Pramana – J. Phys. (2016) 87: 17
DOI 10.1007/s12043-016-1225-7

The $\left(G'/G, 1/G\right)$-expansion method for solving nonlinear space–time fractional differential equations

EMRULLAH YAŞAR* and İLKER BURAK GIRESUNLU

Department of Mathematics, Faculty of Arts and Sciences, Uludag University, 16059, Bursa, Turkey
*Corresponding author. E-mail: eyasar@uludag.edu.tr

MS received 4 July 2015; revised 10 September 2015; accepted 1 October 2015; published online 6 July 2016

Abstract. In this work, we present $\left(G'/G, 1/G\right)$-expansion method for solving fractional differential equations based on a fractional complex transform. We apply this method for solving space–time fractional Cahn–Allen equation and space–time fractional Klein–Gordon equation. The fractional derivatives are described in the sense of modified Riemann–Lioville. As a result of some exact solution in the form of hyperbolic, trigonometric and rational solutions are deduced. The obtained solutions may be used for explaining of some physical problems. The $\left(G'/G, 1/G\right)$-expansion method has a wider applicability for nonlinear equations. We have verified all the obtained solutions with the aid of *Maple*.

Keywords. Exact solution; modified Riemann–Liouville fractional derivative; space–time Cahn–Allen equation; space–time Klein–Gordon equation; $\left(G'/G, 1/G\right)$-expansion method.

PACS Nos 02.30.Jr; 02.70.Wz; 04.20.Jb

1. Introduction

Fractional calculus can be viewed as one of the extensions of classical ordinary calculus. In fact, the roots of fractional calculus dates back to three hundred years ago. There have been many contributions in this area. Oldham and Spanier [1] first considered the fractional differential equations (FDEs). The study of the exact solutions of nonlinear fractional evolution equations (NLFEEs) plays an important role in understanding the nonlinear physical phenomena which are described by these equations. For example, the nonlinear oscillation of an earthquake can be modelled with the fractional derivatives. In reality, a physical phenomenon may depend not only on the time instant but also on the previous time history, which can be successfully modelled by using the theory of derivatives and integrals of fractional order [2–4]. As fractional partial differential equations (FPDEs) appear frequently in diverse fields such as physics, biology, rheology, viscoelasticity control theory, signal processing, systems identification and electrochemistry, they attract considerable interest and recently there has been significant theoretical development [5]. In recent years, several powerful and efficient methods have been developed for finding analytic solutions of NLFEEs. Some of the most important methods found in literature include the exp-function method [6], Adomian decomposition method [7], tanh–sech function method [8], the first integral method [9] and the (G'/G)-expansion method [10] which can be used to construct exact solutions for some time and space FPDEs. Based on these methods, a variety of FPDEs have been investigated and solved.

Very recently, Jumarie [11] suggested a modified Riemann–Liouville derivative. With the help of this fractional derivative and some important formulas, one can convert FDEs into integer-order differential equations by fractional complex transformation [12,13]. The main aim of this work is to extend the application of the $\left(G'/G, 1/G\right)$-expansion method [14,15] to obtain some exact travelling wave solutions to some space–time FDEs.

The first model considered is the space–time fractional Cahn–Allen equation:

$$D_t^{\alpha} u - D_x^{\alpha}\left(D_x^{\alpha} u\right) + u^3 - u = 0. \tag{1}$$

Cahn–Allen equation arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics [16,17].

The second model studied is the space–time fractional Klein–Gordon equation:

$$D_t^{\alpha}\left(D_t^{\alpha}u\right) - D_x^{\alpha}\left(D_x^{\alpha}u\right) + u^3 - u = 0. \tag{2}$$

The nonlinear fractional Klein–Gordon equation models many types of nonlinearities. It plays a significant role in several real-world applications, for example, in solid-state physics, nonlinear optics and quantum field theory [18].

The rest of this work is organized as follows: In §2, we present some basic properties of the modified Riemann–Liouville derivative operator. The main steps of the $\left(G'/G, 1/G\right)$-expansion method are given in §3. In §4 and 5, we illustrate this method in detail with space–time fractional Cahn–Allen equation and space–time fractional Klein–Gordon equations, respectively. In the last section, some conclusions are provided.

2. Description of Jumarie's modified Riemann–Liouville derivative operator and its important property

The Jumarie's modified Riemann–Liouville (RL) derivative is defined as follows [11]:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\mathrm{d}\xi,$$
$$0<\alpha<1, \tag{3}$$

where $f\colon R \to R$, $t \to f(t)$ denotes a continuous function and $f(t) - f(0) = 0$ for $t < 0$. Also, if $f(0)$ in any case is infinity, then specifically one needs to take $f(t) - f(0^+)$ or the finite part of $(f(t) - f(0))$.

As the expression in (3) is also a Caputo derivative, only if $f(t)$ is differentiable, one may state that in this Jumarie's definition of modified RL fractional derivative with offsetting via function value at start point of fractional differentiation, the condition of differentiability of $f(t)$ is mandatory.

For the derivative, we give the following important property which is useful to solve FDEs:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}.$$

3. Algorithm of $(G'/G, 1/G)$-expansion method with fractional complex transform

Before presenting the algorithm, we need the solutions of the auxilary equation (see [14,15]).

Remark 1. If we consider the second-order linear ODE:

$$G''(\xi) + \lambda G(\xi) = \mu \tag{4}$$

and set $\phi = G'/G$, $\psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{5}$$

One can consider the general solution of (4), in two distinct subcases:

Case 1. If $\lambda < 0$, then the general solutions of (4) has the form:

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{6}$$

where A_1, A_2 are constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{7}$$

where $\sigma = A_1^2 - A_2^2$.

Case 2. If $\lambda > 0$, then the general solutions of (4) has the form

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{8}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{9}$$

where $\sigma = A_1^2 + A_2^2$.

The main steps of this method are described as follows:

Step 1. We consider the following general nonlinear FDE of the form:

$$F\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_t^{\alpha}D_t^{\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{\beta}D_x^{\beta}u, \ldots\right) = 0, \tag{10}$$

where F is a polynomial in $u(x,t)$ and its partial fractional derivatives.

The complex wave variable

$$u(x,t) = U(\xi), \quad \xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} + \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{11}$$

where k and c are constants, reduces (10) to an ODE in the form:

$$P\left(U, U', U'', U''', \ldots\right) = 0, \tag{12}$$

where P is a polynomial of $U(\xi)$ and its total derivatives with respect to ξ. If possible, we should integrate (12), term by term one or more times.

Step 2. Assume that the solution of (12) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$U(\xi)=\sum_{i=0}^{N}a_i\phi^i+\sum_{j=1}^{N}b_j\phi^{j-1}\psi, \tag{13}$$

where $a_i(i=0,1,2,\ldots,N)$ and $b_j(j=1,\ldots,N)$ are constants to be determined later.

Determine the positive integer N in (13) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in (12).

Step 3. Substitute (13) into (12) along with (5) and (7), the left-hand side of (12) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not greater than 1. Equating each coefficient of this polynomial to zero, yields a system of algebraic equations which can be solved by using *Maple* to get the values of a_i, b_i, k, c, μ, A_1, A_2 and λ, where $\lambda<0$.

Step 4. Similar to Step 3, substituting (13) into (12), along with (5) and (9) for $\lambda>0$, we obtain the exact solutions of (12) expressed by trigonometric functions.

4. Implementation of $(G'/G, 1/G)$-expansion method for the exact solutions of space–time fractional Cahn–Allen equation

We consider space–time fractional Cahn–Allen equation

$$D_t^{\alpha}u-D_x^{\alpha}\left(D_x^{\alpha}u\right)+u^3-u=0, \tag{14}$$

where $u(x,t)$ is an unknown function.

For our goal, we present the following transformation:

$$u(x,t)=U(\xi),\quad \xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{15}$$

where c and k are constants. Then, by using (15), (14) can be turned into an ODE:

$$-k^2U''+cU'+U^3-U=0, \tag{16}$$

where $U'=\mathrm{d}U/\mathrm{d}\xi$.

Balancing the order of U'' and U^3 in (16), we get $N=1$. Consequently, we get

$$U(\xi)=a_0+a_1\phi+b_1\psi, \tag{17}$$

where a_0, a_1 and b_1 are constants.

Case 1. Hyperbolic function solutions ($\lambda<0$)
If $\lambda<0$, substituting (17) into (16) and using (5) and (7), the left-hand side of eq. (16) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to zero, yields a system of algebraic equations in a_0, a_1, b_1, c and k as follows:

$$\begin{aligned}
&\phi^3:\ \mu^2a_1^2-2\mu^2k^2-3b_1^2\lambda-2k^2\lambda^2\sigma+a_1^2\lambda^2\sigma=0,\\
&\phi^2\psi:\ 3\mu^2a_1^2-2\mu^2k^2+3a_1^2\lambda^2\sigma-2k^2\lambda^2\sigma-b_1^2\lambda=0,\\
&\phi^2:\ -3a_0b_1^2\lambda\mu^2-ca_1\mu^4-ca_1\lambda^4\sigma^2+3a_0a_1^2\lambda^4\sigma^2\\
&\quad -3a_0b_1^2\lambda^3\sigma+6a_0a_1^2\mu^2\lambda^2\sigma-k^2b_1\lambda^3\mu\sigma\\
&\quad -2ca_1\mu^2\lambda^2\sigma-2b_1^3\lambda^2\mu-k^2b_1\lambda\mu^3\\
&\quad +3a_0a_1^2\mu^4=0,\\
&\phi\psi:\ 3k^2a_1\mu^3-\mu^2cb_1+6\mu^2a_0a_1b_1+3\mu k^2a_1\lambda^2\sigma\\
&\quad +6\mu a_1b_1^2\lambda+6a_0a_1b_1\lambda^2\sigma-cb_1\lambda^2\sigma=0,\\
&\phi:\ -\mu^2-2\mu^2k^2\lambda+3\mu^2a_0^2+3a_0^2\lambda^2\sigma-3b_1^2\lambda^2\\
&\quad -\lambda^2\sigma-2k^2\lambda^3\sigma=0,\\
&\psi:\ -b_1^3\lambda^4\sigma+3b_1^3\lambda^2\mu^2+3a_0^2b_1\mu^4-b_1\lambda^4\sigma^2\\
&\quad +ca_1\mu^5+ca_1\mu\lambda^4\sigma^2-2b_1\mu^2\lambda^2\sigma\\
&\quad +6a_0^2b_1\mu^2\lambda^2\sigma+k^2b_1\lambda\mu^4+3a_0^2b_1\lambda^4\sigma^2\\
&\quad +6a_0b_1^2\lambda\mu^3+6a_0b_1^2\lambda^3\mu\sigma-k^2b_1\lambda^5\sigma^2\\
&\quad -b_1\mu^4+2ca_1\mu^3\lambda^2\sigma=0,\\
&\text{cons.}:\ -3a_0b_1^2\lambda^4\sigma-3a_0b_1^2\lambda^2\mu^2-2a_0\lambda^2\sigma\mu^2\\
&\quad +2a_0^3\lambda^2\sigma\mu^2-ca_1\lambda\mu^4-k^2b_1\lambda^2\mu^3-a_0\mu^4\\
&\quad -k^2b_1\lambda^4\mu\sigma-a_0\lambda^4\sigma^2-2ca_1\lambda^3\sigma\mu^2+a_0^3\lambda^4\sigma^2\\
&\quad -ca_1\lambda^5\sigma^2-2b_1^3\lambda^3\mu+a_0^3\mu^4=0.
\end{aligned}$$

On solving the above algebraic equations using *Maple*, we get the following results:

$$\left\{a_0=0,\quad a_1=\mp\frac{1}{\sqrt{-\lambda}},\quad b_1=\mp\frac{\sqrt{\lambda^2\sigma+\mu^2}}{\lambda},\quad c=0,\quad k=\mp\sqrt{\frac{2}{-\lambda}}\right\},$$

$$\left\{a_0=\frac{1}{2},\quad a_1=\mp\frac{1}{2\sqrt{-\lambda}},\quad b_1=\mp\frac{\sqrt{\lambda^2\sigma+\mu^2}}{2\lambda},\quad c=\mp\frac{3}{2\sqrt{-\lambda}},\quad k=\mp\sqrt{\frac{1}{-2\lambda}}\right\},$$

$$\left\{a_0=-\frac{1}{2},\quad a_1=\mp\frac{1}{2\sqrt{-\lambda}},\quad b_1=\mp\frac{\sqrt{\lambda^2\sigma+\mu^2}}{2\lambda},\quad c=\pm\frac{3}{2\sqrt{-\lambda}},\quad k=\mp\sqrt{\frac{1}{-2\lambda}}\right\}.$$

In this case, the exact solutions of (14) are

$$u_{11}(x,t)=\frac{A_1\cosh\left(\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+A_2\sinh\left(\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+A_2\cosh\left(\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

$$u_{12}(x,t)=-\frac{A_1\cosh\left(-\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+A_2\sinh\left(-\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(-\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+A_2\cosh\left(-\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

$$u_{21}(x,t)=\frac{1}{2}+\frac{1}{2}\frac{A_1\cosh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\sinh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\cosh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

$$u_{22}(x,t)=\frac{1}{2}-\frac{1}{2}\frac{A_1\cosh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\sinh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\cosh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

$$u_{31}(x,t)=-\frac{1}{2}+\frac{1}{2}\frac{A_1\cosh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\sinh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\cosh\left(\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}-\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

$$u_{32}(x,t)=-\frac{1}{2}-\frac{1}{2}\frac{A_1\cosh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\sinh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1\sinh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+A_2\cosh\left(-\frac{x^\alpha}{\sqrt{2}\Gamma(1+\alpha)}+\frac{3t^\alpha}{2\Gamma(1+\alpha)}\right)+\frac{\mu}{\lambda}},$$

where

$$\sigma=A_1^2-A_2^2.$$

Case 2. Trigonometric function solutions ($\lambda>0$)
If $\lambda>0$, substituting (17) into (16) and using (5) and (9), the left-hand side of eq. (16) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to zero, yields a system of algebraic equations in a_0, a_1, b_1, c and k as follows:

$$\begin{aligned}\phi^3:\ & -2\mu^2k^2+\mu^2a_1^2-3b_1^2\lambda\\ & -a_1^2\lambda^2\sigma+2k^2\lambda^2\sigma=0,\end{aligned}$$

$$\begin{aligned}\phi^2\psi:\ & -2\mu^2k^2+3\mu^2a_1^2-b_1^2\lambda\\ & +2k^2\lambda^2\sigma-3a_1^2\lambda^2\sigma=0,\end{aligned}$$

$$\begin{aligned}\phi^2:\ & -6a_0a_1^2\lambda^2\sigma\mu^2-2b_1^3\lambda^2\mu-ca_1\lambda^4\sigma^2\\ & +3a_0a_1^2\mu^4-k^2b_1\lambda\mu^3-ca_1\mu^4\\ & +3a_0a_1^2\lambda^4\sigma^2+3a_0b_1^2\lambda^3\sigma-3a_0b_1^2\lambda\mu^2\\ & +2ca_1\mu^2\lambda^2\sigma+k^2b_1\lambda^3\mu\sigma=0,\end{aligned}$$

$$\begin{aligned}\phi\psi:\ & 3k^2a_1\mu^3+6\mu^2a_0a_1b_1-\mu^2cb_1\\ & -3\mu k^2a_1\lambda^2\sigma+6\mu a_1b_1^2\lambda\\ & -6a_0a_1b_1\lambda^2\sigma+cb_1\lambda^2\sigma=0,\end{aligned}$$

$$\begin{aligned}\phi:\ & -\mu^2+3\mu^2a_0^2-2\mu^2k^2\lambda-3b_1^2\lambda^2\\ & +2k^2\lambda^3\sigma-3a_0^2\lambda^2\sigma+\lambda^2\sigma=0,\end{aligned}$$

$$\begin{aligned}\psi:\ & b_1^3\lambda^4\sigma+3b_1^3\lambda^2\mu^2+3a_0^2b_1\mu^4-b_1\lambda^4\sigma^2\\ & +ca_1\mu^5+3a_0^2b_1\lambda^4\sigma^2+ca_1\mu\lambda^4\sigma^2\\ & +6a_0b_1^2\lambda\mu^3-6a_0b_1^2\lambda^3\mu\sigma+k^2b_1\lambda\mu^4\\ & -k^2b_1\lambda^5\sigma^2+2b_1\mu^2\lambda^2\sigma-b_1\mu^4\\ & -2ca_1\mu^3\lambda^2\sigma-6a_0^2b_1\lambda^2\sigma\mu^2=0,\end{aligned}$$

$$\begin{aligned}\text{cons.}:\ & 3a_0b_1^2\lambda^4\sigma-3a_0b_1^2\lambda^2\mu^2+2a_0\lambda^2\sigma\mu^2\\ & -2a_0^3\lambda^2\sigma\mu^2+2ca_1\lambda^3\sigma\mu^2\\ & -k^2b_1\lambda^2\mu^3+a_0^3\lambda^4\sigma^2-ca_1\lambda^5\sigma^2\\ & -ca_1\lambda\mu^4+k^2b_1\lambda^4\mu\sigma-a_0\mu^4+a_0^3\mu^4\\ & -2b_1^3\lambda^3\mu-a_0\lambda^4\sigma^2=0.\end{aligned}$$

On solving the above algebraic equations using *Maple*, we get the following results:

$$\left\{a_0 = 0, \quad a_1 = \mp\frac{1}{\sqrt{-\lambda}}, \quad b_1 = \mp\frac{\sqrt{\mu^2 - \lambda^2\sigma}}{\lambda}, \quad c = 0, \quad k = \mp\sqrt{\frac{2}{-\lambda}}\right\},$$

$$\left\{a_0 = \frac{1}{2}, \quad a_1 = \mp\frac{1}{2\sqrt{-\lambda}}, \quad b_1 = \mp\frac{\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda}, \quad c = \mp\frac{3}{2\sqrt{-\lambda}}, \quad k = \mp\sqrt{\frac{1}{-2\lambda}}\right\},$$

$$\left\{a_0 = -\frac{1}{2}, \quad a_1 = \mp\frac{1}{2\sqrt{-\lambda}}, \quad b_1 = \mp\frac{\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda}, \quad c = \pm\frac{3}{2\sqrt{-\lambda}}, \quad k = \mp\sqrt{\frac{1}{-2\lambda}}\right\}.$$

In this case, the exact solutions of (14) are

$$u_{11}(x,t) = \frac{A_1\cos\left(i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) - A_2\sin\left(i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + A_2\cos\left(i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

$$u_{12}(x,t) = -\frac{A_1\cos\left(-i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) - A_2\sin\left(-i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(-i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + A_2\cos\left(-i\frac{\sqrt{2}x^\alpha}{\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

$$u_{21}(x,t) = \frac{1}{2} + \frac{1}{2}\frac{A_1\cos\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\sin\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\cos\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

$$u_{22}(x,t) = \frac{1}{2} - \frac{1}{2}\frac{A_1\cos\left(\frac{-ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\sin\left(\frac{-ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(\frac{-ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\cos\left(\frac{-ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

$$u_{31}(x,t) = -\frac{1}{2} + \frac{1}{2}\frac{A_1\cos\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\sin\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\cos\left(\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} - \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

$$u_{32}(x,t) = -\frac{1}{2} - \frac{1}{2}\frac{A_1\cos\left(-\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\sin\left(-\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1\sin\left(-\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + A_2\cos\left(-\frac{ix^\alpha}{\sqrt{2}\Gamma(1+\alpha)} + \frac{3it^\alpha}{2\Gamma(1+\alpha)}\right) + \frac{\mu}{\lambda}},$$

where

$$\sigma = A_1^2 + A_2^2.$$

5. Implementation of $(G'/G, 1/G)$-expansion method for the exact solutions of space–time fractional Klein–Gordon equation

We consider space–time fractional Klein–Gordon equation

$$D_t^\alpha\left(D_t^\alpha u\right) - D_x^\alpha\left(D_x^\alpha u\right) + u^3 - u = 0, \tag{18}$$

where $u(x,t)$ is an unknown function. We suppose the following transformation:

$$u(x,t) = U(\xi), \quad xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{19}$$

where c and k are constants. Then putting (19) in (18), we can get an ODE of the form:

$$(c^2 - k^2)U'' + U^3 - U = 0, \tag{20}$$

where

$$U' = \mathrm{d}U/\mathrm{d}\xi.$$

Balancing the order of U'' and U^3 in (20), we get $N = 1$. Consequently, we get

$$U(\xi) = a_0 + a_1\phi + b_1\psi, \tag{21}$$

where a_0, a_1 and b_1 are constants.

Case 1. Hyperbolic function solutions ($\lambda < 0$)

Similarly, solving the algebraic system with *Maple*, we get

$$\left\{a_0 = 0, \quad a_1 = \mp\frac{1}{\sqrt{-\lambda}}, \quad b_1 = \mp\frac{\sqrt{\lambda^2\sigma + \mu^2}}{\lambda}, \quad c = \mp\sqrt{k^2 + \frac{2}{\lambda}}, \quad k = k\right\}.$$

In this case, the exact solutions of (18) are:

$$u_1(x,t) = \frac{A_1 \cosh\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \sinh\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1 \sinh\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \cosh\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\mu}{\lambda}},$$

$$u_2(x,t) = -\frac{A_1 \cosh\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \sinh\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}}{A_1 \sinh\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \cosh\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{-\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\mu}{\lambda}}. \tag{22}$$

Case 2. Trigonometric function solutions ($\lambda > 0$)
In this case, solving the algebraic system with *Maple*, we get

$$\left\{a_0 = 0, \quad a_1 = \mp\frac{1}{\sqrt{-\lambda}}, \quad b_1 = \mp\frac{\sqrt{\mu^2 - \lambda^2\sigma}}{\lambda}, \quad c = \mp\sqrt{k^2 + \frac{2}{\lambda}}, \quad k = k\right\}.$$

The exact solutions of (18) are

$$u_1(x,t) = \frac{i\left\{A_1 \cos\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] - A_2 \sin\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right]\right\} + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1 \sin\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \cos\left[\left(kx^\alpha + \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\mu}{\lambda}},$$

$$u_2(x,t) = -\frac{i\left\{A_1 \cos\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] - A_2 \sin\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right]\right\} + \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda}}{A_1 \sin\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] + A_2 \cos\left[\left(kx^\alpha - \sqrt{k^2+\frac{2}{\lambda}}t^\alpha\right)\frac{\sqrt{\lambda}}{\Gamma(1+\alpha)}\right] + \frac{\mu}{\lambda}}. \tag{23}$$

Remark 2. All solutions of this work have been checked with *Maple* by putting them back into the original eqs (14) and (18).

6. Conclusion

In this work, we have implemented fractional complex transform with the help of $(G'/G, 1/G)$-expansion method for obtaining several travelling wave exact solutions of nonlinear evolution equations, namely, the space–time fractional Cahn–Allen and Klein–Gordon equations. With the fractional complex transform, the equation(s) can be very easily converted to the corresponding ODE. Then, we use the $(G'/G, 1/G)$-expansion method for getting exact travelling wave solutions of the aforementioned equations. The obtained exact solutions would be very useful in various areas of applied mathematics and they can be used to interpret some physical phenomena. From our results obtained in this paper, we conclude that the $(G'/G, 1/G)$-expansion method is powerful, effective and convenient for NLFEEs. This method readily does not use any linearization process, unrealistic ansatzes (see also [7]). In addition, as one can see, this method has more general applications than the other methods.

References

[1] K B Oldham and J Spanier, *The fractional calculus* (Academic Press, New York, USA, 1974)
[2] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, New York, 1993)
[3] S Samko, A A Kilbas and O Marichev, *Fractional integrals and derivatives: Theory and applications* (Gordon and Breach Science, Yverdon, Switzerland, 1993)
[4] R Sahadevan and T Bakkyaraj, *J. Math. Anal. Appl.* **393**, 341 (2012)
[5] Q Huang and R Zhdanov, *Physica A* **409**, 110 (2014)
[6] A Bekir, Ö Güner and A C Çevikel, *Ab. Appl. Anal.* **2013**, 426462 (2013)
[7] E Babolian, A R Vahidi and A Shoja, *Indian J. Pure Appl. Math.* **45(6)**, 1017 (2014)
[8] S S Ray and S Sahoo, *Rep. Math. Phys.* **75**, 63 (2015)
[9] A Bekir, Ö Güner and Ö Ünsal, *J. Comput. Nonlinear Dyn.* **10**, 021020 (2015)
[10] B Zheng, *Commun. Theor. Phys.* **58(5)**, 623 (2012)
[11] G Jumarie, *Comput. Math. Appl.* **51**, 1367 (2006)
[12] Z B Li and J He, *Comput. Appl.* **15(5)**, 970 (2010)
[13] Z B Li and J H He, *Nonlinear Sci. Lett. A* **2**, 121 (2011)
[14] X L Li, Q E Li and L M Wang, *Appl. Math. J. Chin. Univ.* **25**, 454 (2010)
[15] E M E Zayed and M A M Abdelaziz, *Math. Prob. Eng.* **2012**, 725061 (2012)
[16] S M Allen and J W Cahn, *Acta Metall.* **27**, 1085 (1979)
[17] F Taşcan and A Bekir, *Appl. Math. Comput.* **207(1)**, 279 (2009)
[18] H Jafari, H Tajadodi, N Kadkhoda and D Baleanu, *Ab. Appl. Anal.* **2013**, 587179 (2013)