 q = q/(q_0\sqrt{2})$ and $\chi_0 p = p/(p_0\sqrt{2})$, and using the identity

$$H_m\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2^{m/2}} \sum_{k=0}^{[m/2]} \frac{m!}{k!(m-2k)!} (-1)^k H_{m-2k}(x), \qquad (27)$$

the functions $\tilde{J}_m(p)$ and $J_m(q)$, given by (21) and (24), can be expressed as the following linear combinations of the momentum and position eigenfunctions of the $f$-dimensional harmonic oscillators, $\tilde{\varphi}_n(p,t)$ and $\varphi_n(q,t)$, respectively:

$$\tilde{J}_m(p)=\left(+\frac{q_0}{\sqrt{2}}\right)^m \sum_{k=0}^{[m/2]} \frac{m!}{2^k k!\sqrt{(m-2k)!}}\,\tilde{\varphi}_{m-2k}(p) \quad (28)$$

and

$$J_m(q)=\left(+i\frac{p_0}{\sqrt{2}}\right)^m \sum_{k=0}^{[m/2]} \frac{m!}{2^k k!\sqrt{(m-2k)!}} \times(-1)^k\varphi_{m-2k}(q). \quad (29)$$

Now, because the wave functions $\tilde{\varphi}_n(p,t)$ and $\varphi_n(q,t)$ are connected by a Fourier transforms pair, the Fourier transforms of $\tilde{J}_m(p)$ and $J_m(q)$ are given by

$$\mathbb{J}_m(q) := (2\pi\hbar)^{-f/2}\int w(q,p)\tilde{J}_m(p)\mathrm{d}p = \left(+\frac{q_0}{\sqrt{2}}\right)^m \sum_{k=0}^{[m/2]} \frac{m!}{2^k\, k!\sqrt{(m-2k)!}} \times\varphi_{m-2k}(q) \quad (30)$$

and

$$\tilde{\mathbb{J}}_m(p) := (2\pi\hbar)^{-f/2}\int w^\star(q,p)J_m(q)\mathrm{d}q = \left(+i\frac{p_0}{\sqrt{2}}\right)^m \sum_{k=0}^{[m/2]} \frac{m!}{2^k k!\sqrt{(m-2k)!}} \times(-1)^k\tilde{\varphi}_{m-2k}(p). \quad (31)$$

Note that the sums over $k$ on the right side of (28) and (31) differ only by the sign factor $(-1)^k$, which is also the case for eqs (29) and (30). Then, the Fourier transforms of $\tilde{\mathbb{J}}_m(p)$ and $\mathbb{J}_m(q)$ imply that

$$(2\pi\hbar)^{-f/2}\int w(q,p)\tilde{\mathbb{J}}_m(p)\mathrm{d}p = J_m(q) \quad (32)$$

and

$$(2\pi\hbar)^{-f/2}\int w^\star(q,p)\mathbb{J}_m(q)\mathrm{d}q = \tilde{J}_m(p). \quad (33)$$

### 3.4 *Position and momentum expectation values*

By using the completeness relation for the eigenkets $|q\rangle$ and recalling that $-i\hbar\partial/\partial q$ is the momentum operator $\hat{p}$ in position representation, one can see that (for $\eta = 0, 1, 2, \ldots$)

$$\langle\Psi(t)|\,\hat{p}^\eta\,|\Psi(t)\rangle = (-i\hbar)^\eta\int \Psi^\star(q,t)\Psi^{(\eta)}(q,t)\mathrm{d}q = \int p^\eta|\tilde{\psi}(p,t)|^2\mathrm{d}p, \quad (34)$$

where the last equality is based on the completeness relations for the eigenkets $|p\rangle$, and $\tilde{\Psi}(p,t)$ is the momentum wave function.

By following a procedure similar to the foregoing text, one also finds that

$$\langle\Psi(t)|\,\hat{q}^\eta\,|\Psi(t)\rangle = (i\hbar)^\eta\int \tilde{\Psi}^\star(p,t)\tilde{\Psi}^{(\eta)}(p,t)\mathrm{d}p = \int q^\eta\,|\psi(q,t)|^2\,\mathrm{d}q. \quad (35)$$

In particular for $\eta = 0$, if $\langle\Psi(t)|\,\Psi(t)\rangle = 1$, then $\int|\tilde{\psi}(p,t)|^2\mathrm{d}p = \int|\psi(q,t)|^2\mathrm{d}q = 1$.

At this point, it is pertinent to note that, according to (2), the expectation values given by (34) and (35) can also be written in terms of the Wigner function:

$$\langle\Psi(t)|\hat{p}^\eta\,|\Psi(t)\rangle = \int p^\eta\rho_{\mathrm{W}}(q,p,t)\mathrm{d}q\mathrm{d}p, \quad (36)$$

$$\langle\Psi(t)|\hat{q}^\eta\,|\Psi(t)\rangle = \int q^\eta\rho_{\mathrm{W}}(q,p,t)\mathrm{d}p\mathrm{d}q. \quad (37)$$

## 4. Phase-space representation of quantum operators

As mentioned in §1.2, quantum mechanics on phase-space requires the description of position and momentum operators $\hat{q}$ and $\hat{p}$ by means of differential operators $\breve{Q}$ and $\breve{P}$ that act over phase-space functions, $\Psi_\pm(q,p,t)$. In general, consider an operator $\hat{B}(t) := B(\hat{q},\hat{p},t)$ that admits a power-series expansion in $\hat{q}$ and $\hat{p}$, and ask for the representations in terms of the $qp$- and $pq$-coherent states. Unlike other proposal in which $\breve{Q}$ and $\breve{P}$ are parametrized (e.g. [13–15,17–19,21,22]) in this work, no particular assumption about the form of these operators is required.

### 4.1 *Permutation rules*

Consider $F(x,t) = \sum_{n=0}^{\infty} a_n(t)(x-x_0)^n$, centred at the point $x_0$, as the convergent power series defining a given function $F(x)$. Then, given the relation

$$\Psi_-(q,p,t) = w^\star(q,p)\Psi_+(q,p,t),$$

one gets the identity

$$\left(i\hbar\frac{\partial}{\partial p} - x_0\breve{1}\right)^n \Psi_-(q,p,t) = w^\star(q,p)\left(q + i\hbar\frac{\partial}{\partial p} - x_0\breve{1}\right)^n \Psi_+(q,p,t), \quad (38)$$

where $n$ is a multi-index and $\breve{1}$ is the unit operator in the phase-space. One thus obtains

$$F\left(i\hbar\frac{\partial}{\partial p},t\right)\Psi_-(q,p,t) = w^\star(q,p)F\left(q+i\hbar\frac{\partial}{\partial p},t\right)\Psi_+(q,p,t) = w^\star(q,p)F\left(q+i\hbar\frac{\partial}{\partial p},t\right)w(q,p)\Psi_-(q,p,t). \quad (39)$$

Similarly, for a given function $G(y,t)=\sum_{m=0}^{\infty} b_m(t)(y-y_0)^m$, one gets

$$G\left(-i\hbar\frac{\partial}{\partial q},t\right)\Psi_+(q,p,t) = w(q,p)G\left(p-i\hbar\frac{\partial}{\partial q},t\right)\Psi_-(q,p,t) = w(q,p)G\left(p-i\hbar\frac{\partial}{\partial q},t\right)w^\star(q,p) \times \Psi_+(q,p,t). \quad (40)$$

In (39) and (40), the first equalities can be interpreted as intertwining relations between the functions $\Psi_-(q,p,t)$ and $\Psi_+(q,p,t)$, whereas the second ones establish an equivalence between phase-space operators.

### 4.2 *Functions of the operators $\hat{q}$ and $\hat{p}$*

From the power series expansion of $F(x,t)$, one defines the Hilbert-space operator $F(\hat{q},t)$. In this way, using the eigenvalue equation $\hat{q}|q'\rangle = q'|q'\rangle$, the completeness relation of the eigenkets $|q'\rangle$, the scalar product $\langle q'|p,q\rangle = w(q',p)M(q'-q)$, the identity $F(q',t)w^\star(q',p) = F(i\hbar\partial/\partial p,t)w^\star(q',p)$, and the relations (19) and (39), one obtains

$$\langle p,q|F(\hat{q},t)|\Psi(t)\rangle = \int F(q',t)w^\star(q',p)M(q'-q)\Psi(q',t)\mathrm{d}q' \quad (41a)$$

$$= F\left(i\hbar\frac{\partial}{\partial p},t\right)\int w^\star(q',p)M(q'-q)\Psi(q',t)\mathrm{d}q' \quad (41b)$$

$$= F\left(i\hbar\frac{\partial}{\partial p},t\right)\Psi_-(q,p,t) = w^\star(q,p)F\left(q+i\hbar\frac{\partial}{\partial p},t\right)\Psi_+(q,p,t). \quad (41c)$$

From (12), because $|p,q\rangle = w(q,p)|q,p\rangle$ and $\langle q,p| = w(q,p)\langle p,q|$, then multiplication of (41a)–(41c) on the left by the factor $w(q,p)$ gives (i.e., for $\theta=1$)

$$\langle q,p|F(\hat{q},t)|\Psi\rangle = w(q,p)F\left(i\hbar\frac{\partial}{\partial p},t\right)\Psi_-(q,p,t) = F\left(q+i\hbar\frac{\partial}{\partial p},t\right)\Psi_+(q,p,t). \quad (42)$$

Similarly, using the eigenkets $|p'\rangle$ of the momentum operator $\hat{p}$, one gets ($\theta=1$)

$$\langle q,p|G(\hat{p},t)|\Psi(t)\rangle = G\left(-i\hbar\frac{\partial}{\partial q},t\right)\Psi_+(q,p,t) = w(q,p)G\left(p-i\hbar\frac{\partial}{\partial q},t\right)\times\Psi_-(q,p,t) = w(q,p)G\left(p-i\hbar\frac{\partial}{\partial q},t\right)\times\Psi_-(q,p,t). \quad (43)$$

This expression can be rewritten as ($\theta=0$)

$$\langle p,q|G(\hat{p},t)|\Psi(t)\rangle = w^\star(q,p)G\left(-i\hbar\frac{\partial}{\partial q},t\right)\times\Psi_+(q,p,t) = G\left(p-i\hbar\frac{\partial}{\partial q},t\right)\times\Psi_-(q,p,t). \quad (44)$$

4.2.1 *Product of operators*. The preceding equations can be generalized for dealing with products of operators. For example, consider $\langle p,q|F(\hat{q},t)G(\hat{p},t)|\Psi(t)\rangle$. Defining $|\Phi(t)\rangle := G(\hat{p},t)\,|\Psi(t)\rangle$, one writes

$$\langle p,q|F(\hat{q},t)G(\hat{p},t)|\Psi(t)\rangle = \langle p,q|F(\hat{q},t)|\Phi(t)\rangle = F(i\hbar\partial/\partial p,t)\Phi_-(q,p,t),$$

where the relation (41c) was used. Afterwards, as

$$\Phi_-(q,p,t) = \langle p,q|G(\hat{p},t)|\Phi(t)\rangle,$$

eq. (44) implies ($\theta=1$)

$$\langle p,q|F(\hat{q},t)G(\hat{p},t)|\Psi(t)\rangle = F\left(i\hbar\frac{\partial}{\partial p},t\right)G\left(p-i\hbar\frac{\partial}{\partial q},t\right)\Psi_-(q,p,t). \quad (45)$$

Performing a similar procedure, one also finds ($\theta=0$)

$$\langle q,p|G(\hat{p},t)F(\hat{q},t)|\Psi(t)\rangle = G\left(-i\hbar\frac{\partial}{\partial q},t\right)F\left(q+i\hbar\frac{\partial}{\partial p},t\right)\Psi_+(q,p,t). \quad (46)$$

Table 1 summarizes the results obtained. The *qp*-representation of the position and momentum operators corresponds to the quantization mapping introduced in 1951 by L Van Hove in the research on the problem of getting a quantum formulation of a system starting from its classical description. These operators were also obtained in [33] via considerations based on the unitary implementation of the Galilei group.

In general, for any operator

$$B(\hat{q}, \hat{p}, t) = \sum_{n=0}^{\infty} a_n(t) \hat{A}_n(t),$$

with real or complex valued coefficients $a_n(t)$, and operators $\hat{A}_n(t)$ of the form

$$\hat{A}_n(t) = F_0(\hat{q}, t) G_0(\hat{p}, t)\, F_1(\hat{q}, t) G_1(\hat{p}, t) \cdots \times F_n(\hat{q}, t) G_n(\hat{p}, t), \tag{47}$$

and for any coherent state $|\theta, q'', p''\rangle$, one finds

$$\begin{aligned} &\langle \theta, q'', p'' | B(\hat{q}, \hat{p}, t) | \Psi(t) \rangle \\ &= \underset{\rightarrow}{B}(\breve{Q}\left(\theta | q'', p''\right), \breve{P}\left(\theta | q'', p''\right), t) \\ &\quad \times \Psi\left(\theta | q'', p'', t\right) \end{aligned} \tag{48}$$

with phase-space operators (compare with table 1)

$$\begin{aligned} \breve{Q}\left(\theta | q'', p''\right) &:= \theta q'' + i\hbar \frac{\partial}{\partial p''}, \\ \breve{P}\left(\theta | q'', p''\right) &:= (1-\theta) p'' - i\hbar \frac{\partial}{\partial q''}, \end{aligned} \tag{49}$$

where the order of the factors must be strictly preserved as given in (47). Naturally, before the application of rule (48), the operator $\hat{A}_n(t)$ may be rearranged in a particular ordering, either the antistandard or the standard ordering.

For future use, it is convenient to consider the adjoint of (48), namely,

$$\begin{aligned} &\langle \Psi(t) | B^{+}(\hat{q}, \hat{p}, t) | \theta, q'', p'' \rangle \\ &= \Psi^{\star}\left(\theta | q'', p'', t\right) \underset{\leftarrow}{B}^{+}(\breve{Q}\left(\theta | q'', p''\right), \\ &\quad \times \breve{P}\left(\theta | q'', p''\right), t). \end{aligned} \tag{50}$$

In (48) and (50), the notation $\underset{\rightarrow}{B}(\cdots)$ and $\underset{\leftarrow}{B}(\cdots)$ denote that the phase-space operators act to the right and left, respectively. For a Hermitian operator, $B^{+}(\hat{q}, \hat{p}, t) = B(\hat{q}, \hat{p}, t)$, one has $\underset{\leftarrow}{B}^{+}(\breve{Q}\left(\theta | q'', p''\right), \breve{P}\left(\theta | q'', p''\right), t) = \underset{\leftarrow}{B}(\breve{Q}^{\star}(\theta | q'', p''), \breve{P}^{\star}(\theta | q'', p''), t)$.

Some additional comments:

(a) In (48), consider in explicit form the cases $\theta = 0$ and $\theta = 1$, use the relations $|p'', q''\rangle = w\left(q'', p''\right)|q'', p''\rangle$, $\langle q'', p''| = w(q'', p'')\langle p'', q''|$ and $\Psi_{-}(q'', p'', t) = w^{\star}(q'', p'')\Psi_{+}(q'', p'', t)$, and compare the results. Then one arrives at the following rule linking the $pq$ and $qp$ representations of the operator $\hat{B}(\hat{q}, \hat{p}, t)$:

$$\begin{aligned} &B\left(i\hbar \frac{\partial}{\partial p},\ p - i\hbar \frac{\partial}{\partial q}, t\right) \\ &= w^{\star}(q,p) B\left(q + i\hbar \frac{\partial}{\partial p}, -i\hbar \frac{\partial}{\partial q}, t\right) w(q, p)\,. \end{aligned} \tag{51}$$

(b) In terms of the shorthands $\breve{Q}(\theta) = \breve{Q}(\theta | q, p)$ and $\breve{P}(\theta) = \breve{P}(\theta | q, p)$, one writes the commutation relations $[\breve{Q}(0), \breve{P}(0)] = i\hbar \breve{1}$, $[\breve{Q}(1), \breve{Q}(1)] = i\hbar \breve{1}$, $[\breve{P}(0), \breve{P}(1)] = 0$ and $[\breve{Q}(0), \breve{Q}(1)] = 0$, where $\breve{1}$ is the unit operator in the phase-space. Therefore, these results are consistent with the Heisenberg uncertainty principle.

(c) Comparison of table 1 with the expressions $\breve{Q}\Psi(q, p) = (\alpha q + i\hbar\beta\partial/\partial p)\Psi(q, p)$ and $\breve{P}\Psi(q, p) = (\gamma p + i\hbar\delta\partial/\partial q)\Psi(q, p)$ given in [14], eqs (2.4) and (2.5) leads to the following results: (i) in the $pq$-representation, one has $\alpha = 0$, $\beta = 1$, $\gamma = 1$ and $\delta = -1$, and (ii) in the $pq$-representation, it follows that $\alpha = 1$, $\beta = 1$, $\gamma = 0$ and $\delta = -1$. In both cases $\beta\gamma - \alpha\delta = 1$, but the values of the parameters are different from the canonical values given in [14], eqs (4.4) and (4.5).

## 5. Quantum dynamics

In this section, the Schrödinger and the quantum Liouville equations are written using the $pq$ and $qp$ phase-space representations given by eqs (12).

### 5.1 *Schrödinger equation*

Consider a quantum system with Hamiltonian $\hat{H}(t) = H(\hat{q}, \hat{p}, t)$, represented at time $t_0$ by the state $|\Psi(t_0)\rangle$, with the time evolution of the state $|\Psi(t)\rangle$ governed by the Schrödinger equation,

$$\begin{aligned} &i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi(t)\rangle = H(\hat{q}, \hat{p}, t) |\Psi(t)\rangle, \\ &\lim_{t \to t_0} |\Psi(t)\rangle = |\Psi(t_0)\rangle. \end{aligned} \tag{52}$$

The solution of this equation can be written in terms of the time evolution operator $\hat{U}(t, t_0)$, which transforms the initial state $|\Psi(t_0)\rangle$ to the state $|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi(t_0)\rangle$. Thus, using the notation (7), one has $\Psi(\theta | q, p, t) = \langle \theta, q, p || \hat{U}(t, t_0) | \Psi(t_0) \rangle$

and, for time-independent $(q, p)$, the Schrödinger equation (52) takes the form

$$i\hbar\frac{\partial}{\partial t}\Psi\,(\theta|\,q,\,p,t) = \underset{\rightarrow}{H}(\breve{Q}\,(\theta|q,\,p)\,,\,\breve{P}\,(\,\theta|q,\,p)\,,t)\Psi(\,\theta|q,\,p,t)\,, \quad (53)$$

with the initial condition $\lim_{t\to t_0}\Psi(\theta|q,\ p,\ t) = \Psi(\theta|q,\ p,\ t_0)$. Assuming a Hermitian Hamiltonian, and using (50), the complex conjugate of (53) can be written as

$$-i\hbar\frac{\partial}{\partial t}\Psi^{\star}(\theta|\,q,p,t) = \underset{\rightarrow}{H}(\breve{Q}^{\star}(\theta|\,q,\,p)\,,\ \breve{P}^{\star}(\theta|\,q,\,p)\,,t)\,\Psi^{\star}(\theta|q,p,t)\,. \quad (54)$$

Hereafter, in (53) and (54) the arrow below $\underset{\rightarrow}{H}(\cdots)$ will be suppressed.

### 5.2 *Quantum Liouville equation*

Consider a mixture of $\mathcal{N}$ independent states $|\Psi_{[k]}(t)\rangle$ listed by the index $k = 1, 2, \ldots, \mathcal{N}$, where each state is normalized to unity $\langle\Psi_{[k]}(t)|\Psi_{[k]}(t)\rangle = 1$ and has associated a statistical weight $0 < W_{[k]} \leq 1$ and $\sum_{k=1}^{\mathcal{N}} W_{[k]} = 1$. In the Hilbert space, the density operator describing the mixture is then defined as

$$\hat{\rho}(t) := \sum_{k=1}^{\mathcal{N}} W_{[k]}|\Psi_{[k]}(t)\rangle\langle\Psi_{[k]}(t)|, \quad (55)$$

where the square bracket $[k]$ is used as a mnemonic that states $|\Psi_{[k]}(t)\rangle$ is an element of a statistical mixture. Given the initial mixture at time $t_0$, described by $\hat{\rho}(t_0)$, the time development of $\hat{\rho}(t)$ is ruled by the the quantum Liouville equation,

$$i\hbar\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = \big[H(\hat{q},\,\hat{p},\,t),\,\hat{\rho}(t)\big], \quad \lim_{t\to t_0}\hat{\rho}(t) = \hat{\rho}(t_0), \quad (56)$$

where $[\hat{H}, \hat{\rho}]$ is the commutator between the Hermitian Hamiltonian $\hat{H}(t) = H\left(\hat{q}, \hat{p}, t\right)$ and the density operator $\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}^{-1}(t, t_0)$.

#### 5.2.1 *A generalized $(q_a, p_a, q_b, p_b)$ representation.*

In order to get the phase-space representations of the Liouville equation (56), consider any pair of phase-space points, $(q_a, p_a)$ and $(q_b, p_b)$, the complex variables $z_a = \kappa_0 q_a + i\chi_0 p_a$ and $z_b = \kappa_0 q_b + i\chi_0 p_b$, the Glauber coherent states $|z_a\rangle$ and $|z_b\rangle$ and the corresponding *pq* and *qp* coherent states constructed by using (12). Afterwards, define the matrix elements of $\hat{\rho}(t)$ in the *pq* and *qp* representations,

$$\rho(\theta|q_a, p_a, q_b, p_b, t) := \langle\theta|q_b, p_b|\hat{\rho}(t)|\,\theta|\,q_a,\,p_a\rangle. \quad (57)$$

In particular, for a pure state $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$, one has

$$\rho(\theta|q_a,\,p_a,\,q_b,\,p_b,\,t) = \big[\Psi(\theta|q_a,\,p_a,\,t)\big]^{\star}\,\Psi(\theta|q_b,\,p_b,\,t)$$

and, by convention, $[\Psi(\cdots)]^{\star}$ is positioned at the left of $\Psi(\cdots)$. This convention explains the order of the points $(q_a, p_a)$ and $(q_b, p_b)$ adopted in the left and right sides of (57).

Inserting the expression (55) for the density operator into (57), one gets

$$\rho\,(\theta|\,q_a,\,p_a,\,q_b,\,p_b,\,t) = \sum_{k=1}^{\mathcal{N}} W_{[k]}\left[\Psi_{[k]}\,(\theta|\,q_a,\,p_a,\,t)\right]^{\star} \times \Psi_{[k]}\,(\theta|\,q_b,\,p_b,\,t)\,, \quad (58)$$

where $\rho(\theta|\,q_a,\,p_a,\,q_b,\,p_b,\,t)$ has $4f$ independent variables besides $t$, namely, $q_a, p_a, q_b$ and $p_b$. Then, one may derive the equation of motion for the density operator using the fact that the wave functions $\left[\Psi_{[k]}(\theta|q_a,\,p_a,\,t)\right]^{\star}$ and $\Psi_{[k]}(\theta|q_b,\,p_b,\,t)$ satisfy the Schrödinger equations (54) and (53), with $(q,\,p) \to (q_a,\,p_a)$ and $(q,\,p) \to (q_b,\,p_b)$, respectively.

In accordance with (49), one associates with the independent variables $(q_b, p_b)$ and $(q_a, p_a)$, the phase-space operators $\breve{Q}\,(\theta|\,q_b,\,p_b)$, $\breve{P}\,(\theta|\,q_b,\,p_b)$, $\breve{Q}(\theta|\,q_a,\,p_a)$ and $\breve{P}(\theta|\,q_a,\,p_a)$. Consequently, the phase-space representations of the Liouville equation can be written as

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\rho\,(\theta|\,q_a,\,p_a,\,q_b,\,p_b,\,t) &= \Big\{H\left(\breve{Q}\,(\theta|q_b,\,p_b)\,,\,\breve{P}\,(\theta|q_b,\,p_b)\,,\,t\right) \\ &\quad - H\Big(\breve{Q}^{\star}\,(\theta|q_a,\,p_a)\,,\,\breve{P}^{\star}\,(\theta|q_a,\,p_a)\,,\,t\Big)\Big\} \\ &\quad \times \rho\,(\theta|q_a,\,p_a,\,q_b,\,p_b,\,t)\,, \end{aligned} \quad (59)$$

with initial condition $\lim_{t\to t_0}\rho(\theta|q_b,\,p_b,\,q_a,\,p_a,\,t) = \rho(\theta|q_b,\,p_b,\,q_a,\,p_a,\,t_0)$.

At this point, using the expression for $\Psi\,(\theta|\,q,\,p,\,t)$ given in (7), one may write

$$\begin{aligned} &\left[\Psi_{[k]}\,(\theta|q_a,\,p_a,\,t)\right]^{\star}\Psi_{[k]}\,(\theta|q_b,\,p_b,\,t) \\ &\quad = \mathcal{F}\left[\langle z_a|\Psi_{[k]}(t)\rangle\right]^{\star}\langle z_b|\Psi_{[k]}(t)\rangle. \end{aligned}$$

Except by the phase factor $\mathcal{F} = \exp\left(i(2\theta - 1)(q_b p_b - q_a p_a)/(2\hbar)\right)$, the $k$th summand in (58) is the product of the probability amplitudes of finding simultaneously the system in the Glauber coherent states

$|z_a\rangle = |\kappa_0 q_a + i\chi_0 p_a\rangle$ and $|z_b\rangle = |\kappa_0 q_b + i\chi_0 p_b\rangle$, when the system is in the pure state $|\Psi_{[k]}(t)\rangle$. Thus, $\rho(\theta|\, q_a, p_a, q_b, p_b, t)$ measures the weighted contributions of the pure states taking part in the mixture (58).

5.2.2 *Parametric representation of Liouville equation.* Let me start this subsection with figure 1. Given any two points $(q_a, p_a)$ and $(q_b, p_b)$ in the phase-space, one can introduce the transformation

$$q_b = q' + \frac{1}{2}q, \quad p_b = p' + \frac{1}{2}p,$$
$$q_a = q' - \frac{1}{2}q, \quad p_a = p' - \frac{1}{2}p. \tag{60}$$

From (60), one finds new phase-space points $(q', p')$ and $(q, p)$ given by

$$q' = \frac{1}{2}(q_b + q_a), \quad p' = \frac{1}{2}(p_b + p_a),$$
$$q = q_b - q_a, \quad p = p_b - p_a, \tag{61}$$

which can be reinterpreted geometrically: the components of $(q', p')$ are formed by the mean position and momentum of the points $(q_b, p_b)$ and $(q_a, p_a)$, whereas $(q, p)$ measures the separation between those points along the position and momentum axes, respectively. Thus, one may contemplate (61) as defining a $2f$-dimensional cell in phase-space, which is centred at the point $(q', p')$, and has edge lengths $q$ and $p$. However, when the system is in a coherent state, the position and momentum uncertainties are given by $\Delta q = (q_0/\sqrt{2})^f$ and $\Delta p = (p_0/\sqrt{2})^f$, respectively. Therefore, the coherent states associated with

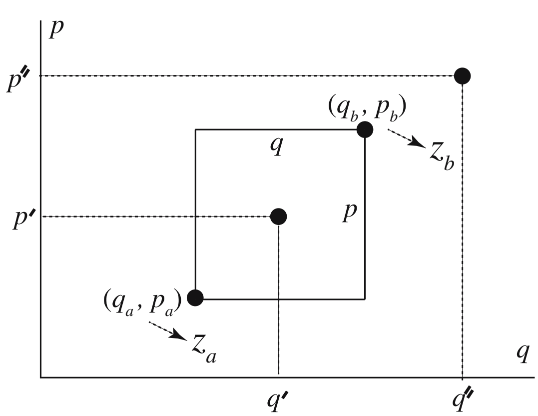


**Figure 1.** The phase-space points $(q_b, p_b)$ and $(q_a, p_a)$ defined by (60) are located on a $2f$-dimensional hypercell represented in the graphics by the box with the central point $(q', p')$ and the edges $q$ and $p$. The role of an arbitrary phase-space point $(q'', p'')$ will be explicit in (71) and (72).

the points on the border induce an uncertainty equivalent to $\Delta q \Delta p = (\frac{1}{2}\hbar)^f$ in the volume of the cell.

Now, consider the set $\{(q_a, p_a), (q_b, p_b)\}$ of all points $(q_a, p_a)$ and $(q_b, p_b)$ having the same middle point $(q', p')$, so that this point can be treated as a parameter. The transformations (60) imply that

$$\partial/\partial q_b = \frac{1}{2}\partial/\partial q, \quad \partial/\partial p_b = \frac{1}{2}\partial/\partial p,$$
$$\partial/\partial q_a = -\frac{1}{2}\partial/\partial q, \quad \partial/\partial p_a = -\frac{1}{2}\partial/\partial p,$$

and operators involved in the Liouville equation (59) can be rewritten as

$$\breve{Q}_b(\theta, q'|q, p) := \breve{Q}(\theta|q_b, p_b),$$
$$\breve{P}_b(\theta, p'|q, p) := \breve{P}(\theta|\, q_b, p_b),$$
$$\breve{Q}_a(\theta, q'|q, p) := \breve{Q}(\theta|q_a, p_a)$$

and

$$\breve{P}_a(\theta, p'|q, p) := \breve{P}(\theta|q_a, p_a).$$

Therefore, with the help of (60) and (49), it follows that

$$\breve{Q}_b(\theta, q'|q, p) = \theta q' + \frac{1}{2}\breve{Q}(\theta|q, p),$$
$$\breve{P}_b(\theta, p'|q, p) = (1-\theta)p' + \frac{1}{2}\breve{P}(\theta|q, p),$$
$$\breve{Q}^{\star}_a(\theta, q'|q, p) = \theta q' - \frac{1}{2}\breve{Q}(\theta|q, p),$$
$$\breve{P}^{\star}_a(\theta, p'|q, p) = (1-\theta)p' - \frac{1}{2}\breve{P}(\theta|q, p). \tag{62}$$

Based on the above, one can define a function

$$\rho(\theta, q', p'|q, p, t) := \rho\left(\theta|q' - \frac{1}{2}q, p' - \frac{1}{2}p, q' + \frac{1}{2}q, p' + \frac{1}{2}p, t\right)$$
$$= \begin{cases} \langle p' + \frac{1}{2}p,\ q' + \frac{1}{2}q|\hat{\rho}(t)|p' - \frac{1}{2}p,\ q' - \frac{1}{2}q\rangle \\ \quad \text{if } \theta = 0 \\ \langle q' + \frac{1}{2}q,\ p' + \frac{1}{2}p|\hat{\rho}(t)|q' - \frac{1}{2}q,\ p' - \frac{1}{2}p\rangle \\ \quad \text{if } \theta = 1, \end{cases} \tag{63}$$

which exhibits a parametric dependence on $\theta$ and $(q', p')$ and, apart from $t$, has $2f$ independent variables: $q$ and $p$. Thus, from the point of view of an observer located at the point $(q', p')$, the Liouville equation (59) can be rewritten as

$$i\hbar\frac{\partial}{\partial t}\rho(\theta, q', p'|q, p, t)$$
$$= [H_b(\theta, q', p'|q, p, t) - H_a(\theta, q', p'|q, p, t)]$$
$$\times\ \rho(\theta, q', p'|q, p, t), \tag{64}$$

with initial condition $\lim_{t\to t_0} \rho(\theta, q', p'|q, p, t) = \rho(\theta, q', p'|q, p, t_0)$, and operators

$$\left.\begin{array}{l} H_b(\theta, q', p'|q, p, t) \\ H_a(\theta, q', p'|q, p, t) \end{array}\right\} := H\left(\theta\, q' \pm \frac{1}{2}\breve{Q}(\theta|q, p),\right.$$
$$\left.(1-\theta)p' \pm \frac{1}{2}\breve{P}(\theta|q, p),\ t\right). \tag{65}$$

These operators are associated with the Hamiltonian of the system, and the phase-space points $(q_b, p_b)$ and $(q_a, p_a)$, respectively. Both points contribute to the dependence of $\rho(\theta, q', p'|q, p, t)$ on the variables $q$ and $p$, as seen by writting (63) in explicit form:

$$\rho\left(\theta|\,q' - \frac{1}{2}q, p' - \frac{1}{2}p, q' + \frac{1}{2}q, p' + \frac{1}{2}p, t\right)$$
$$= \sum_{k=1}^{\mathcal{N}} W_{[k]}\left[\Psi_{[k,a]}\left(\theta|\,q' - \frac{1}{2}q, p' - \frac{1}{2}p, t\right)\right]^{\star}$$
$$\times \Psi_{[k,b]}\left(\theta|\,q' + \frac{1}{2}q, p' + \frac{1}{2}p, t\right). \tag{66}$$

The subscripts have been complemented, $[k] \to [k, a]$ and $[k] \to [k, b]$, in order to trace the genealogy of the functions's arguments due to the transforms (60). Thus, similar to what was noted with respect to (59), the action of $H_b(\theta, q', p'|q, p, t)$ and $H_a(\theta, q', p'|q, p, t)$ on $\rho(\theta|q' - \frac{1}{2}q, p' - \frac{1}{2}p, q' + \frac{1}{2}q, p' + \frac{1}{2}p, t)$, in (66), is restricted to the functions denoted by the subindices $[k, b]$ and $[k, a]$, respectively.

5.2.3 *Some comments*. Bopp in 1961 [34] interpreted the quantum formalism as an extension of classical statistical mechanics by describing the disturbance the observer causes on a system as an ensemble in phase-space, i.e., as a cloud of points in phase-space with a density represented by a distribution function $\varrho(q, p, t)$. Similarly, Kubo in 1964 [35] used Wigner representation to study the motion of electrons in a magnetic field. In these works, operators denoted by Bopp [34] as $\breve{Q} = q + \frac{1}{2}i\hbar\partial/\partial p$, $\breve{P} = p - \frac{1}{2}i\hbar\partial/\partial q$, $\breve{Q}^{\star} = q - \frac{1}{2}i\hbar\partial/\partial p$ and $\breve{P}^{\star} = p + \frac{1}{2}i\hbar\partial/\partial q$ allowed to write the equation of motion for $\rho(q, p, t)$ in the form $i\hbar\, \mathrm{d}\varrho(q, p, t)/\mathrm{d}t = [H(\breve{Q}, \breve{P}) - H(\breve{Q}^{\star}, \breve{P}^{\star})]\varrho(q, p, t)$.

Despite the apparent similarity between the results of Bopp and Kubo and the ones obtained in this paper, one notes that: (i) $\breve{Q}_b(\theta, q'|q, p)$, $\breve{P}_b(\theta, p'|q, p)$, $\breve{Q}^{\star}_a(\theta, q'|q, p)$ and $\breve{P}^{\star}_a(\theta, p'|q, p)$ given in (62) differ from Bopp's operators. This can be seen by the parametric role of $(q',\ p')$, and the fact that the *pq* and *qp* representations are selected by choosing $\theta = 0$ and $\theta = 1$, respectively. (ii) Equation (63) provides information on the allowed functions for solving the quantum Liouville equation (64), which has no analogues in the Bopp's and Kubo's formulations. Thus, the present treatment provides a novel method for the description of quantum dynamics in the phase-space.

### 5.3 *On the matrix elements* $\rho(\theta|q_a, p_a, q_b, p_b, t)$ *and the Husimi function*

In order to gain insight about the functions $\rho(\theta|q_a, p_a, q_b, p_b, t)$ and $\rho(\theta, q', p'|q, p, t)$, defined by (57) and (63), consider a pure state $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ and note that (57) reduces to $\rho(\theta|q_a, p_a, q_b, p_b, t) = [\Psi(\theta|q_a, p_a, t)]^{\star}\Psi(\theta|q_b, p_b, t)$. Then, in the general case of a mixed state, using (23) and (20) one arrives at the expansions

$$\rho(0|q_a, p_a, q_b, p_b, t) = (2\pi\hbar)^f \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{1}{\mu!\,\eta!}$$
$$\times\left[\sum_{k=1}^{\mathcal{N}} W_{[k]}\left(\tilde{\Psi}^{(\mu)}_{[k]}(p_a, t)\right)^{\star} \tilde{\Psi}^{(\eta)}_{[k]}(p_b, t)\right] J^{\star}_{\mu}(q_a) J_{\eta}(q_b) \tag{67}$$

and

$$\rho(1|q_a, p_a, q_b, p_b, t) = (2\pi\hbar)^f \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{1}{\mu!\,\eta!}$$
$$\times\left[\sum_{k=1}^{\mathcal{N}} W_{[k]}\left(\Psi^{(\mu)}_{[k]}(q_a, t)\right)^{\star} \Psi^{(\eta)}_{[k]}(q_b, t)\right] \tilde{J}^{\star}_{\mu}(p_a)\, \tilde{J}_{\eta}(p_b), \tag{68}$$

where $J_{\mu}(\cdots)$ and $\tilde{J}_{\mu}(\cdots)$ are reference functions associated with the $f$-dimensional harmonic oscillator (§3.3), whereas the contributions of the specific system under study are the ones enclosed within the square brackets.

The Husimi function at the point $(q',\ p')$ can be obtained as a particular case from (67) and (68), when the points $(q_a, p_a)$ and $(q_b, p_b)$ are arbitrarily close. In fact, from (60), (57) and (63), it follows that

$$\lim_{\substack{q\to 0\\ p\to 0}} \rho(\theta|\,q_a, p_a, q_b, p_b, t) = \lim_{\substack{q\to 0\\ p\to 0}} \rho\left(\theta, q', p'|q, p, t\right)$$
$$= \rho_{\mathrm{H}}(q', p', t). \tag{69}$$

Finally, because the expectation value of an observable $\hat{B}(t)$ is given by $\langle\langle\hat{B}\rangle\rangle(t) = \mathrm{Tr}(\hat{B}(t)\hat{\rho}(t))$, and the trace of any operator may be expressed as an integral over the $pq$ or the $qp$ coherent states, then

$$\langle\langle\hat{B}\rangle\rangle(t) = \frac{1}{(2\pi\hbar)^f}\int B(\breve{Q}(\theta), \breve{P}(\theta), t)\rho_{\mathrm{H}}\left(q', p', t\right) \times \mathrm{d}q'\mathrm{d}p', \tag{70}$$

where $\breve{Q}(\theta)$ and $\breve{P}(\theta)$ are given by (49), with the substitution $(q, p) \to (q', p')$.

## 6. Relationship between *pq* and *qp* representations, and Wigner function

The purpose of this section is to connect the treatment in this paper with the Wigner function. After recalling the transformations (60) and (63), one may multiply both sides of (67) and (68) by $w^\star(q, p'')w(q'', p)$ and integrate the resulting expressions over the $f$-dimensional variables $p$ and $q$. Because the point $(q', p')$ is a parameter, it is convenient to define a double Fourier transform $(p'', q'') \leftarrow (q, p)$ through the relation (figure 1)

$$W(\theta, q', p'|q'', p'', t) := \frac{1}{(2\pi\hbar)^f}\int w(q'', p)w^\star(q, p'') \times\rho(\theta, q', p'|q, p, t)\mathrm{d}q\mathrm{d}p, \tag{71}$$

whose inverse $(p'', q'') \to (q, p)$ is given by

$$\rho(\theta, q', p'|q, p, t) = \frac{1}{(2\pi\hbar)^f}\int w^\star(q'', p)w(q, p'') \times W(\theta, q', p'|q'', p'', t)\mathrm{d}q''\mathrm{d}p''. \tag{72}$$

As a particular situation, in (71) and (72), a coalescence of $(q'', p'')$ with $(q', p')$ can happen, so that $W(\theta, q', p'|q'', p'', t)$ reduces to

$$W(\theta|q', p', t) := W(\theta, q', p'|q', p', t) = \frac{1}{(2\pi\hbar)^f}\int w(q', p)w^\star(q, p') \times\rho(\theta, q', p'|q, p, t)\mathrm{d}q\mathrm{d}p. \tag{73}$$

In this case, $W(\theta|\, q', p', t)$ depends on $(q', p')$ as a blend of two contributions, the first one is associated with the role of $(q', p')$ as parameter, whereas the second one arises from the Fourier transform $(p', q') \leftarrow (q, p)$. In what follows, it will be noted that the function $W(\theta|\, q', p', t)$ is directly related to the Wigner function.

### 6.1 *Structure of $W(\theta|q', p', t)$, when $\theta = 0$*

From (67), (63) and (60), one arrives at

$$W(0|q', p', t) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\rho_{\mu\eta}(q', p', t) \times\tilde{\mathcal{J}}_{\mu\eta}\left(q', p'\right) \tag{74}$$

with (see Appendix A)

$$\begin{aligned}\rho_{\mu\eta}(q',\ p',\ t) &:= \sum_{k=1}^{\mathcal{N}} W_{[k]}\int w(q',\ p) \\ &\quad\times\left[\tilde{\Psi}^{(\mu)}_{[k]}\left(p'-\frac{1}{2}p, t\right)\right]^\star \tilde{\Psi}^{(\eta)}_{[k]}\left(p'+\frac{1}{2}p, t\right)\mathrm{d}p \\ &= \left(+\frac{i}{\hbar}\right)^\mu\left(-\frac{i}{\hbar}\right)^\eta\int w^\star(q, p')\left(q'-\frac{1}{2}q\right)^\mu \\ &\quad\times\left(q'+\frac{1}{2}q\right)^\eta\left\langle q'+\frac{1}{2}q\mid\hat{\rho}(t)\,|q'-\frac{1}{2}q\right\rangle\mathrm{d}q\end{aligned} \tag{75}$$

and

$$\begin{aligned}\tilde{\mathcal{J}}_{\mu\eta}(q',\ p') &:= \int w^\star\left(q, p'\right)\left[J_\mu\left(q'-\frac{1}{2}q\right)\right]^\star \\ &\quad\times J_\eta\left(q'+\frac{1}{2}q\right)\mathrm{d}q \\ &= \int w(q', p)\left[\tilde{\mathbb{J}}_\mu\left(p'-\frac{1}{2}p\right)\right]^\star \\ &\quad\times\tilde{\mathbb{J}}_\eta\left(p'+\frac{1}{2}p\right)\mathrm{d}p.\end{aligned} \tag{76}$$

For getting the last equality, one uses (32) to express $J_\mu(q'-\frac{1}{2}q)$ and $J_\eta(q'+\frac{1}{2}q)$ as Fourier transforms of $\tilde{\mathbb{J}}_\mu(y)$ and $\tilde{\mathbb{J}}_\eta(Y)$, performs the integration over $q$ and reorganizes the resulting integral over $y$, introducing a change of variable of the form $\frac{1}{2}p = p' - y$.

Using the expression

$$\delta(x-x') = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} w(x-x', y)\mathrm{d}y$$

for the integral representation of the Dirac delta, it follows that

$$\frac{1}{(2\pi\hbar)^f}\int\rho_{\mu\eta}(q', p', t)\mathrm{d}q' = \sum_{k=1}^{\mathcal{N}} W_{[k]}\left[\tilde{\Psi}^{(\mu)}_{[k]}\left(p', t\right)\right]^\star \times\tilde{\Psi}^{(\eta)}_{[k]}(p', t) \tag{77}$$

and

$$\frac{1}{(2\pi\hbar)^f}\int\rho_{\mu\eta}\left(q',\ p',\ t\right)\mathrm{d}p' = (+i)^{\mu-\eta}\left(\frac{q'}{\hbar}\right)^{\mu+\eta} \times\langle q'|\,\hat{\rho}(t)|q'\rangle. \tag{78}$$

Similarly, one finds that

$$\frac{1}{(2\pi\hbar)^f}\int \tilde{\mathcal{J}}_{\mu\eta}\left(q',\ p'\right)\mathrm{d}p' = [J_\mu(q')]^\star J_\eta(q') \tag{79}$$

and

$$\frac{1}{(2\pi\hbar)^f}\int \tilde{\mathcal{J}}_{\mu\eta}(q',\ p')\mathrm{d}q' = [\tilde{\mathbb{J}}_\mu(p')]^\star\, \tilde{\mathbb{J}}_\eta(p'). \tag{80}$$

### 6.2 *Structure of $W(\theta|\,q',p',t)$, when $\theta = 1$*

Similarly, from (68), (63) and (60), the function $W(1|\,q',p',t)$ can be written as

$$W(1|q',p',t) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\tilde{\rho}_{\mu\eta}\left(q',p',t\right)\mathcal{J}_{\mu\eta}\left(q',p'\right), \tag{81}$$

with (see Appendix A)

$$\begin{aligned}\tilde{\rho}_{\mu\eta}(q',p',t) &:= \sum_{k=1}^{\mathcal{N}} W_{[k]}\int w^\star(q,p')\left[\Psi^{(\mu)}_{[k]}\left(q'-\frac{1}{2}q,t\right)\right]^\star \\ &\quad\times\Psi^{(\eta)}_{[k]}\left(q'+\frac{1}{2}q,t\right)\mathrm{d}q \\ &= \left(-\frac{i}{\hbar}\right)^\mu\left(+\frac{i}{\hbar}\right)^\eta\int w(q',p)\left(p'-\frac{1}{2}p\right)^\mu \\ &\quad\times\left(p'+\frac{1}{2}p\right)^\eta\left\langle p'+\frac{1}{2}p|\,\hat{\rho}(t)\,|p' \right.\\ &\quad\left. -\frac{1}{2}p\right\rangle\mathrm{d}p \end{aligned} \tag{82}$$

and

$$\begin{aligned}\mathcal{J}_{\mu\eta}(q',p') &:= \int w\left(q',\ p\right)\left[\tilde{J}_\mu\left(p'-\frac{1}{2}p\right)\right]^\star \\ &\quad\times\tilde{J}_\eta\left(p'+\frac{1}{2}p\right)\mathrm{d}p \\ &= \int w^\star(q,p')\left[\mathbb{J}_\mu\left(q'-\frac{1}{2}q\right)\right]^\star \\ &\quad\times\mathbb{J}_\eta\left(q'+\frac{1}{2}q\right)\mathrm{d}q. \end{aligned} \tag{83}$$

Equation (82) implies that

$$\frac{1}{(2\pi\hbar)^f}\int\tilde{\rho}_{\mu\eta}(q',p',t)\mathrm{d}p' = \sum_{k=1}^{\mathcal{N}} W_{[k]}\left[\Psi^{(\mu)}_{[k]}(q',t)\right]^\star \times\Psi^{(\eta)}_{[k]}(q',t) \tag{84}$$

and

$$\frac{1}{(2\pi\hbar)^f}\int\tilde{\rho}_{\mu\eta}(q',p',t)\mathrm{d}q' = (-i)^{\mu-\eta}\left(\frac{p'}{\hbar}\right)^{\mu+\eta}\times\langle p'|\,\hat{\rho}(t)|p'\rangle. \tag{85}$$

Similarly, from (83), the following properties of $\mathcal{J}_{\mu\eta}(q',p')$ can be derived:

$$\frac{1}{(2\pi\hbar)^f}\int\mathcal{J}_{\mu\eta}(q',p')\mathrm{d}q' = [\tilde{J}_\mu(p')]^\star\tilde{J}_\eta(p') \tag{86}$$

and

$$\frac{1}{(2\pi\hbar)^f}\int\mathcal{J}_{\mu\eta}(q',p')\mathrm{d}p' = [\mathbb{J}_\mu(q')]^\star\mathbb{J}_\eta(q'). \tag{87}$$

### 6.3 *Reorganization of $W(\theta|q',p',t)$*

From (12), one notes that $|p,q\rangle = w(q,p)\,|q,p\rangle$ and, therefore, the matrix elements involved in (57) are related as

$$\begin{aligned}\langle p_b,q_b|\hat{\rho}(t)|p_a,q_a\rangle &= w^\star(q_b,p_b)w(q_a,p_a) \\ &\quad\times\langle q_b,p_b|\hat{\rho}(t)|q_a,p_a\rangle.\end{aligned} \tag{88}$$

Consequently, using (60) and (63) one may write

$$\begin{aligned}\rho(0,q',p'|q,p,t) &= w^\star(q,p')w^\star(q',p) \\ &\quad\times\rho(1,q',p'|q,p,t).\end{aligned} \tag{89}$$

Therefore, the integrands of $W(\theta|\,q',p',t)$ in the left hand sides of (74) and (81) can be written as $w^\star(q,2p')\rho(1,q',p'|q,p,t)$ and $w(2q',p)$ $\rho(0,q',p'|q,p,t)$, respectively. Thus, inserting (68) and (67) into the integrals $W(\theta|q',p',t)$, one finds

$$W(0|q',p',t) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\tilde{\rho}_{\mu\eta}(q',2p',t)\mathcal{J}_{\mu\eta}(0,p') \tag{90}$$

and

$$W(1|q',p',t) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\rho_{\mu\eta}(2q',p',t)\tilde{\mathcal{J}}_{\mu\eta}(q',0). \tag{91}$$

A comparison between (74) and (90), and between (81) and (91), shows a change in the value of the function's arguments together with an interchange in the role of the functions, namely: (i) for $\theta = 0$, $\rho_{\mu\eta}(q',\ p',\ t) \leftrightarrow \tilde{\rho}_{\mu\eta}(q',\ 2p',\ t)$ and $\tilde{\mathcal{J}}_{\mu\eta}(q',\ p') \leftrightarrow \mathcal{J}_{\mu\eta}(0,\ p')$ and (ii) for $\theta = 1$, $\tilde{\rho}_{\mu\eta}(q',\ p',\ t) \leftrightarrow \rho_{\mu\eta}(2q',\ p',\ t)$ and $\mathcal{J}_{\mu\eta}(q',\ p') \leftrightarrow \tilde{\mathcal{J}}_{\mu\eta}(q',\ 0)$.

### 6.4 *Some comments on the above defined functions*

At this point one may conclude

(1) In (75) and (82), when $\mu = \eta = 0$, the functions $\rho_{00}(q',\ p',\ t)$ and $\tilde{\rho}_{00}(q', p', t)$ coincide with the standard definition of the Wigner function $\rho_{\mathrm{W}}(q',\ p',\ t)$, given by (1) in the case of a pure state, $|\Psi(t)\rangle$.
(2) Comparing (76) and (83), it follows that the change of the *pq*- to the *qp*-representation (i.e., from $\theta = 0$ to $\theta = 1$) is associated with a change of the roles of the functions $J_m(q)$ and $\mathbb{J}_m(q)$, and $\tilde{\mathbb{J}}_m(p)$ and $\tilde{J}_m(p)$, and vice versa.
(3) The functions $J_m(q)$, $\mathbb{J}_m(q)$, $\tilde{\mathbb{J}}_m(p)$ and $\tilde{J}_m(p)$ have a universal nature, in the sense that they are independent of the particular quantum system under study, except by the number of degrees of freedom, *f*. These functions allow the calculation of $\tilde{\mathcal{J}}_{\mu\eta}(q',\ p')$ and $\mathcal{J}_{\mu\eta}(q',\ p')$, which escort the cross-Wigner transforms, $\rho_{\mu\eta}(q',\ p',\ t)$ and $\tilde{\rho}_{\mu\eta}(q', p', t)$, in the expansions (74), (81), (90) and (91).
(4) The general transforms (75) and (82) belong to the class of the so-called cross-Wigner transforms [36], in this case for all the pairs $(\tilde{\Psi}^{(\mu)}_{[k]}, \tilde{\Psi}^{(\eta)}_{[k]})$ and $(\Psi^{(\mu)}_{[k]}, \Psi^{(\eta)}_{[k]})$ induced by the pure states involved in the density operator $\hat{\rho}(t)$.

According to [36], the physical interpretation of the cross-Wigner transform of a pair $(\phi, \psi)$ is that of an interference term in the Wigner distribution of the sum $\phi+\psi$. From this point of view, in order to get a glimpse of the meaning of the cross-Wigner transforms (75) and (82), let us consider the $k$th pure state in $\hat{\rho}(t)$, the Taylor series expansion

$$\tilde{\Psi}_{[k]}\left(p''+p'\pm\frac{1}{2}p,t\right)=\sum_{\ell=0}^{\infty}\frac{(p'')^{\ell}}{\ell!}\,\tilde{\Psi}^{(\ell)}_{[k]}\left(p'\pm\frac{1}{2}p,t\right), \tag{92}$$

and refer to $\tilde{\Psi}^{(\ell)}_{[k]}(p' \pm \frac{1}{2}p, t)$ as the $\ell$th component of $\tilde{\Psi}_{[k]}(p'' + p' \pm \frac{1}{2}p, t)$ around the point $p''$. These components remain unmodified when the phase-space point $(q'',\ p'')$ moves up or down along the momentum axis $p$ (see figure 1). A similar feature exhibits the components of the position wave function $\Psi_{[k]}(q'' + q' \pm \frac{1}{2}q, t)$, when $(q'',\ p'')$ is moved left or right along the position axis.

When the components $\tilde{\Psi}^{(\mu)}_{[k]}(p' - \frac{1}{2}p, t)$ and $\tilde{\Psi}^{(\eta)}_{[k]}(p' + \frac{1}{2}p, t)$ overlap, at a given time $t$, they give rise to an interference phenomenon and, because each pure state has an associated weight $W_{[k]}$, the function $\rho_{\mu\eta}(q',\ p',\ t)$ given in (75) may be interpreted as a weighted measure of the interference pattern generated at the point $(q',\ p')$ and time $t$ by the cumulative contributions of all $\mathcal{N}$ pure states in the mixture $\hat{\rho}(t)$. Using (92), one can calculate the average on the statistical mixture of states, namely:

$$\sum_{k=1}^{\mathcal{N}} W_{[k]}|\tilde{\Psi}_{[k]}\left(p''+p',t\right)|^2 = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{(p'')^{\mu+\eta}}{\mu!\,\eta!}\left(\sum_{k=1}^{\mathcal{N}} W_{[k]}[\tilde{\Psi}^{(\mu)}_{[k]}\left(p',t\right)]^{\star} \times \tilde{\Psi}^{(\eta)}_{[k]}(p',t)\right). \tag{93}$$

The sum inside the parenthesis is nothing but the right side of expression (77), i.e., the product of the probability amplitudes for observing the system at time $t$ with momentum $p'$, due to the components $\tilde{\Psi}^{(\mu)}_{[k]}(p', t)$ and $\tilde{\Psi}^{(\eta)}_{[k]}(p', t)$ of the momentum wave function $\tilde{\Psi}_{[k]}(p'' + p', t)$, averaged over the statistical mixture of states.

In complete analogy to the above process with the momentum wave function one deals with the position wave function $\Psi_{[k]}(q'' + q' \pm \frac{1}{2}q, t)$, the cross-Wigner function $\tilde{\rho}_{\mu\eta}\left(q', p', t\right)$ given by (82) and the expression (84).

## 7. Summary and conclusion

As a general conclusion, the phase-space formulation of quantum mechanics based on the Glauber coherent states constitutes an appropriate and consistent method of quantum theory, and it is able to bridge the formulations *à la* Bargmann and *à la* Wigner summarized in §1. When the system is in a pure state then, in similarity to the position and momentum wave functions of the Schrödinger picture, the proposed method uses the probability amplitudes (phase-space wave functions) $\Psi_{\pm}(q, p, t)$ defined by (7):

(1) The functions $\Psi_{\pm}(q, p, t)$ are closely related to those of the Bargmann formulation of quantum mechanics, but they are not elements of the Bargmann space of the entire analytic functions.
(2) When a suitable integration is carried out, the functions $\Psi_{\pm}(q, p, t)$ give the position and momentum probability amplitudes together with the derivatives of the wave functions, $\Psi^{(\eta)}(q, t)$ and $\tilde{\Psi}^{(\eta)}(p, t)$.

(3) As a consequence of working with the wave functions $\Psi_{\pm}(q, p, t)$, it is possible to get the mapping of quantum operators in the *pq* and *qp* representations (table 1).
(4) The functions $\Psi(\theta|\, q, p, t)$ allow the formulation of the Schrödinger equation in the phase-space in terms of the $pq(\theta = 0)$ and $qp(\theta = 1)$ representations. The resulting equations describe the evolution of probability amplitudes in a form having high similarity with the usual coordinate and momentum representations in quantum mechanics.

When the system is in a mixed state described by the density operator $\hat{\rho}(t)$, the method is easily generalized in such a way that the state of the system is represented by the function $\rho(\theta|\, q_a, p_a, q_b, p_b, t)$ defined in (58), which depends on two sets of variables, $(q_a, p_a)$ and $(q_b, p_b)$, unlike the standard position representation in which the matrix element $\langle q''|\hat{\rho}(t)|q'\rangle$ relies on two position variables. In this case,

(1) The quantum Liouville equation is formulated in terms of the *pq* and *qp* representations, eq. (59), in terms of a two-point function in phase-space, namely: $(q_a, p_a)$ and $(q_b, p_b)$.
(2) By using the transforms (60), the state of the system can be represented by a reduced function $\rho(\theta, q', p'|q, p, t)$, defined in (63), which is a one-point function in phase-space: $(q, p)$ are independent variables and $(q', p')$ is a parametric point.
(3) The quantum Liouville equation is reformulated in terms of the reduced function $\rho(\theta, q', p'|\, q, p, t)$, eq. (64).
(4) The relationship between the method in this work and the Wigner function is shown by using the fact that the *pq* and *qp* phase-space representations of the density operator are given by novel expansions involving the position and momentum wave functions and their derivatives, as seen in expansions (67)–(68) and in §6. From these expansions, we can see that the Wigner function arises in a very natural way by means of the double Fourier transformation described in §6.

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## Appendix A. Proof of (75) and (82)

To demonstrate the second equality in (75), consider a pure state described by the momentum wave function $\tilde{\Psi}(p, t)$. The Fourier transform of the $\mu$th partial derivative of $\tilde{\Psi}(p, t)$ with respect to $p$ is denoted as $\Phi_\mu(q, t)$ and given by

$$\Phi_\mu(q, t) := (2\pi\hbar)^{-f/2} \int w(q, p)\tilde{\Psi}^{(\mu)}(p, t)\,\mathrm{d}p = \left(-\frac{i}{\hbar}q\right)^{\mu} \Psi(q, t), \tag{A.1}$$

which is simply related to the position wave function $\Psi(q, t)$. Using the inverse Fourier transform of (A.1), the functions $\tilde{\Psi}^{(\mu)}(p' - \frac{1}{2}p, t)$ and $\tilde{\Psi}^{(\eta)}(p' + \frac{1}{2}p, t)$ are written as Fourier transforms of $\Phi_\mu(x, t)$ and $\Phi_\eta(X, t)$, and inserted in (75), in the definition of $\rho_{\mu\eta}(q', p', t)$. One performs the integration over $p$ by invoking the integral representation of the Dirac delta function, and reorganizes the resulting integral over $x$ by introducing a change of variable of the form $\frac{1}{2}q = q' - x$ (see figure 1). Afterwards, the second equality in (A.1) is used, and the resulting expression is reorganized by means of the identity

$$\Psi^{\star}\left(q' - \frac{1}{2}q, t\right)\Psi\left(q' + \frac{1}{2}q, t\right) = \left\langle q' + \frac{1}{2}q \,\middle|\, \Psi(t)\right\rangle \times \left\langle \Psi(t) \,\middle|\, q' - \frac{1}{2}q\right\rangle. \tag{A.2}$$

Finally, by considering all the pure states involved in the density operator $\hat{\rho}(t)$, given by (55), one obtains the second equality in (75).

A similar procedure allows to demonstrate the second equality in (82). Instead of (A.1), one now uses the expression

$$\tilde{\Phi}_\mu(p, t) := (2\pi\hbar)^{-f/2} \int w^{\star}(q, p)\Psi^{(\mu)}(q, t)\,\mathrm{d}q = \left(+\frac{i}{\hbar}p\right)^{\mu} \tilde{\Psi}(p, t) \tag{A.3}$$

and the identity

$$\tilde{\Psi}^{\star}\left(p' - \frac{1}{2}p, t\right)\tilde{\Psi}\left(p' + \frac{1}{2}p, t\right) = \left\langle p' + \frac{1}{2}p \,\middle|\, \Psi(t)\right\rangle \times \left\langle \Psi(t) \,\middle|\, p' - \frac{1}{2}p\right\rangle. \tag{A.4}$$

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