*Pramana – J. Phys.* (2016) 87: 18
DOI 10.1007/s12043-016-1227-5


# First integrals and analytical solutions of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient

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MS received 7 February 2015; revised 21 May 2015; accepted 20 October 2015; published online 6 July 2016

**Abstract.** Fin materials can be observed in a variety of engineering applications. They are used to ease the dissipation of heat from a heated wall to the surrounding environment. In this work, we consider a nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient. The equation(s) under study are highly nonlinear. Both the thermal conductivity and the heat transfer coefficient are given as arbitrary functions of temperature. Firstly, we consider the Lie group analysis for different cases of thermal conductivity and the heat transfer coefficients. These classifications are obtained from the Lie group analysis. Then, the first integrals of the nonlinear straight fin problem are constructed by three methods, namely, Noether's classical method, partial Noether approach and Ibragimov's nonlocal conservation method. Some exact analytical solutions are also constructed. The obtained result is also compared with the result obtained by other methods.

**Keywords.** Fin equation; Lie symmetry; first integrals; exact solutions.

**PACS Nos 02.20.Tw; 02.30.Hq**

## 1. Introduction

It is well known that, to obtain physical meanings of the equation considered, conservation laws which has familiar physical laws such as energy, momentum, Hamiltonian, are the key instruments. Conservation laws are used for the numerical schemas and for mathematical analysis, in particular, of existence, uniqueness and Lyapunov stability. In addition, they can give rise to some new integrable systems via reciprocal transformations [1]. The conserved quantity is called first integral, which is analogous to models of conservation laws for ordinary differential equation (ODE) [2].

As stated in [3,4], in the engineering applications, fins are used to facilitate the dissipation of heat from a heated wall to the surrounding environment. They are used in diverse fields such as air conditioning, air-cooled craft engines, refrigeration, cooling of computer processors, cooling of oil carrying pipe line, and so on. The heat conducted through a fin material is removed via convective and/or radiative processes. In the field of heat transfer, an analysis of this conduction– convection/radiation system is quite important because of its practical importance and many studies on fin analysis have been done.

Much effort has been made on the construction of exact and numerical solutions of the fin equation. In cases of constant heat transfer coefficient and constant thermal conductivity, the analytical solution can be easily obtained. Nowadays, many studies have been conducted to obtain analytical and numerical solutions of the nonlinear fin problem with a temperature-dependent thermal conductivity and/or heat transfer coefficient. Due to the nonlinearity of the problem, it is not easy to obtain exact solutions. Nevertheless, this problem has been studied by using various analytical and numerical methods, such as perturbation method [5], Adomian decomposition method [6], variational iteration method [7] and Lie group analysis [8,9].

In this work, first integrals of the nonlinear fin with a temperature-dependent heat transfer coefficient and a temperature-dependent thermal conductivity [10] and [11] with respect to Lie group classification are proposed. In §2, we present some fundamental definitions and operators of Lie symmetry method for ODEs of the Euler–Lagrange operator. Then, we present the

Noether's [12], partial Noether's [13] and non-local conservation methods [14–17] for obtaining the first integral of ODEs.

In §3, we present the symmetry analysis, first integrals and exact analytical solutions of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient by the aforementioned three methods. Concluding remarks are summarized in §4.

## 2. Preliminaries

We first present notation to be used and recall the definitions that appear in [2,13,14,18].

Consider a second-order ODE

$$E(x, y, y', y'') = 0, \tag{1}$$

where $x$ is an independent variable and $y$ is a dependent variable.

The total derivative with respect to $x$ is

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots . \tag{2}$$

We shall use the following basic operators defined in $\mathcal{A}$, the vector space of differential functions.

The Lie–Bäcklund operator $X$ is defined as

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \sum_{s\geq 1}\zeta_s\frac{\partial}{\partial y^{(s)}}, \tag{3}$$

where

$$\zeta_s = D_x(\zeta_{s-1}) - y^{(s)}D_x(\xi), \qquad s \geq 1 \tag{4}$$

in which $\zeta_0 = \eta$.

The Euler operator is given by

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \sum_{s\geq 1}(-D_x)^s\frac{\partial}{\partial^{(s)}}. \tag{5}$$

The Noether operator associated with a Lie–Bäcklund operator $X$ is

$$N = \xi + W\frac{\delta}{\delta y'} + \sum_{s\geq 1} D_x^s(W)\frac{\delta}{\delta y^{(s+1)}}, \tag{6}$$

where $W$ is the Lie characteristic function and is defined by

$$W = \eta - \xi y' \tag{7}$$

and

$$\frac{\delta}{\delta y'} = \frac{\partial}{\partial y'} + \sum_{s\geq 1}(-D_x^s)\frac{\partial}{\partial y^{(s+1)}}. \tag{8}$$

A first integral of eq. (1) is a differential function $I \in \mathcal{A}$, such that

$$D_x I \underset{E=0}{|} = 0. \tag{9}$$

We use the following propositon for obtaining the classical Lagrangians of second-order ODEs.

PROPOSITION 1

*The equation of motion*

$$y'' + a(x, y)y'^2 + b(x, y)y' + c(x, y) = 0 \tag{10}$$

*admits standard Lagrangians*

$$L = \frac{1}{2}P(x, y)y'^2 + Q(x, y)y' + R(x, y) \tag{11}$$

*if and only if*

$$b_y = 2a_x. \tag{12}$$

*The Euler–Lagrange equation*

$$\frac{\delta L}{\delta y} = 0 \tag{13}$$

*corresponding to eq.* (11) *is*

$$y'' + \frac{P_y}{2P}y'^2 + \frac{P_x}{P}y' + \frac{Q_x - R_y}{P} = 0. \tag{14}$$

*There exist some special cases:*

(1) $P = P(x)$ *and* $Q \equiv 0$:

$$y'' + b(x)y' + c(x, y) = 0 \;\Rightarrow L = \left(\frac{1}{2}y'^2 - \int^y c(x, \xi)\mathrm{d}\xi\right)\mathrm{e}^{\int^x b(\tau)\mathrm{d}\tau}. \tag{15}$$

(2) $P = P(y)$ *and* $R \equiv 0$:

$$y'' + a(y)y'^2 + c(x, y) = 0 \;\Rightarrow L = \left(\frac{1}{2}y'^2 + y'\int^x c(\tau, x)\mathrm{d}\tau\right). \tag{16}$$

(3) $P = P(y)$ *and* $Q \equiv 0$:

$$y'' + a(y)y'^2 + c(x, y) = 0 \;\Rightarrow L = \frac{1}{2}y'^2\mathrm{e}^2\int^y a(\xi)\mathrm{d}\xi - \int^y c(x, \xi)\mathrm{e}^{2\int^\xi a(z)\mathrm{d}z}\mathrm{d}\xi. \tag{17}$$

### 2.1 *Noether's method*

A Lie–Bäcklund operator $X$ is a Noether symmetry generator associated with a Lagrangian $L$ of the Euler–Lagrange differential eq. (13), if there exists a gauge function $B$ such that

$$X(L) + LD_x(\xi) = D_x(B). \tag{18}$$

For each Noether symmetry generator $X$ associated with a given Lagrangian $L$ corresponding to the Euler–Lagrange differential equation, there corresponds a function $I$ known as a first integral and is defined by

$$I = B - \xi L - W\frac{\delta L}{\delta y'} - \sum_{s\geq 1} D_x^s(W)\frac{\delta L}{\delta y^{(s+1)}}.$$

The above formulas for the second-order ODEs are

$$I = B - \xi L - W\left[\frac{\partial L}{\partial y'} - D_x\left(\frac{\partial L}{\partial y''}\right)\right] - D_x(W)\frac{\partial L}{\partial y''}. \tag{19}$$

### 2.2 *Partial Noether method*

This method can be applied when the differential equation does not have a known Lagrangian. Suppose that the second-order ODE (1) can be expressed as

$$E = E_0 + E_1 = 0. \tag{20}$$

A function $L = L(x, y, y')$ is known as a partial Lagrangian of eq. (20) if

$$\frac{\delta L}{\delta y} = fE_1 \tag{21}$$

provided $E_1 \neq 0$ and $f \neq 0$.

The operator $X$ satisfying

$$X(L) + LD_x(\xi) = D_x(B) + W\frac{\delta L}{\delta y} \tag{22}$$

is a partial Noether operator corresponding to the partial Lagrangian $L$.

After finding the partial Lagrangian $L$, the partial Noether operator $X$ and gauge function $B$, one can use eq. (19) for getting the first integrals of the considered equation.

### 2.3 *Nonlocal conservation method*

This method which was developed in 2007 by Ibragimov [14] does not need classical or partial Lagrangian of the underlying equations or systems. It can be applied to equations of any order. Nevertheless, nowadays the method is extended to fractional differential equations [16]. Now, we present a brief description of the method.

Let $v = v(x)$ be the new dependent variable. The adjoint equation to the second-order ODE (1) is defined by

$$E^*(x, y, v, y', v', y'', v'') = 0, \tag{23}$$

where

$$E^*(x, y, v, y', v', y'', v'') = \frac{\delta(vE)}{\delta y}. \tag{24}$$

Ibragimov [14] showed that the adjoint eq. (24) inherits all the symmetries of eq. (1). In addition, he showed again in [14], every Lie point, Lie–Bäcklund and nonlocal symmetry of eq. (1) yields a first integral consisting of (1) and the adjoint eq. (24). Let $L$ be the formal Lagrangian defined by

$$L = vE. \tag{25}$$

Then, the first integrals are given by eq. (19) with the gauge function $B = 0$,

$$I = \xi L + W\left[\frac{\partial L}{\partial y'} + D_x\left(\frac{\partial L}{\partial y''}\right)\right] + D_x(W)\frac{\partial L}{\partial y''}. \tag{26}$$

The first integrals constructed from (26) contain arbitrary solutions $v$ of the adjoint eq. (24) and, thus, for each solution $v$, one has first integrals. To get the local first integrals, one can eliminate such variables, if the original equation is self-adjoint [17].

DEFINITION 2

Equation (1) is said to be strictly self-adjoint, if the adjoint eq. (24) becomes equivalent to the original eq. (1) upon the substitution $v = y$. Equation (1) is said to be nonlinearly self-adjoint, if the adjoint eq. (24) is satisfied for all solutions $y$ of eq. (1) upon substituting

$$v = \phi(x, y) \tag{27}$$

such that

$$\phi(x, y) \neq 0. \tag{28}$$

In other words, the following equation holds:

$$E^*(x, y, \phi, y', \phi', y'', \phi'') = \lambda E(x, y, y', y''), \tag{29}$$

where $\lambda$ is the undetermined coefficient.

## 3. First integrals and exact solutions of the fin equation

First, we construct mathematical models of the considered problem. We consider a straight fin with a cross-sectional area $A_c$. The perimeter and length are given by $P$ and $L$, respectively. The fin material is attached to a fixed base surface of temperature $T_b$ and extends into a fluid of temperature $T_a$. We assume that the thermal conductivity $k$ and the heat transfer coefficient $h$ are dependent on the temperature only, i.e., $k = k(T)$ and $h = h(T)$. The one-dimensional steady-state heat balance equation in dimensional form can be written as [3,4]

$$A_c\frac{\mathrm{d}}{\mathrm{d}X}\left(k(T)\frac{\mathrm{d}T}{\mathrm{d}X}\right) - Ph(T)(T - T_a) = 0, \qquad 0 < X < L.$$

For this equation, the related boundary conditions are posed by

$$T(L) = T_b, \quad \left.\frac{\mathrm{d}T}{\mathrm{d}X}\right|_{X=0} = 0.$$

Resorting to the dimensionless variables

$$x = \frac{X}{L}, \quad y = \frac{T - T_a}{T_b - T_a}, \quad H(y) = \frac{h(T)}{h_b},$$
$$K(y) = \frac{k(T)}{k_a}, \quad M^2 = \frac{P h_b L^2}{k_a A_c},$$

where $k_a = k(T_a),\ h_b = h(T_b)$. The energy equation reads as

$$y'' + \frac{1}{K}\frac{\mathrm{d}K}{\mathrm{d}y}(y')^2 = \frac{H}{K} \tag{30}$$

and the boundary conditions become

$$y(1) = 1, \qquad y'(0) = 0,$$

where $y = y(x)$ is the temperature function and $x$ is the dimensional spatial variable.

The Lie group analysis of eq. (30) for some special cases was carried out by the authors of [4] and [8]. We consider the most general cases of $K(y) = ky^n$ and $H(y) = cy^{n+1} + d$. The corresponding governing equations and Lie point generators are the following.

*Case* 1. $c = d = 0$

For this case, the equation reads as follows:

$$yy'' + n(y')^2 = 0. \tag{31}$$

The Lie point generators of eq. (31) for this case are as follows:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = y\frac{\partial}{\partial y}, \qquad X_3 = y^{-n}\frac{\partial}{\partial y},$$
$$X_4 = x\frac{\partial}{\partial x}, \qquad X_5 = y^{n+1}\frac{\partial}{\partial x}, \qquad X_6 = xy^{-n}\frac{\partial}{\partial y},$$
$$X_7 = (n+1)x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$
$$X_8 = (n+1)xy^{n+1}\frac{\partial}{\partial x} + y^{n+2}\frac{\partial}{\partial y}. \tag{32}$$

*Case* 2. $c = 0,\ \ d \neq 0,\ \ n \neq -1$

For this case, the equation reads as follows:

$$ky^n y'' + kny^{n-1}(y')^2 - d = 0. \tag{33}$$

The Lie point generators of eq. (33) for this case are as follows;

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = y^{-n}\frac{\partial}{\partial y}, \qquad X_3 = xy^{-n}\frac{\partial}{\partial y},$$
$$X_4 = ((n+1)dx^2y^{-n} - 2ky)\frac{\partial}{\partial y},$$
$$X_5 = x\frac{\partial}{\partial x} + \frac{d}{k}x^2y^{-n}\frac{\partial}{\partial y},$$
$$X_6 = y^{n+1}\frac{\partial}{\partial x} - \frac{d}{4k^2}x((n+1)dx^2y^{-n} - 6ky)$$
$$X_7 = \frac{1}{2}x^2\frac{\partial}{\partial x} + \frac{x}{4k(n+1)}((n+1)dx^2y^{-n} + 2ky)\frac{\partial}{\partial y}$$
$$X_8 = \frac{1}{2k}x(-(n+1)dx^2 + 2ky^{n+1})\frac{\partial}{\partial x}$$
$$- \frac{1}{4k^2(n+1)}((n+1)^2d^2x^4y^{-n} - 4k^2y^{n+2})\frac{\partial}{\partial y}. \tag{34}$$

*Case* 3. $c \neq 0,\ \ d = 0,\ \ n \neq -1$

For this case, the equation reads as follows:

$$ky^n y'' + kny^{n-1}(y')^2 - cy^{n+1} = 0. \tag{35}$$

The Lie point generators of eq. (35) for this case are as follows:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = y\frac{\partial}{\partial y}, \qquad X_3 = y^{-n}\mathrm{e}^{\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$
$$X_4 = y^{-n}\mathrm{e}^{-\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$
$$X_5 = \mathrm{e}^{2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} + \sqrt{\frac{c}{k(n+1)}}y\mathrm{e}^{2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$
$$X_6 = \mathrm{e}^{-2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} - \sqrt{\frac{c}{k(n+1)}}y\mathrm{e}^{-2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$
$$X_7 = y^{n+1}\mathrm{e}^{\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} + \sqrt{\frac{c}{k(n+1)}}y^{n+2}\mathrm{e}^{\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$
$$X_8 = y^{n+1}\mathrm{e}^{-\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} - \sqrt{\frac{c}{k(n+1)}}y^{n+2}\mathrm{e}^{-\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y}. \tag{36}$$

*Case* 4. $c \neq 0,\ \ d \neq 0,\ \ n \neq -1$

For this case, the equation reads as follows:

$$ky^n y'' + kny^{n-1}(y')^2 - cy^{n+1} - d = 0. \tag{37}$$

The Lie point generators of eq. (37) for this case are as follows:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \left(dy^{-n} + cy\right)\frac{\partial}{\partial y},$$

$$X_3 = y^{-n}\cosh\sqrt{\frac{c(n+1)}{k}}x\frac{\partial}{\partial y}, \quad X_4 = y^{-n}\sinh\sqrt{\frac{c(n+1)}{k}}x\frac{\partial}{\partial y},$$

$$X_5 = (cy^{n+1} + d)\mathrm{e}^{\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} + (cy^{n+2} + 2dy)\sqrt{\frac{c}{k(n+1)}}\mathrm{e}^{\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$

$$X_6 = (cy^{n+1} + d)\mathrm{e}^{-\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} - (cy^{n+2} + 2dy)\sqrt{\frac{c}{k(n+1)}}\mathrm{e}^{-\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial y},$$

$$X_7 = \mathrm{e}^{2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} + \frac{cy\mathrm{e}^{2\sqrt{\frac{c(n+1)}{k}}x} + dy^{-n}\left(\cosh\left(2\sqrt{\frac{c(n+1)}{k}}x\right) + \sinh\left(2\sqrt{\frac{c(n+1)}{k}}x\right)\right)}{\sqrt{ck(n+1)}}\frac{\partial}{\partial y},$$

$$X_8 = \mathrm{e}^{-2\sqrt{\frac{c(n+1)}{k}}x}\frac{\partial}{\partial x} - \frac{cy\mathrm{e}^{-2\sqrt{\frac{c(n+1)}{k}}x} + dy^{-n}\left(\cosh\left(2\sqrt{\frac{c(n+1)}{k}}x\right) - \sinh\left(2\sqrt{\frac{c(n+1)}{k}}x\right)\right)}{ck(n+1)}\frac{\partial}{\partial y}. \tag{38}$$

### 3.1 *Application of the Noether's approach to eq.* (30)

We first apply classical Noether's approach with detail for Case 4. The results for the other cases can be easily obtained. Using eq. (16), the standard Lagrangian of eq. (37), is as follows:

$$L = \frac{1}{2}y^{2n}y'^2 - \frac{c}{k}xy^{2n+1}y' - \frac{d}{k}xy^n y'. \tag{39}$$

We note that one can obtain the other distinct Lagrangian function

$$L = \frac{1}{2}y^{2n}y'^2 + \frac{c}{k(2n+2)}y^{2n+2} + \frac{d}{k(n+1)}y^{n+1}, \tag{40}$$

using eq. (17). We have not considered eq. (40) as the Lagrangian function of eq. (16) because of its complexness.

We note that only $X_1$, $X_3$ and $X_4$ Lie symmetry generators from (38) satisfy the divergence condition (18) associated with Lagrangian function (39). In this case, the corresponding gauge functions of $X_1$, $X_3$ and $X_4$ and Lie characteristic functions are as follows:

$$B_1 = -\frac{c}{k(2n+2)}y^{2n+2} - \frac{d}{k(n+1)}y^{n+1},$$

$$B_3 = (ey^{n+1} + d) \times \left[\frac{\sinh\left(\sqrt{\frac{c(n+1)}{k}}x\right)}{\sqrt{ck(n+1)}} - \frac{x}{k}\cosh\left(\sqrt{\frac{c(n+1)}{k}}x\right)\right],$$

$$B_4 = (cy^{n+1} + d)\left[\frac{\cosh\left(\sqrt{\frac{c(n+1)}{k}}x\right)}{\sqrt{ck(n+1)}} - \frac{x}{k}\sinh\left(\sqrt{\frac{c(n+1)}{k}}x\right)\right],$$

$$W_1 = -y',$$

$$W_3 = y^{-n}\cosh\left(\sqrt{\frac{c(n+1)}{k}}x\right),$$

$$W_4 = y^{-n}\sinh\left(\sqrt{\frac{c(n+1)}{k}}x\right). \tag{41}$$

Substituting eqs (39) and (41) into eq. (19), we yield the following first integrals:

$$I_1 = \frac{1}{2}y^{2n}y'^2 - \frac{y^{n+1}}{k(n+1)}\left(\frac{c}{2}y^{n+1} + d\right),$$

$$\begin{aligned} I_3 &= -y^n y' \cosh\left(\sqrt{\frac{c(n+1)}{k}}x\right) \\ &\quad +\frac{cy^{n+1}+d}{\sqrt{ck(n+1)}}\sinh\left(\sqrt{\frac{c(n+1)}{k}}x\right), \\ I_4 &= -y^n y' \sinh\left(\sqrt{\frac{c(n+1)}{k}}x\right) \\ &\quad +\frac{cy^{n+1}+d}{\sqrt{ck(n+1)}}\cosh\left(\sqrt{\frac{c(n+1)}{k}}x\right). \end{aligned} \tag{42}$$

The first integral $I_1$ from (42) with $n = 1$ coincides exactly with eq. (112) in [9].

*Case* 1. We consider for this case the generators in (32). The corresponding Lagrangian, gauge functions and first integrals are as follows:

$$L = \frac{1}{2}y^{2n}y'^2, \tag{43}$$

$$B_1 = 0, \qquad B_3 = 0, \qquad B_6 = \frac{y^{n+1}}{n+1},$$

$$B_7 = \frac{y^{2n+2}}{2n+2}, \tag{44}$$

$$I_1 = \frac{1}{2}y^{2n}y'^2, \quad I_3 = -y^n y',$$

$$I_6 = \frac{y^{n+1}}{n+1} - xy^n y',$$

$$I_7 = \frac{y^{2n+2}}{2n+2} + \frac{(n+1)}{2}x^2 y^{2n}y'^2 - xy^{2n+1}y'. \tag{45}$$

The first integral $I_6$ from (45) with $n = 1$ coincides exactly with eq. (83) in [9].

*Case* 2. We consider for this case the generators in (34). The corresponding Lagrangian, gauge functions and first integrals are as follows:

$$L = \frac{y^{2n}}{2}y'^2 - \frac{d^2}{k^2}xy^n y', \tag{46}$$

$$B_1 = -\frac{d}{k(n+1)}y^{n+1}, \qquad B_2 = 0,$$

$$B_3 = \frac{y^{n+1}}{n+1} - \frac{d}{2k}x^2,$$

$$\begin{aligned} B_7 &= \frac{1}{4(n+1)^2}y^{2n+2} - \frac{dx^2}{4k(n+1)}y^{n+1} \\ &\quad -\frac{3d^2x^4}{16k^2}, \end{aligned} \tag{47}$$

$$\begin{aligned} I_1 &= \frac{1}{2}y^{2n}y'^2 - \frac{d}{k(n+1)}y^{n+1}, \\ I_2 &= -y^n y' + \frac{d}{k}x, \\ I_3 &= -xy^n y' + \frac{y^{n+1}}{n+1} + \frac{d}{2k}x^2, \\ I_7 &= \frac{x^2}{4}y^{2n}y'^2 - \left(\frac{d^2}{2k}x^2 + \frac{y^{n+1}}{n+1}\right)\frac{x}{2}y^n y' \\ &\quad +\frac{d^2}{16k^2}x^4 + \frac{d}{4k(n+1)}x^2y^{n+1} \\ &\quad +\frac{y^{2n+2}}{4(n+1)^2}. \end{aligned} \tag{48}$$

*Case* 3. We consider for this case the generators in (36). The corresponding Lagrangian, gauge functions and first integrals are as follows:

$$L = \frac{1}{2}y^{2n}y'^2 - \frac{c^2}{k^2}xy^{2n+1}y', \tag{49}$$

$$\begin{aligned} B_1 &= -\frac{c}{2k(n+1)}y^{2n+2}, \\ B_3 &= \left(\sqrt{\frac{c}{k(n+1)}} - \frac{c}{k}x\right)y^{n+1}e^{\sqrt{\frac{c(n+1)}{k}}x} \\ B_4 &= \left(-\sqrt{\frac{c}{k(n+1)}} - \frac{c}{k}x\right)y^{n+1}e^{\sqrt{\frac{c(n+1)}{k}}x} \end{aligned} \tag{50}$$

$$\begin{aligned} I_1 &= -\frac{c^2}{k^2(2n+2)}y^{2n+2} + \frac{1}{2}y^{2n}y'^2, \\ I_3 &= \left(\sqrt{\frac{c}{k(n+1)}}y^{n+1} - y^n y'\right)e^{\sqrt{\frac{c(n+1)}{k}}x}, \\ I_4 &= -\left(\sqrt{\frac{c}{k(n+1)}}y^{n+1} + y^n y'\right)e^{-\sqrt{\frac{c(n+1)}{k}}x}. \end{aligned} \tag{51}$$

The first integral $I_4$ from (51) with $n = 1$ coincides exactly with eq. (103) in [9].

### 3.2 *Application of the partial Noether approach to eq.* (30)

Orhan *et al* [9] obtained the first integrals of Cases 1–3 of the classification by the partial Noether approach. Let us consider the general case eq. (37). Equation (37) admits the partial Lagrangian $L = (k/2)y'^2$ and the corresponding partial Euler–Lagrangian equation is

$$\frac{\delta L}{\delta y} = -ky'' = \frac{kn}{y}y'^2 - cy - \frac{d}{y^n}, \tag{52}$$

where

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x\frac{\partial}{\partial y'} + D_x^2\frac{\partial}{\partial y''} - \cdots. \tag{53}$$

The partial Noether operators $X = \xi(\partial/\partial x) + \eta(\partial/\partial y)$ corresponding to $L$ satisfy eq. (22); that is,

$$\left[\eta_x + y'\eta_y - y'\left(\xi_x + y'\xi_y\right)\right]\left(ky'\right) + \frac{ky'^2}{2}\left(\xi_x + y'\xi_y\right) = B_x + y'B_y + \left(\eta - y'\xi\right)\left(\frac{kny'^2}{y} - cy - \frac{d}{y^n}\right). \tag{54}$$

The usual separation by derivatives of $y$ gives

$$\begin{aligned}
&\left(y'\right)^3: \xi = a(x)y^{2n},\\
&\left(y'\right)^2: \eta = \frac{a'(x)y^{2n+1}}{2n+2} + b(x)y^n,\\
&\left(y'\right)^1: k\eta_x = B_y + cy\xi + \frac{d}{y^n},\\
&\left(y'\right)^0: B_x - cy\eta - \frac{d}{y^n}\eta = 0.
\end{aligned} \tag{55}$$

Equation (55) yields

$$\begin{aligned}
B &= \frac{ka''(x)}{(2n+2)^2}y^{2n+2} + \frac{kb'(x)}{n+1}y^{n+1} - \frac{ca(x)}{2n+2}y^{2n+2} - \frac{da(x)}{n+1}y^{n+1} + c(x),\\
\xi &= \left(c_1 + c_2 e^{2\sqrt{\frac{c(n+1)}{k}}x} + c_3 e^{-2\sqrt{\frac{c(n+1)}{k}}x}\right)y^{2n},\\
\eta &= \frac{a'(x)}{2n+2}y^{2n+1} + b(x)y^n.
\end{aligned} \tag{56}$$

Formula (19) with $\xi$, $\eta$ and $B$ from eq. (56) yields the first integrals as follows:

$$\begin{aligned}
I_1 &= -\frac{cy^{2n+2}}{2n+2} - \frac{dy^{n+1}}{n+1} + \frac{ky^{2n}y'^2}{2},\\
I_2 &= e^{2\sqrt{\frac{c(n+1)}{k}}x}\left(\frac{cy^{2n+2}}{2n+2} + \frac{dy^{n+1}}{n+1} + \frac{d^2}{2c(n+1)} + \frac{ky^{2n}y'^2}{2} - \frac{\sqrt{kc}y^{2n+1}y'}{\sqrt{n+1}} - \frac{d\sqrt{k}y^n y'}{\sqrt{c(n+1)}}\right)\\
I_3 &= e^{-2\sqrt{\frac{c(n+1)}{k}}x}\left(\frac{cy^{2n+2}}{2n+2} + \frac{dy^{n+1}}{n+1} + \frac{d^2}{2c(n+1)} + \frac{ky^{2n}y'^2}{2} + \frac{\sqrt{kc}y^{2n+1}y'}{\sqrt{n+1}} + \frac{d\sqrt{k}y^n y'}{\sqrt{c(n+1)}}\right),\\
I_4 &= e^{\sqrt{\frac{c(n+1)}{k}}x}\left(\frac{\sqrt{kc}y^{n+1}}{\sqrt{n+1}} + \frac{d\sqrt{k}}{\sqrt{c(n+1)}} - ky^n y'\right),\\
I_5 &= e^{-\sqrt{\frac{c(n+1)}{k}}x}\left(\frac{\sqrt{kc}y^{n+1}}{\sqrt{n+1}} + \frac{d\sqrt{k}}{\sqrt{c(n+1)}} + ky^n y'\right).
\end{aligned} \tag{57}$$

To the best of our knowledge, the above obtained results have not been reported elsewhere.

### 3.3 *Application of the nonlocal conservation method to eq.* (30)

We again present the method for Case 4. The adjoint equation for eq. (37) is

$$E_\alpha^*(x, y, v, y', v', y'', v'') = \frac{\delta}{\delta y}[v(ky^n y'' + kny^{n-1}y'^2 - cy^{n+1} - d)] \tag{58}$$

and this yields

$$kv'' - c(n+1)v = 0. \tag{59}$$

Substituting $v = y$ in eq. (59), we conclude that eq. (58) is not self-adjoint. Let $v = \phi(x, y)$, then $v' = \phi_x + \phi_y y'$ and $v'' = \phi_{xx} + 2\phi_{xy}y' + \phi_y y''$. Substituting these expressions of $v$ and $v''$ into (29) and equating the coefficients of $y$, $y'$ and $y''$ to zero, one obtains

$$v = \phi(x, y) = cy^{n+1} + d. \tag{60}$$

The formal Lagrangian for the system consisting of (37) and (59) is

$$L = v(ky^n y'' + kny^{n-1}y'^2 - cy^{n+1} - d). \tag{61}$$

The formal Lagrangian $L$ satisfies

$$\begin{aligned}
\frac{\delta L}{\delta y} &= kv'' - c(n+1)v = 0\\
\frac{\delta L}{\delta v} &= ky^n y'' + kny^{n-1}y'^2 - cy^{n+1} - d.
\end{aligned} \tag{62}$$

Using eq. (26), we obtain the following first integral:

$$I_1 = kc(n+1)y^{2n}y'^2 - (cy^{n+1} + d)^2 \tag{63}$$

for the Lie symmetry generator $X = \partial/\partial x$ from (38) (with $\xi = 1$, $\eta = 0$ and $W = -y'$) and Lagrangian (61). For the other Lie symmetry generators, we obtain the following first integrals:

$$\begin{aligned}
I_2 &= 0,\\
I_3 &= -kc(n+1)y^n y' \cosh\left(\frac{\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x\right) + \sqrt{k}\sqrt{c}\sqrt{n+1}(cy^{n+1} + d)\sinh\left(\frac{\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x\right),
\end{aligned}$$

$$I_4 = -kc(n+1)y^n y' \sinh\left(\frac{\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x\right) + \sqrt{k}\sqrt{c}\sqrt{n+1}(cy^{n+1}+d)\cosh\left(\frac{\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x\right),$$
$$I_5 = d^2 e^{\frac{\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x}(\sqrt{k}\sqrt{c}\sqrt{n+1}y^n y' - cy^{n+1} - d),$$
$$I_6 = -d^2 e^{\frac{-\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x}(\sqrt{k}\sqrt{c}\sqrt{n+1}y^n y' + cy^{n+1} + d),$$
$$I_7 = e^{\frac{2\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x}(kc(n+1)y^{2n}(y')^2 - 2\sqrt{k}\sqrt{c}\sqrt{n+1}y^n y'(cy^{n+1}+d) + (cy^{n+1}+d)^2),$$
$$I_8 = e^{\frac{-2\sqrt{c}\sqrt{n+1}}{\sqrt{k}}x}(kc(n+1)y^{2n}(y')^2 + 2\sqrt{k}\sqrt{c}\sqrt{n+1}y^n y'(cy^{n+1}+d) + (cy^{n+1}+d)^2). \tag{64}$$

We note that in this case, the first integrals of $I_3$ and $I_4$ coincide exactly with the first integrals of $I_3$ and $I_4$ which is obtained by Noether's method. In addition, $I_5$, $I_6$, $I_7$ and $I_8$ coincide exactly with the first integrals of $I_4$, $I_5$, $I_2$ and $I_3$, which are obtained by the partial Lagrangian approach, respectively.

If one considers for instance, the first integral of $I_5$ from (64) (with $n = 1$), we deduce that the following exact analytical solution (we assume that the first integration constant specifically equals to zero)

$$y(x) = \pm\frac{\sqrt{c\left(C_1 c\exp(\frac{\sqrt{2c}x}{\sqrt{k}}) - d\right)}}{c}, \tag{65}$$

where $C_1$ is a constant. In the following, we demonstrate only the results corresponding to other cases:

*Case* 1.

$$I_1 = I_3 = 0,$$
$$I_2 = (n+1)y^n y',$$
$$I_4 = -y^n y',$$
$$I_5 = -(n+1)y^{2n}y'^2,$$
$$I_6 = 1,$$
$$I_7 = -(n+1)xy^n y' + y^{n+1},$$
$$I_8 = (n+1)y^{2n+1} - (n+1)^2 xy^{2n}y'^2. \tag{66}$$

If we take the first integral of $I_7$ from (66), we obtain the following exact analytical solution (we assume that the first integration constant specifically equals zero)

$$y(x) = C_1 x^{1/(n+1)}, \tag{67}$$

where $C_1$ is a constant.

*Case* 2.

$$I_1 = -d,$$
$$I_2 = 0,$$
$$I_3 = k,$$
$$I_4 = -2k(n+1)(ky^n y' - dx),$$
$$I_5 = -ky^n y' + dx,$$
$$I_6 = -k(n+1)y^{2n}(y')^2 + \frac{3(n+1)dxy^n y'}{2} + \frac{dy^{n+1}}{2} - \frac{3d^2(n+1)x^2}{4k},$$
$$I_7 = -\frac{kxy^n y'}{2} + \frac{ky^{n+1}}{2(n+1)} + \frac{dx^2}{4},$$
$$I_8 = -k(n+1)xy^{2n}(y')^2 + ky^{2n+1}y' + \frac{3(n+1)dx^2y^n y'}{2} - \frac{(n+1)d^2x^3}{2k} - dxy^{n+1}. \tag{68}$$

If we take the first integral of $I_8$ from (68) (with $n = 1$), we get the following exact analytical solution (we assume that the first integration constant specifically equals zero).

$$y(x) = \pm\frac{\sqrt{k(dx^2 + C_1 k)}}{k}, \tag{69}$$

where $C_1$ is a constant.

*Case* 3.

$$I_1 = ky^{2n}y'^2 - \frac{cy^{2n+2}}{n+1},$$
$$I_2 = I_7 = I_8 = 0,$$
$$I_3 = e^{\sqrt{\frac{c(n+1)}{k}}x}\left(-ky^n y' + \frac{\sqrt{kc}y^{n+1}}{\sqrt{n+1}}\right),$$
$$I_4 = e^{-\sqrt{\frac{c(n+1)}{k}}x}\left(-ky^n y' - \frac{\sqrt{kc}y^{n+1}}{\sqrt{n+1}}\right),$$
$$I_5 = e^{2\sqrt{\frac{c(n+1)}{k}}x}\left(ky^{2n}y'^2 - \frac{2\sqrt{kc}y^{2n+2}y'}{\sqrt{n+1}} + \frac{cy^{2n+2}}{n+1}\right),$$
$$I_6 = e^{-2\sqrt{\frac{c(n+1)}{k}}x}\left(ky^{2n}y'^2 + \frac{2\sqrt{kc}y^{2n+2}y'}{\sqrt{n+1}} + \frac{cy^{2n+2}}{n+1}\right). \tag{70}$$

If we take the first integral of $I_5$ from (70) (with $n = 1$), we yield the following exact analytical solution (we assume that the first integration constant specifically equals zero):

$$y(x) = \exp\left(\frac{\frac{1}{2}\sqrt{2c}(x - C_1)}{\sqrt{k}}\right). \tag{71}$$

where $C_1$ is a constant.

## 4. Concluding remarks

First integrals of the one-dimensional nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient were constructed by using Noether, partial Noether and nonlocal conservation methods. As one can see, the considered problem is highly nonlinear. We have also obtained some exact analytical solutions corresponding to the first integrals. As stated in [3], the obtained solutions are readily applicable to various fin-like diffusion problems.

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