Pramana – J. Phys. (2016) 87: 23
DOI 10.1007/s12043-016-1241-7

A Bohmian approach to the perturbations of non-linear Klein–Gordon equation

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MS received 20 March 2015; revised 14 October 2015; accepted 18 November 2015; published online 13 July 2016

Abstract. In the framework of Bohmian quantum mechanics, the Klein–Gordon equation can be seen as representing a particle with mass m which is guided by a guiding wave $\phi(x)$ in a causal manner. Here a relevant question is whether Bohmian quantum mechanics is applicable to a non-linear Klein–Gordon equation? We examine this approach for $\phi^4(x)$ and sine-Gordon potentials. It turns out that this method leads to equations for quantum states which are identical to those derived by field theoretical methods used for quantum solitons. Moreover, the quantum force exerted on the particle can be determined. This method can be used for other non-linear potentials as well.

Keywords. Bohmian quantum mechanics; quantum potential; soliton; scalar field theory.

PACS Nos 03.65.Ta; 03.65.Db; 03.65.Sq; 45.20.Jj; 03.65.Ge

1. Introduction

According to the quantum field theory, all fermions and bosons are considered as fields. This theory is an extension of quantum mechanics in which instead of considering dynamical variables like energy, momentum and position as operators, the field components are considered as operators. In other words, the quantum field theory is the limit of the quantum mechanics when the degrees of freedom approach infinity. In this theory, particles are treated as excited states of a field.

In classical field theory there are special solutions of fields that have particle-like behaviour. They are called solitary waves. A solitary wave maintains its shape. Its energy is localized in space and time. The solitary waves that maintain their shape in collisions, are called solitons. The usual method used for studying quantum solitons is the semiclassical method, for which we shall have a short review here [1–3].

Bohmian quantum mechanics is a deterministic theory that describes atomic world in a causal manner. We shall now explain a little about non-relativistic and relativistic Bohmian mechanics. In non-relativistic Bohmian quantum mechanics $\psi(\mathbf{x}, t)$ is a guiding wave that has a real origin and guides particles on trajectories. In fact, in this theory, unlike the standard quantum mechanics, we attribute dynamical variables to a particle, like the case of classical mechanics. The only difference is that here particles move on trajectories due to quantum potential.

In Bohmian quantum mechanics, we believe in a real fundamental object, which can be considered as a particle (particle ontology) or a field (field ontology). This theory does not have a satisfactory structure for fermions, in the relativistic domain, yet. In other words, there is no comprehensive Bohmian quantum field theory, due to some conceptual and technical problems. Generally, a particle is guided by the wave $\psi(\mathbf{x}, t)$ and a field by a superwave function or functional $\Psi[\phi(\mathbf{x}), t]$ [4–6].

We want to follow a particle ontology according to the relativistic Bohmian quantum mechanics. In other words, in quantum domain, we consider a particle that is guided by a relativistic wave function in the presence of a non-linear potential. In Bohmian relativistic quantum mechanics, there are difficulties to define a trajectory for a single particle. The 4-current of a particle is not always time-like and changes its sign at some region of space–time. It may also vanish at some points in space–time. We have seen such inconsistencies in the usual relativistic quantum mechanics. Now, we are assuming the Bohmian ansatz $\phi(x^\mu) = R(x^\mu)\mathrm{e}^{is(x^\mu)}$ for the relativistic wave function in spite of difficulties in the single-particle interpretation of such a wave function, and we examine perturbative solutions in the case of the $\phi^4(x)$ and sine-Gordon potentials in the light of such ansatz. We shall demonstrate that the eigenvalue equations of the quantum solitons, which are usually derived through a field theoretical approach, can be obtained through the relativistic Bohmian quantum mechanics. In this framework, the quantum potential and the quantum force exerted on a particle in an arbitrary state are determined. We shall see how the classical mass of a particle is modified in the presence of a nonlinear potential. We shall only consider the bound states of a particle.

2. Review of Bohmian quantum mechanics

Bohmian quantum mechanics (BQM), unlike standard quantum mechanics (SQM), is a deterministic theory, i.e., like the case of classical mechanics, the motion of a particle at the instance t causes its motion at $t + \mathrm{d}t$. There is a difference here: the quantum potential, which can be derived from the guiding wave $\psi(\mathbf{x}, t)$, affects the particle, with no classical analogue.

By writing the wave function in a polar form $\psi(\mathbf{x}, t) = R(\mathbf{x}, t)\exp(i\frac{S(\mathbf{x},t)}{\hbar})$ and substituting it in Schrödinger's equation, we obtain the generalized Hamilton–Jacobi equation and the continuity equation:

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}) + Q(\mathbf{x}) = 0, \tag{1}$$

$$\frac{\partial R^2}{\partial t} + \frac{1}{m}\nabla \cdot (R^2\nabla S) = 0. \tag{2}$$

The position of the particle is obtained from the following equation:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \left(\frac{\nabla S(\mathbf{x}, t)}{m}\right)_{\mathbf{X}=\mathbf{x}(t)}, \tag{3}$$

where $\nabla S(\mathbf{x}, t)$ is the momentum of the particle. By knowing the initial position $\mathbf{x}_0$ and wave function $\psi(\mathbf{x}_0, t_0)$, the future of the system is obtained. The expression $\mathbf{X} = \mathbf{x}(t)$ means that among all possible trajectories, in an ensemble of particles, we choose one of them. The probabilistic character of the Bohmian mechanics is due to the environment's effects on the system, which does not allow us to know the initial values with infinite precision.

The quantity Q in (1) is called quantum potential and is given by

$$Q = -\frac{\hbar^2\nabla^2 R(\mathbf{x}, t)}{2mR(\mathbf{x}, t)}. \tag{4}$$

If we multiply the amplitude R, by a real constant factor α, the value of the quantum potential will not change. This means that this potential does not necessarily decrease with distance. This is an indication of the non-local behaviour of the quantum potential (see [4]).

Relativistic Bohmian quantum mechanics: Following Bohm, the substitution of the polar form of the wave function into the Klein–Gordon equation to derive the quantum mechanical Hamilton–Jacobi equation for a relativistic spinless particle, leads us to

$$\partial_\mu S\partial^\mu S = m^2(1 + Q) \equiv M^2, \tag{5}$$

$$Q = \frac{\hbar^2}{m^2}\frac{\Box R}{R}. \tag{6}$$

Equation (5) indicates that the rest mass of the particle, M, is not a constant in the rest frame of the particle; rather, it depends on the quantum potential and consequently on the wave function. The 4-momentum of the particle is obtained from:

$$p^\mu = Mu^\mu = m(\sqrt{1 + Q})u^\mu = -\partial^\mu S \tag{7}$$

and the 4-current of the particle and its continuity equation are

$$j^\mu = -\frac{R^2}{m}\partial^\mu S \quad \text{and} \quad \partial_\mu j^\mu = 0. \tag{8}$$

One problem with this method is that the expression $j^\mu j_\mu$ is not always positive definite, and thus can cause some ambiguities about the trajectories of the particle. Of course, there are prescriptions for removing these ambiguities [7,8].

3. Quantum solitons of ϕ^4 theory through a semiclassical method in (1 + 1) dimensions

The names solitary waves and solitons refer to certain special solutions of non-linear wave equations. The classical solutions of a non-linear wave equation behave like an extended particle. We specially focus on the solutions for ϕ^4 theory. Here, we review this topic shortly [1–3]. Consider a scalar Lagrangian in (1 + 1) dimensions:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U(\phi). \tag{9}$$

We consider ϕ^4 potential as

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2, \tag{10}$$

where λ and m^2 are positive constants and $\pm m/\sqrt{\lambda}$ are degenerate minima of $U(\phi)$ or trivial solutions of the field. The equation of motion is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = m^2\phi - \lambda\phi^3. \tag{11}$$

The static $((\partial^2\phi/\partial t^2) = 0)$ non-trivial solutions of this equation are

$$\phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{mx}{\sqrt{2}}\right). \tag{12}$$

These are kink (antikink) solutions of classical ϕ^4 theory. For these solutions, $\phi(x)$ approaches $\pm m/\sqrt{\lambda}$, when $x \to \pm\infty$. This is a condition which is needed to have localized and particle-like behaviour. Thus, the kink (antikink) solutions and energy density remain localized. The energy density of the kink is localized and is

$$\varepsilon(x) = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi) = \frac{m^4}{2\lambda}\mathrm{sech}^4\left(\frac{mx}{\sqrt{2}}\right). \tag{13}$$

The classical kink mass is given by

$$M_{\mathrm{cl}} = \int_{-\infty}^{+\infty}\varepsilon(x)\mathrm{d}x = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{14}$$

This is the mass of an extended particle.

Kink quantization: Scientists usually consider quantum solutions as perturbations around classical non-trivial solutions. For the kink quantization, they usually expand the functional potential of the field around a classical solution, as a perturbation. In fact, the field is considered as an infinite number of harmonic oscillators with infinite degrees of freedom. Then, they consider perturbations around the stable classical vacuum as quantum solitons. It is done in a field theoretical approach. The Lagrangian of a scalar field is

$$\begin{aligned} L &= \int \mathrm{d}x\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right] \\ &= T[\phi] - U[\phi], \end{aligned} \tag{15}$$

where

$$T[\phi] = \int \mathrm{d}x\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 \tag{16}$$

and

$$V[\phi] = \int \mathrm{d}x\frac{1}{2}[(\nabla\phi)^2 + U(\phi)]. \tag{17}$$

The quantities, $T[\phi]$ and $V[\phi]$, are the kinetic and potential energies of the field. By expanding $V[\phi]$ around the classical field distribution ϕ_0, and considering $(\delta V[\phi]/\delta\phi)_{\phi=\phi_0} = 0$ as a stability condition, we have up to the first approximation

$$\begin{aligned} V[\phi] = {} & V[\phi_0] + \int \mathrm{d}x\frac{1}{2} \\ & \times\left\{\eta(x)\left[-\nabla^2 + \left(\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right)_{\phi_0}\right]\eta(x) + \cdots\right\}, \end{aligned} \tag{18}$$

where $\eta(x) = \phi(x) - \phi_0(x)$. The expression in the brackets is the harmonic oscillator operator. Thus, we have

$$\left[-\nabla^2 + \left(\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right)_{\phi_0}\right]\eta_i(x) = \omega_i^2\eta_i(x) \tag{19}$$

with

$$E_i = V[\phi_0] + \hbar\Sigma_i\left(n_i + \frac{1}{2}\right)\omega_i, \tag{20}$$

where n_i is the excitation number of the ith mode and $V[\phi_0]$ is the classical energy of the kink (antikink). By using relations (10), (12) and (19) for ϕ^4 kink, we get

$$\left[-\frac{\partial^2}{\partial x^2} - m^2 + 3m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)\right]\eta(x)_n = \omega_n^2\eta_n(x). \tag{21}$$

By solving this eigenvalue equation, a series of solutions are obtained and interesting topics for mass renormalization of pions is followed [1].

This argument was based on standard quantum mechanics and field theoretical methods. Now, we want to investigate whether the same results can be obtained through the relativistic Bohmian quantum mechanics? Does the relativistic Bohmian quantum mechanics has the capability for solving such non-linear problems? We do not call what we obtain 'Bohmian solitons', as we are considering a point-particle in a non-linear potential.

4. Using Bohmian approach for the perturbations of ϕ^4 potential

In Bohmian quantum mechanics, we can work in a particle ontology. When we substitute the polar form of the wave function in the Klein–Gordon equation, we are in fact, attributing particle aspect to the equation i.e., a particle which is guided by wave $\phi(x)$. We generalize this for a particle with mass m which is guided by $\phi(x)$, where additional potential ϕ^4 is present.

There are justifications for particle ontology in the relativistic domain of quantum mechanics in the framework of Bohmian approach. The main inconsistency of particle ontology is that the 4-current j^μ of such a particle is not always time-like. So j^0 can take negative values or may vanish at some point in space-time and this leads to superluminal velocities or leads to the creation or annihilation of particles and the concept of Bohmian trajectories is destroyed. But the trajectory problem has justifications, one of which is mentioned in [7].

The Lagrangian (9) in $(1+3)$ dimensions becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\phi^4 + \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\frac{m^4}{\lambda}. \tag{22}$$

So the classical equation of motion becomes

$$\Box\phi - m^2\phi + \lambda\phi^3 = 0. \tag{23}$$

It has some classical solutions or soliton solutions which we represent by ϕ_c. This is not the solution for the free Klein–Gordon equation, rather, these solutions show a particle-like behaviour of a classical field or soliton solutions. The ϕ_c's are the minimums of the nonlinear potential $U(\phi)$ and because of uncertainty principle it is usually said that quantum effects are perturbations around the classical value, i.e., the field is not in its classical minimum state exactly but has small oscillations around it. In standard quantum field theory, in order to solve the above equation with semi-classical methods, people consider quantum effects as small perturbations about classical value, but ϕ is a field, not a wave function [1]. In the Bohmian quantum mechanics, it has been shown that quantum energy comes from quantum potential and this is equal to the quantum energy which is justified by using the uncertainty principle in the standard quantum mechanics [4]. Following this line, we assume that quantum states are perturbations around a classical value. So, we assume that

$$\xi(x) = \phi(x) - \phi_c(x), \tag{24}$$

for using a perturbative method. It may be similar to the previous argument of §3, but notice that we are following Bohm's particle ontology and not the concepts of the field theory. At the classical limit, we treat the equation of motion as a field, like in the case of Klein–Gordon equation, but when we want to investigate quantum mechanical equations, we use particle ontology, instead of field ontology. So, by using eqs (23) and (24), the equation of motion becomes

$$\Box\phi_c - m^2\phi_c + \lambda\phi_c^3 + \Box\xi - m^2\xi + \lambda\xi^3 + 3\lambda\phi_c^2\xi + 3\lambda\phi_c\xi^2 = 0. \tag{25}$$

For quantum purposes, the classical field is discarded, i.e., the first three terms become zero together, because they satisfy the classical motion. After considering only first-order powers of ξ, the equation of motion becomes

$$\Box\xi - m^2\xi + 3\lambda\phi_c^2\xi = 0. \tag{26}$$

This is a wave equation for the wave function ξ which involves a classical functional factor in the perturbation potential. After substituting the Bohmian ansatz $\xi = R(x)e^{is(x)}$ into (26) and separating real and imaginary parts, we have the quantum Hamilton–Jacobi equation

$$\Box R - R\partial^\mu S\partial_\mu S - m^2 R + 3\lambda\phi_c^2 R = 0 \tag{27}$$

and the continuity equation

$$\partial_\mu(R^2\partial^\mu S) = 0, \tag{28}$$

for a particle with mass m. From relation (27), we get quantum Hamilton–Jacobi equation for a particle in a ϕ^4 potential, which is affected by a guiding wave $\phi(x)$.

$$\partial_\mu S\partial^\mu S = E^2 - \mathbf{p}^2 = m^2\left(-1 + \frac{\Box R}{m^2 R} + \frac{3\lambda}{m^2}\phi_c^2\right) = \mathcal{M}^2. \tag{29}$$

For a point-particle in the ϕ^4 potential, with classical solutions (12), the quantum mass in the framework of the Bohmian particle ontology becomes

$$\mathcal{M} = m\left(\frac{\Box R}{m^2 R} - 1 + 3\tanh^2\left(\frac{mx}{\sqrt{2}}\right)\right)^{1/2} = m\left(Q - 1 + 3\tanh^2\left(\frac{mx}{\sqrt{2}}\right)\right)^{1/2}. \tag{30}$$

This is an interesting result which shows how the classical mass of a particle has been modified in a nonlinear equation to give quantum mass. If we consider the trivial vacuum $\phi_c = m/\sqrt{2}$, we get the modified mass:

$$\mathcal{M} = m\left(2 + \frac{\Box R}{m^2 R}\right)^{1/2}. \tag{31}$$

It may be asked why in relation (5), the coefficient of 1 is positive but in (30) it is negative? This is because the Lagrangian (22) is written so that we have suitable classical kink (antikink) solutions. If instead of (22), we work with formal Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{32}$$

then, as this Lagrangian has only classical vacuum $\phi_0 = 0$, we shall only get relation (5). Notice that in the parenthesis of relation (30), we do not have an ordinary summation of the terms, because the amplitude of the wave function (R) and consequently quantum potential (Q) of the system are influenced by the presence of the classical vacuum.

Equations of motion: In Bohmian quantum mechanics, the dynamical variables are not operators in a Hilbert space. They are actual quantities like in classical mechanics. Therefore, a particle can be at rest in a potential well, with its energy being produced by quantum potential.

We are interested in the static solutions, i.e., $(\partial R/\partial t) = 0$. For having bound states, we use the condition $\mathbf{p} = \nabla S = 0$. In other words, the solutions should be real functions. This is a case that has no analogous in the standard quantum mechanics. In fact, a particle can be at rest in an external potential, but having energy due to the quantum potential, while as in the standard quantum mechanics, this energy is justified by uncertainty principle [4,9–11]. We shall see that only in this case ($\mathbf{p} = 0$), we can gain the eigenvalue equations of standard quantum mechanics. By these considerations, eq. (29) becomes

$$E_n^2 R_n(\mathbf{x}) = (-m^2 - \nabla^2 + 3\lambda\phi_c^2)R_n(\mathbf{x}) \tag{33}$$

because for a stationary state we can write $S(\mathbf{x}, t) = W(\mathbf{x}) - Et$, where we have used $\Box = (\partial^2/\partial t^2) - \nabla^2$ and $((\partial R/\partial t) = 0)$. Now, we have a static equation in an arbitrary spatial interval. We consider this equation for simplicity in $(1 + 1)$ dimensions:

$$\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_c^2\right)R_n(\mathbf{x}) = E_n^2 R_n(\mathbf{x}). \tag{34}$$

We conclude that this eigenvalue equation is a special case of a general equation in which $\mathbf{p} \neq 0$. The eigenvalue eq. (34) is the same as that of semiclassical method, i.e., eq. (21) when $E_n = \omega_n$ ($\hbar = 1$); but now it has been obtained through the relativistic Bohmian mechanics.

Quantum potential and the quantum force of states: A stationary quantum state in the Bohmian quantum mechanics deals with a particle which is at rest, with its energy coming from the quantum potential. The quantum force acting on a particle in a potential ϕ^4 can be obtained. The solutions of eq. (34) or (21) are [1]:

$$R_0 = \frac{1}{\cosh^2(mx/\sqrt{2})}, \quad \mathcal{M}_{\text{static},0} = 0, \quad n = 0, \tag{35}$$

$$R_1 = \frac{\sinh(mx/\sqrt{2})}{\cosh^2(mx/\sqrt{2})}, \quad \mathcal{M}_{\text{static},1} = \sqrt{\frac{3}{2}}m, \quad n = 1 \tag{36}$$

and

$$R_q = \mathrm{e}^{iq(mx/\sqrt{2})}\left(3\tanh^2\left(\frac{mx}{\sqrt{2}}\right) - 1 - q^2 - 3iq\tanh\left(\frac{mx}{\sqrt{2}}\right)\right),$$
$$\mathcal{M}_{\text{static},q} = m\left(\frac{1}{2}q^2 + 2\right)^{1/2}, \quad n \geq 2. \tag{37}$$

In the above equations,

$$\mathcal{M}_{\text{static},n} = \mathcal{M}((\partial R/\partial t) = 0, R = R_n(x)).$$

According to what we mentioned, the condition $\mathbf{p} = \nabla S = 0$ is necessary. For solutions (35) and (36), bound-state condition is satisfied because the solutions are real and $S(\mathbf{x}) = 0$. But the solutions (37) are complex functions. Thus, we should consider the complete form of (37). For this purpose, we should add negative-momentum solutions to (37) for having a net zero momentum [10]. Thus, a real solution can be considered as

$$R_q^{\text{real}} = \frac{1}{2}(R_q + R_q^*) \tag{38}$$

or

$$R_q^{\text{real}} = \frac{1}{2i}(R_q - R_q^*). \tag{39}$$

With some calculations it can be shown that the quantum potentials for R_q, R_q^* and R_q^{real} are equal.

The quantum potentials for these solutions, in $(1+1)$ dimensions, are

$$Q_{\text{static},0} = -\frac{\nabla^2 R_0}{m^2 R_0} = 1 - 3\tanh^2\left(\frac{mx}{\sqrt{2}}\right), \tag{40}$$

$$Q_{\text{static},1} = -\frac{\nabla^2 R_1}{m^2 R_1} = \frac{5}{2} - 3\tanh^2\left(\frac{mx}{\sqrt{2}}\right) \tag{41}$$

and

$$Q_{\text{static},q} = -\frac{\nabla^2 R_q}{m^2 R_q} = \frac{q^2}{2} + 3 - 3\tanh^2\left(\frac{mx}{\sqrt{2}}\right). \tag{42}$$

If we substitute, for example $Q_{\text{static},1}$ in relation (30), we find that the x-dependent part of $Q_{\text{static},1}$ cancels out the classical part in (30) and only the quantum part remains. This is true for all states. In other words, the quantum energy totally comes from the quantum potential (figures 1 and 2).

We can obtain the quantum force for an arbitrary case. For example, for the state $n = 1$, it is

$$f_{\text{static},1} = -\frac{\mathrm{d}Q_{\text{static},1}}{\mathrm{d}x} = \frac{3\sqrt{2}\sinh^2(mx/\sqrt{2})}{\cosh^3(mx/\sqrt{2})}. \tag{43}$$

The quantum potential for $n = 0$ and $n = 1$ depends on m and for $n \geq 2$ in addition to m, it also depends on q (wave number), but the quantum force in this case is also independent of q (figure 3).

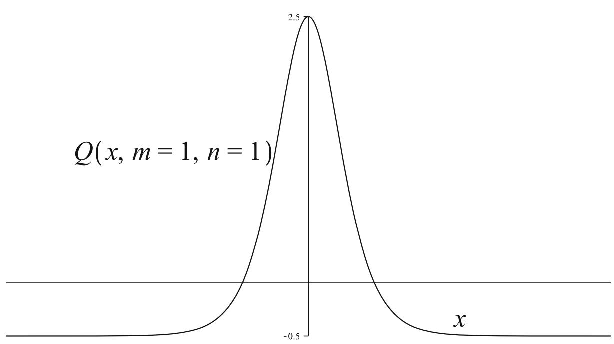

Figure 1. The quantum potential for the state $n = 1$ with $m = 1$.

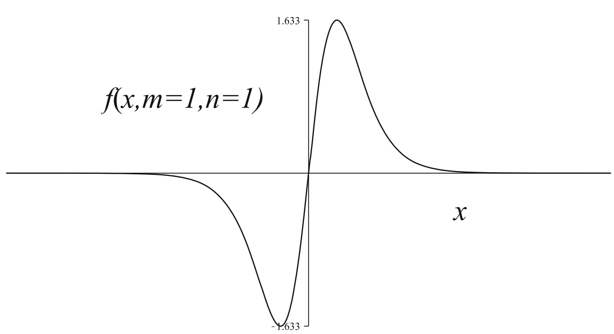

Figure 2. The quantum force for the state $n = 1$ with $m = 1$.

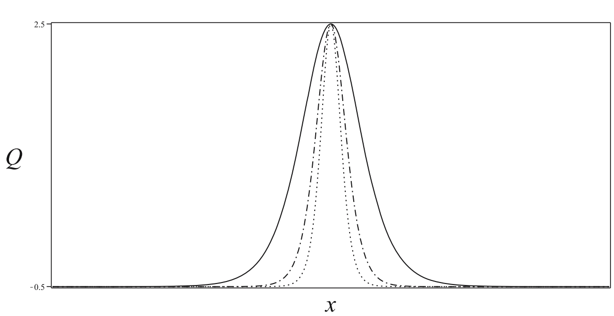

Figure 3. The behaviour of the quantum potential for the state $n = 1$ with increasing classical rest mass.

We can depict the quantum potential for an arbitrary state, for example $n = 1$, with different masses. But in this approach, as the classical potential depends on mass implicitly, so this special feature also affects the behaviour of quantum potential; because, as the rest mass of the particle increases, the width of the potential decreases.

This method can be used for different non-linear potentials. For example, the study of quantum perturbations of sine-Gordon solitons around their classical values, using Bohmian approach, can be interesting. In the following section, we use our method for deriving eigenvalue equations of quantum sine-Gordon equation through the relativistic Bohmian quantum mechanics.

5. Using Bohmian approach for the purturbations of sine-Gordon equation

The classical Lagrangian density of sine-Gordon equation is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^4}{\lambda}\left\{\cos\left[\left(\frac{\sqrt{\lambda}}{m}\right)\phi\right] - 1\right\}, \tag{44}$$

where

$$U(\phi) = \frac{m^4}{\lambda}\left\{1 - \cos\left[\left(\frac{\sqrt{\lambda}}{m}\right)\phi\right]\right\} \tag{45}$$

is the potential of the field. The equation of motion is

$$\Box\phi + \frac{m^3}{\sqrt{\lambda}}\sin\left[\left(\frac{\sqrt{\lambda}}{m}\right)\phi\right] = 0. \tag{46}$$

This equation has classical solutions which lay in topological sectors. Hence, these are called topological solitons [1]. The static solutions are

$$\phi(x) = \pm 4\frac{m}{\sqrt{\lambda}}\tan^{-1}[\exp(mx)]. \tag{47}$$

The solution with positive sign is called soliton and that with minus sign is called antisoliton. These soliton and antisoliton have moving solutions

$$\phi(x) = \pm 4\frac{m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(\frac{m(x-ut)}{\sqrt{1-u^2}}\right)\right], \tag{48}$$

where u is the velocity of a reference frame.

In addition to these solutions, there are exact solutions which vary with time and are not static even in their rest frame. These solutions are, in fact, the result of non-linear superposition of single solitons.

$$\phi_{SA}(x) = 4\frac{m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{\sinh\left(\frac{mut}{\sqrt{1-u^2}}\right)}{u\cosh\left(\frac{mx}{\sqrt{1-u^2}}\right)}\right) \tag{49}$$

and

$$\phi_{SS}(x) = 4\frac{m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{u\sinh\left(\frac{mx}{\sqrt{1-u^2}}\right)}{\cosh\left(\frac{mut}{\sqrt{1-u^2}}\right)}\right). \tag{50}$$

If we convert the real parameter u in $\phi_{SA}(x)$ to an imaginary one by $u \to iv$, we shall have the solution

$$\phi_v(x) = 4\frac{m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{\sin\left(\frac{mvt}{\sqrt{1+v^2}}\right)}{v\cosh\left(\frac{mx}{\sqrt{1+v^2}}\right)}\right). \tag{51}$$

This can be interpreted as a bound solution of a soliton–antisoliton pair and is called breather. We shall not discuss these solutions and their properties here. We return to our discussion, i.e., the application of Bohm's approach for such non-linear equations.

Similar to §3, we should linearize the classical equation of motion around ϕ_c. The classical equation is

$$\Box\phi(x) + \frac{\delta U[\phi(x)]}{\delta\phi(x)} = 0. \tag{52}$$

By differentiation from (45) with respect to ϕ, we get

$$\begin{aligned}\frac{\delta U[\phi(x)]}{\delta\phi(x)} &= \frac{m^4}{\lambda}\frac{\sqrt{\lambda}}{m}\left(\sin\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right)\\ &= \frac{m^4}{\lambda}\frac{\sqrt{\lambda}}{m}\left(\frac{\sqrt{\lambda}}{m}\phi - \frac{(\sqrt{\lambda})^3}{3!m^3}\phi^3\right.\\ &\qquad\left. + \frac{(\sqrt{\lambda})^5}{5!m^5}\phi^5 - \cdots\right).\end{aligned} \tag{53}$$

Now, we evaluate this relation at $\phi = \phi_c + \xi$ and hold only the first-order powers of ξ,

$$\begin{aligned}&\frac{\delta U[\phi(x)]}{\delta\phi(x)} - (\text{classical parts})\\ &\simeq m^2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)^{2n}\right)\xi(x)\\ &= m^2\cos\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)\xi(x).\end{aligned} \tag{54}$$

Now, the equation of motion for the perturbations of the field becomes

$$\left(\Box + m^2\cos\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)\right)\xi(x) = 0. \tag{55}$$

By substituting the polar form of $\xi(x)$ in eq. (55), we get the quantum Hamilton–Jacobi equation of the particle (figure 4)

$$\begin{aligned}\partial_\mu S\partial^\mu S &= m^2\left(\frac{\Box R}{m^2 R} + \cos\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)\right)\\ &= E^2 - \mathbf{p}^2 = \mathcal{M}^2.\end{aligned} \tag{56}$$

The quantum mass of the particle, in this case, is given by

$$\mathcal{M} = m\left(\frac{\Box R}{m^2 R} + \cos\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)\right)^{1/2}. \tag{57}$$

For a particle at rest, ($\mathbf{p} = 0$), from eq. (56), we get the eigenvalue equation for the static solutions ($\partial R/\partial t = 0$) in $(1+1)$ dimensions:

$$\left(-\frac{\partial^2}{\partial x^2} + m^2\cos\left(\frac{\sqrt{\lambda}\phi_c}{m}\right)\right)R_n(x) = E_n^2 R_n(x). \tag{58}$$

This is the same eigenvalue equation which was derived by field theoretical methods in standard approach for the quantum sine-Gordon solitons [1]. One can solve this equation and obtain its solution. Then by using those solutions, quantum potential and quantum force for different states R_n are obtained; like the case we had for ϕ^4 theory.

The eigenvalue equation for a breather solution is also determined; but in that case, the term $\partial^2 R/\partial t^2$ does not vanish in relation (56). It has been argued

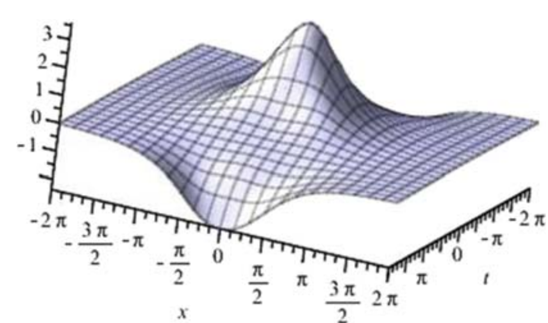

Figure 4. The ground state of the quantum breather for $m = 1$ and $v = 0.5$.

in [1] that the ground state of a quantum breather ($\omega_0 = 0$) is given by

$$\xi_0 = \frac{\partial \phi_v}{\partial t} = \frac{4m^2}{\sqrt{\lambda}\sqrt{v^2+1}} \left(\frac{\cos\left(\frac{mvt}{\sqrt{v^2+1}}\right)}{\cos\left(\frac{mx}{\sqrt{v^2+1}}\right)} \right) \times \left(1 + \frac{\sin^2\left(\frac{mvt}{\sqrt{v^2+1}}\right)}{v^2\left(\cosh^2\left(\frac{mx}{\sqrt{v^2+1}}\right)\right)} \right)^{-1}. \quad (59)$$

We calculated the quantum potential ($Q = (\Box\xi_0/m^2\xi_0)$) and the quantum force of the ground state ($f = -(\mathrm{d}Q/\mathrm{d}x)$). After substituting quantum potential of the ground state in relation (57), we found that $\mathcal{M} = 0$ for the ground state and this shows the validity of our method. Here, we depict the ground state of the quantum breather and its quantum potential and quantum force for arbitrary values $m = 1$ and $v = 0.5$. Figures 5 and 6 give a good understanding of what happens to the particle in a non-linear field.

Our aim from this discussion is not to get a differential equation and its solutions. Rather, we want to demonstrate that it is possible to have a different view of quantum perturbations through a deterministic theory

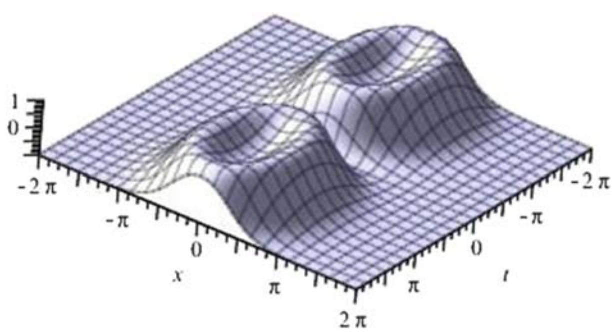

Figure 5. The quantum potential of the ground state for $m = 1$ and $v = 0.5$.

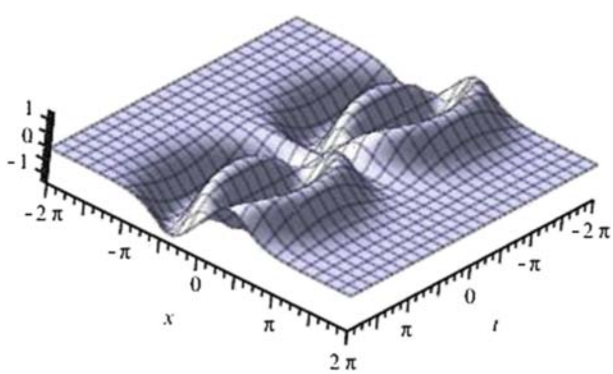

Figure 6. The quantum force exerted on a particle when it is in ground state for $m = 1$ and $v = 0.5$.

like Bohm's quantum theory. These results are obtainable in a deterministic theory which is based on the existence of reality and causality, like in classical mechanics. In this way, we have used the particle ontology of Bohm's theory, i.e., we have considered a deterministic relativistic quantum mechanics, and not the Bohmian scalar field theory or standard quantum field theory. Scientists usually abandon particle description of relativistic quantum mechanics and use quantum field theory to overcome the problems of relativistic quantum mechanics such as creation or annihilation of particles, but here our aim was simply to apply Bohm's theory for the non-linear Klein–Gordon equation. These arguments show that Bohmian approach for relativistic quantum mechanics is applicable for dealing with quantum behaviour of non-linear fields.

6. Conclusion

We have argued that the Bohm's particle ontology can be used for a non-linear Klein–Gordon equation. However, we could also use functional methods in the framework of Bohmian scalar field theory to deal with this issue. But we are interested in the system 'particle+wave' more than a field for which excited states are associated with particles. We derived static equations for evolution of quantum states of ϕ^4 and sine-Gordon theory through the relativistic Bohmian quantum mechanics, and not through a field theoretical one. The results are compatible with those of the standard quantum field theory but here the quantum force exerted on a particle and its quantum potential is determinable and this gives a more ontological understanding than the standard quantum field theory. Obtaining quantum potential and quantum force for different states of a non-linear equation specially for sine-Gordon equation can be interesting and useful to study the behaviour of quantum solitons in the framework of Bohmian quantum mechanics. Our method unfolds a deterministic approach for dealing with problems in the domain of non-linear equations.

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