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The modified simple equation method for solving some fractional-order nonlinear equations

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Abstract. Nonlinear fractional differential equations are encountered in various fields of mathematics, physics, chemistry, biology, engineering and in numerous other applications. Exact solutions of these equations play a crucial role in the proper understanding of the qualitative features of many phenomena and processes in various areas of natural science. Thus, many effective and powerful methods have been established and improved. In this study, we establish exact solutions of the time fractional biological population model equation and nonlinear fractional Klein–Gordon equation by using the modified simple equation method.

Keywords. Fractional differential equation; fractional complex transform; modified simple equation method; modified Riemann–Liouville derivative.

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1. Introduction

Fractional differential equations (FDEs) are generalizations of the previous differential equations of integer order to non-integer one, through the application of fractional calculus. It has been shown that the new fractional-order models are more adequate than the previously used integer-order models because fractional-order models can describe the nonlinear phenomena more exactly. As FDEs appear more and more frequently in various research and engineering applications, such as signal processing, control theory, viscoelasticity, biology, physics and electrochemistry (see refs [1–6] and references therein), they attract considerable interest and there has been a significant theoretical development recently (for example, see refs [2,4,5]). Also many powerful methods, for example, exponential function method [7,8], (G'/G)-expansion method [9,10], first integral method [11,12], subequation method [13,14], functional variable method [15,16], modified simple equation method [17], modified trial equation method [18] and so on [19] have been proposed to obtain exact solutions of fractional differential equations.

The aim of this paper is to find new exact travelling wave solutions of FDEs by using modified simple equation (MSE) method. The rest of the paper is organized as follows: In §2, the definition of modified Riemann–Liouville derivative and some of its properties are given. In §3, fractional complex transformation and MSE method are introduced. In §4, exact solutions of fractional biological population model equation and nonlinear fractional Klein–Gordon equation are verified by using the mentioned method. Finally, some conclusions are given.

2. Modified Riemann–Liouville derivative and some of its properties

In literature, some alternative definitions of fractional derivatives, such as the Caputo, Grünwald–Letnikov, Weyl and Riesz fractional derivatives [3,6, 20–22] are considered. Each fractional derivative presents some advantages and disadvantages. For instance, the Riemann–Liouville derivative of a constant is not zero. The Caputo derivative of a constant is zero, but it is defined only for differentiable functions, while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense [3].

The Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{Bmatrix} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-\xi)^{-\alpha-1}[f(\xi)-f(0)]\mathrm{d}\xi, \\ \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t (t-\xi)^{-\alpha}[f(\xi)-f(0)]\mathrm{d}\xi, \\ 0<\alpha<1 \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, \quad n \le \alpha < n+1, \quad n \ge 1 \end{Bmatrix}, \tag{2.1}$$

where $f\colon R \to R$, $t \to f(t)$ and $f(t)$ is a continuous function. Some significant properties of the modified Riemann–Liouville derivative can be summarized as follows [23,24]:

$$D_t^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0 \tag{2.2}$$

$$\int (\mathrm{d}\xi)^\alpha = \xi^\alpha, \tag{2.3}$$

$$\Gamma(1+\alpha)\mathrm{d}t = \mathrm{d}^\alpha t. \tag{2.4}$$

3. Fractional complex transformation and MSE method

Suppose that a fractional partial differential equation, say in the independent variables t, x is given by

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, ...) = 0, \quad 0 < \alpha, \beta < 1, \tag{3.1}$$

where u is an unknown function, P is a polynomial in u and their various partial derivatives include fractional derivatives.

The main steps of the MSE method are summarized as follows:

Step 1. To find the exact solution of eq. (3.1), we introduce the following fractional complex transformation:

$$u(x,t) = u(\xi), \qquad \xi = \frac{kx^\beta}{(1+\beta)} + \frac{ct^\alpha}{(1+\alpha)}. \tag{3.2}$$

Employing eq. (3.2), we can rewrite eq. (3.1) in the following nonlinear ODE:

$$Q(u, u', u'', u''', ...) = 0, \tag{3.3}$$

where the prime denotes the derivation with respect to ξ. Equation (3.2) is then integrated as many times as possible, setting the constant of integration to be zero.

Step 2. The solution of eq. (3.2) can be expressed by a polynomial in $(\phi'(\xi)/\phi(\xi))$, i.e.,

$$u(\xi) = \sum_{k=0}^{m} a_k \left[\frac{\phi'(\xi)}{\phi(\xi)}\right]^k, \tag{3.4}$$

where $a_k (k = 0, 1, 2, ..., m)$ are arbitrary constants to be determined such that $a_m \neq 0$, and $\phi(\xi)$ is an unknown function to be determined later. In the tanh-function method, (G'/G)-expansion method, exp-function method, etc., the solution is represented in terms of some predefined functions, but in the modified simple method, ϕ is not predefined or not a solution of any predefined equation. Therefore, some fresh solutions may be found by this method. This is the distinction of the MSE method [25–27].

Step 3. The positive integer m can be determined by considering homogeneous balance between the highest-order derivative term with the highest-order nonlinear term appearing in eq. (3.2).

Step 4. Substitute eq. (3.4) into eq. (3.3). As a result of this substitution, a polynomial of $\phi^{-j}(\xi)$ is verified with the derivatives of $\phi(\xi)$. We equate all the coefficients of $\phi^{-j}(\xi)$ to zero, where $j \ge 0$. This operation yields a system which can be solved to find $a_k (k = 0, 1, 2, ..., m)$ and $\phi(\xi)$. Substituting the values of a_k and $\phi(\xi)$ into eq. (3.4) completes the determination of the solution of eq. (3.1)

4. Applications

In this section, we construct exact solutions of the following two nonlinear fractional partial differential equations using the proposed method of §2.

4.1 *Fractional biological population model equation*

We first consider the fractional biological population model equation of the form [28]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) + h(u^2 - r), \quad t > 0, x, y \in \mathbb{R}, \tag{4.1}$$

where u denotes the population density and $h(u^2 - r)$ represents the population supply due to births and deaths. To find the exact solution we first introduce the following travelling wave transformation:

$$u = u(\xi), \qquad \xi = kx + ly + ct + \xi_0. \tag{4.2}$$

So, eq. (4.1) is reduced to the following nonlinear fractional ODE:

$$c^\alpha D_\xi^\alpha - h(u^2 - r) = 0, \quad l = ik, \quad i^2 = -1. \tag{4.3}$$

Balancing D_ξ^α and u^2 terms in eq. (4.3), the balance number $m = 1$ is found. Then the solution function (3.3) takes the form

$$u(\xi) = a_0 + a_1\left(\frac{\phi'(\xi)}{\phi(\xi)}\right), \tag{4.4}$$

where a_0, a_1 ($a_1 \neq 0$) are constants and $\phi(\xi)$ is a function to be determined later. We substitute eq. (4.4) into eq. (4.3) and collect all the terms with the same power of ϕ^{-j} ($j = 0, 1, 2$)

$$h(r - a_0^2) + \frac{-2ha_0a_1\phi'(\xi) + c^\alpha a_1\phi''(\xi)}{\phi(\xi)} + \frac{-a_1(ha_1 + c^\alpha)(\phi'(\xi))^2}{\phi^2(\xi)} = 0. \tag{4.5}$$

Equating each coefficient to zero yields a set of the following algebraic equations:

$$\begin{aligned}
&\phi^0(\xi):\ h(r - a_0^2) = 0,\\
&\phi^{-1}(\xi):\ -2ha_0a_1\phi'(\xi) + c^\alpha a_1\phi''(\xi) = 0,\\
&\phi^{-2}(\xi):\ -a_1(ha_1 + c^\alpha)(\phi'(\xi))^2 = 0.
\end{aligned} \tag{4.6}$$

Solving this set of algebraic equations by using *Maple*, we get

$$a_0 = \pm\sqrt{r}, \qquad a_1 = -\frac{c^\alpha}{h}. \tag{4.7}$$

Then

$$\phi(\xi) = c_1 + c_2 e^{(2\sqrt{r}h/c^\alpha)\xi} \tag{4.8}$$

is verified, where c_1 and c_2 are arbitrary constants. Substituting eqs (4.7) and (4.8) into eq. (4.4), we have the exact solution of fractional-order biological population model equation (4.1) as follows:

$$u(\xi) = \sqrt{r} - \frac{2c_2\sqrt{r}\left(\cosh\left(\frac{2\sqrt{r}h}{c^\alpha}\xi\right) + \sinh\left(\frac{2\sqrt{r}h}{c^\alpha}\xi\right)\right)}{c_1 + c_2\left(\cosh\left(\frac{2\sqrt{r}h}{c^\alpha}\xi\right) + \sinh\left(\frac{2\sqrt{r}h}{c^\alpha}\xi\right)\right)},$$

where $\xi = kx + ly + ct + \xi_0$.

Note that our solutions are new and more extensive than the ones given in [29]. When the parameters are given special values, the solitary waves are derived from the travelling waves (figure 1).

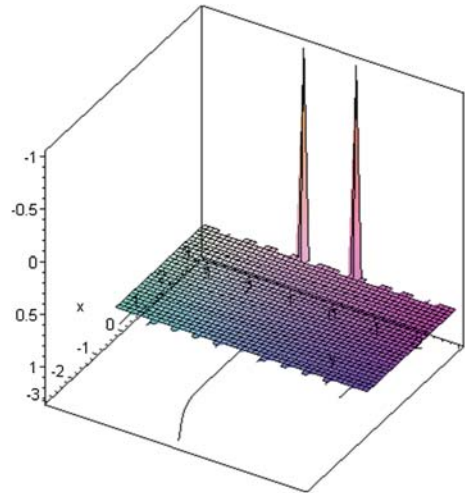

Figure 1. Graph of $u(x, t)$ corresponding to the value $\alpha = 0.5$ when $r = 1$, $h = -1$, $c = 1$, $c_1 = 2$, $c_2 = -2$, $k = 3$, $l = 3i$, $\xi_0 = 0$, $t = 10$.

4.2 *Nonlinear fractional Klein–Gordon equation*

Secondly, we consider nonlinear fractional Klein–Gordon equation [30]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2} + \theta_1 u + \theta_2 u^3, \quad t > 0, \qquad 0 < \alpha \leq 1, \tag{4.9}$$

where θ_1 and θ_2 are arbitrary constants. Let us now solve eq. (4.9) using the proposed method of §2. To this end we suppose that

$$u(x, t) = y(\xi), \qquad \xi = lx - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha. \tag{4.10}$$

Then by using eq. (4.10), eq. (4.9) can be turned into the following ODE with integer order:

$$(\lambda^2 - l^2)y'' - \theta_1 y - \theta_2 y^3 = 0. \tag{4.11}$$

Here $'$ denotes derivative with respect to ξ. Balancing the highest order of derivative term y'' and highest nonlinear term y^2 in eq. (4.11), we have the balance number as $m = 1$. Therefore, the solution (3.3) takes the form eq. (4.4). We substitute eq. (4.4) into eq. (4.11) and collect all the terms with the same power of ϕ^{-j} ($j = 0$, 1, 2, 3). Equating each coefficient to zero yields a set of the following algebraic equations:

$$\begin{aligned}
&\phi^0(\xi):\ -\theta_1 a_0 - \theta_2 a_0^3 = 0,\\
&\phi^{-1}(\xi):\ a_1(\lambda^2 - l^2)\phi'''(\xi)\\
&\qquad -a_1(\theta_1 + 3\theta_2 a_0^2)\phi'(\xi) = 0,\\
&\phi^{-2}(\xi):\ a_1(3l^2 - 3\lambda^2)\phi''(\xi)\,\phi'(\xi)\\
&\qquad -3\theta_2 a_0 a_1^2(\phi'(\xi))^2 = 0,\\
&\phi^{-3}(\xi):\ 2a_1(\lambda^2 - l^2)(\phi'(\xi))^3\\
&\qquad -\theta_2 a_1^3(\phi'(\xi))^3 = 0.
\end{aligned} \tag{4.12}$$

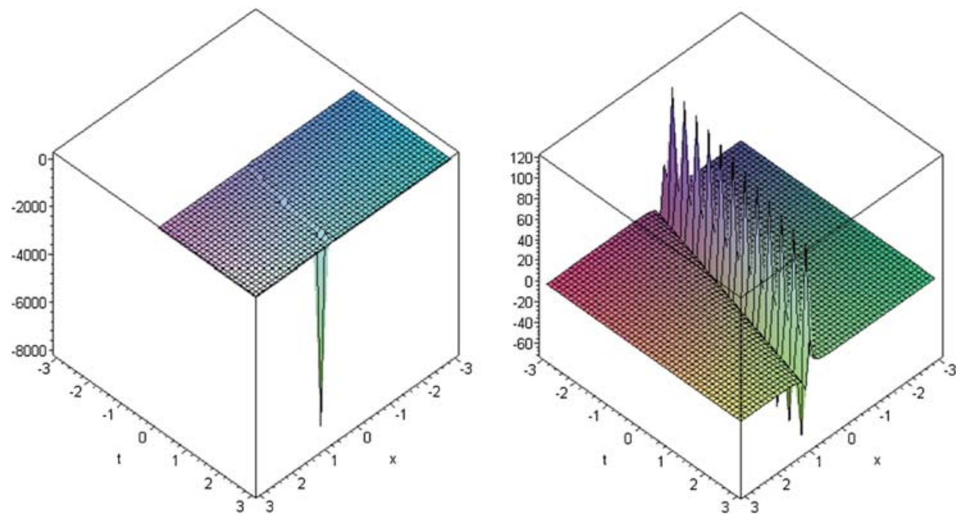

Figure 2. Graph of $u(x,t)$ corresponding to the values $\alpha = 0.5$, 1 from left to right when $\theta_1 = 8$, $\theta_2 = -1$, $c_1 = -1$, $c_2 = 2$, $\lambda = 2$, $l = 8$.

Solving eqs (4.12), we get

$$a_0 = \sqrt{-\frac{\theta_1}{\theta_2}}, \qquad a_1 = \sqrt{\frac{2(\lambda^2 - l^2)}{\theta_2}} \tag{4.13}$$

and

$$\phi(\xi) = c_1 + c_2 \mathrm{e}^{\sqrt{(2(\lambda^2 - l^2)/\theta_2)}\xi}. \tag{4.14}$$

Substituting eqs (4.13) and (4.14) into eq. (4.4), we have the exact solution of nonlinear fractional-order Klein–Gordon equation (4.9) as follows:

$$u(\xi) = \sqrt{-\frac{\theta_1}{\theta_2}}\left(\frac{c_1 - c_2\cosh\left(\sqrt{\frac{2\theta_1}{(l^2-\lambda^2)}}\xi\right) - c_2\sinh\left(\sqrt{\frac{2\theta_1}{(l^2-\lambda^2)}}\xi\right))}{c_1 + c_2(\cos h\left(\sqrt{\frac{2\theta_1}{(l^2-\lambda^2)}}\xi\right) + c_2\sinh\left(\sqrt{\frac{2\theta_1}{(l^2-\lambda^2)}}\xi\right))}\right). \tag{4.15}$$

Here $\xi = lx - [\lambda/\Gamma(1+\alpha)]t^{\alpha}$.

Note: Comparing our solutions with [30,31], it can be seen that by choosing suitable values for the parameters, similiar solutions can be verified (figure 2).

5. Conclusions

In this paper, we have successfully applied the modified simple equation method to solve two nonlinear fractional partial differential equations. This method is also a standard, direct and computerizable method, which allows us to do complicated and tedious algebraic calculations. So, the method we dealt with can be extended to solve many nonlinear fractional partial differential equations which are arising in the theory of solitons and other areas of mathematical physics and engineering. All solutions in this paper have been verified using *Maple* packet program. Thus, we conclude that the proposed method can be extended to solve nonlinear fractional problems which arise in the theory of solitons and other areas.

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