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Quantum mechanics of $\mathcal{PT}$ and non-$\mathcal{PT}$-symmetric potentials in three dimensions

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Abstract. With a view of exploring new vistas with regard to the nature of complex eigenspectra of a non-Hermitian Hamiltonian, the quasi-exact solutions of the Schrödinger equation are investigated for a shifted harmonic potential under the framework of extended complex phase-space approach. Analyticity property of the eigenfunction alone is found sufficient to throw light on the nature of the eigenvalues and eigenfunctions of a system. Explicit expressions of eigenvalues and eigenfunctions for the ground state as well as excited state including their $\mathcal{PT}$-symmetric version are worked out.

Keywords. Ansatz; complex potential; $\mathcal{PT}$-symmetry; eigenvalues and eigenfunctions.

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1. Introduction

The study of complex potentials has generated a lot of interest [1–5] for the better theoretical understanding of some newly discovered phenomena in various streams of science like optical model of a nucleus, delocalization transitions in condensed matter systems such as a vortex flux line depinning in type-II superconductors, population biology, in the description of a Bose system of hard spheres, energy spectra of complex Toda lattice, quantum cosmology, quantum field theory, supersymmetric quantum mechanics etc. [6,7] and phenomena pertaining to resonance scattering in atomic, molecular and nuclear physics and some chemical reactions [8,9]. In addition to this, complex Hamiltonian has been studied in several other theoretical contexts also, e.g., the studies of complex trajectories with regard to the calculation of a semiclassical coherent state propagator in the path integral method have drawn peculiar interest in laser physics [10]. Besides some general studies of a complex Hamiltonian in a non-linear domain [4,5], some efforts have been made to study both classical as well as quantum aspects of a complex Hamiltonian system [11–13]. In the classical context, the Hamiltonian becomes the function of complex variables and the use of Cauchy–Riemann condition for the analyticity of $H(x, y, z, p_x, p_y, p_z) = H_1(x_1, p_1, x_2, p_2, x_3, p_3) + iH_2(x_1, p_1, x_2, p_2, x_3, p_3)$ leads to several interesting features regarding the integrability of the associated real system. In the quantum context, on the other hand, analyticity of $H(x, y, z, p_x, p_y, p_z)$ gets translated into that of the complex potential. A complex Hamiltonian is no longer Hermitian and ordinarily does not guarantee for real eigenvalues. However, in $\mathcal{PT}$-symmetric form, the system is found to exhibit real eigenvalue spectrum [14]. The reality of the spectrum is a direct consequence of the combined action of parity and time-reversal invariance of Hamiltonian [15,16]. Thus, the $\hat{\mathcal{P}}\hat{\mathcal{T}}$-operation is defined as

$$\hat{\mathcal{P}}\hat{\mathcal{T}} : (x, y, z, p_x, p_y, p_z, i) \rightarrow (-x, -y, -z, p_x, p_y, p_z, -i), \tag{1}$$

where $\hat{\mathcal{P}}^2\hat{\mathcal{T}}^2 = 1$. In literature, various methods are available for complexifying a given Hamiltonian system [10,11]. Here we use the scheme given by Xavier and de-Aguiar [17] to transform the potential in extended complex phase-space approach (ECPSA) characterized by

$$x = x_1 + ip_4, \quad y = x_2 + ip_5, \quad z = x_3 + ip_6$$
$$p_x = p_1 + ix_4, \quad p_y = p_2 + ix_5, \quad p_z = p_3 + ix_6. \tag{2}$$

The presence of variables, x_4, x_5, x_6, p_1, p_2, p_3, in the above transformations represents coordinate–momentum interactions of a dynamical system. Note that in this complexifying scheme, the degrees of freedom of the underlying system become double and (x_1, p_1), (x_2, p_2), (x_3, p_3), (x_4, p_4), (x_5, p_5) and (x_6, p_6) turn to be canonical pairs. Similar transformations to eq. (2) have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma [12]. Some authors have also obtained the solutions of SE using the extended complex phase-space approach (ECPSA), but their studies are confined to the two-dimensional systems. No effort has been made towards higher dimensions to explore the possibilities of finding more applications. With this motivation and to extend the domain of the applications, we extend the study of complex coupled shifted harmonic potential in three dimensions.

The paper is organized as follows: in §2, we develop a simple mathematical prescription to obtain the general results under the elegance of extended complex phase-space approach. Exploiting the same results, the eigenspectra of a coupled shifted harmonic potential and its variant including the $\mathcal{PT}$-symmetric version are worked out in §3. Finally, concluding remarks are made in §4.

2. General results

The SE (for $\hbar = m = 1$) for a three-dimensional system is written as

$$\hat{H}(x, y, z, p_x, p_y, p_z)\psi(x, y, z) = E\psi(x, y, z), \quad (3)$$

where,

$$\hat{H}(x, y, z, p_x, p_y, p_z) = -\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\right) + V(x, y, z). \quad (4)$$

Equation (2) yields

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_4}\right), \quad \frac{\mathrm{d}}{\mathrm{d}p_x} = \frac{1}{2}\left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}y} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_5}\right), \quad \frac{\mathrm{d}}{\mathrm{d}p_y} = \frac{1}{2}\left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial p_6}\right), \quad \frac{\mathrm{d}}{\mathrm{d}p_z} = \frac{1}{2}\left(\frac{\partial}{\partial p_3} - i\frac{\partial}{\partial x_6}\right). \quad (5)$$

The corresponding real and imaginary parts of $V(x, y, z)$, $\psi(x, y, z)$ and $E(x, y, z)$ are written as

$$V(x, y, z) = V_r(x_1, p_4, x_2, p_5, x_3, p_6) + iV_i(x_1, p_4, x_2, p_5, x_3, p_6),$$
$$\psi(x, y, z) = \psi_r(x_1, p_4, x_2, p_5, x_3, p_6) + i\psi_i(x_1, p_4, x_2, p_5, x_3, p_6),$$
$$E = E_r + iE_i. \quad (6)$$

On substituting eqs (4)–(6) in eq. (3) and then separating the real and imaginary parts, we have

$$-\frac{1}{8}\big(\psi_{r,x_1x_1} - \psi_{r,p_4p_4} + 2\psi_{i,x_1p_4} + \psi_{r,x_2x_2} - \psi_{r,p_5p_5} + 2\psi_{i,x_2p_5} + \psi_{r,x_3x_3} - \psi_{r,p_6p_6} + 2\psi_{i,x_3p_6}\big) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \quad (7)$$

$$-\frac{1}{8}\big(\psi_{i,x_1x_1} - \psi_{i,p_4p_4} - 2\psi_{r,x_1p_4} + \psi_{i,x_2x_2} - \psi_{i,p_5p_5} - 2\psi_{r,x_2p_5} + \psi_{i,x_3x_3} - \psi_{i,p_6p_6} - 2\psi_{r,x_3p_6}\big) + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r, \quad (8)$$

where, subscripts r and i denote the real and imaginary parts of the corresponding quantities and other subscripts to these quantities separated by comma will denote the partial derivatives of the quantity concerned. For the wave function $\psi(x, y, z)$, Cauchy–Riemann condition suggests

$$\psi_{r,x_1} = \psi_{i,p_4}, \quad \psi_{r,x_2} = \psi_{i,p_5}, \quad \psi_{r,x_3} = \psi_{i,p_6},$$
$$\psi_{r,p_4} = -\psi_{i,x_1}, \quad \psi_{r,p_5} = -\psi_{i,x_2}, \quad \psi_{r,p_6} = -\psi_{i,x_3}. \quad (9)$$

After imposing condition (9) on eqs (7) and (8), we have

$$-\frac{1}{2}(\psi_{r,x_1x_1} + \psi_{r,x_2x_2} + \psi_{r,x_3x_3}) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \quad (10)$$

$$-\frac{1}{2}(\psi_{i,x_1x_1} + \psi_{i,x_2x_2} + \psi_{i,x_3x_3}) + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r. \quad (11)$$

For the eigenfunction $\psi(x, y, z)$, we make an ansatz as [15,16]

$$\psi(x, y, z) = \phi(x, y, z)\exp\big[g(x, y, z)\big], \quad (12)$$

where, $\phi = \phi_r + i\phi_i$ and $g = g_r + ig_i$. The real and imaginary parts of the wave function $\psi(x, y, z)$ becomes

$$\psi_r = e^{g_r}(\phi_r \cos g_i - \phi_i \sin g_i),$$
$$\psi_i = e^{g_r}(\phi_i \cos g_i + \phi_r \sin g_i). \quad (13)$$

The analyticity condition (9) for the functions g_r and g_i turns out to be

$$g_{r,x_1} = g_{i,p_4}, \quad g_{r,x_2} = g_{i,p_5}, \quad g_{r,x_3} = g_{i,p_6},$$
$$g_{i,x_1} = -g_{r,p_4}, \quad g_{i,x_2} = -g_{r,p_5}, \quad g_{i,x_3} = -g_{r,p_6}. \quad (14)$$

Substituting eqs (13) and (14) in eqs (10) and (11), one finds

$$\begin{aligned} &g_{r,x_1x_1} + g_{r,x_2x_2} + g_{r,x_3x_3} + (g_{r,x_1})^2 + (g_{r,x_2})^2 \\ &+(g_{r,x_3})^2 - (g_{i,x_1})^2 - (g_{i,x_2})^2 - (g_{i,x_3})^2 + \frac{1}{(\phi_r^2 + \phi_i^2)} \\ &\times[\phi_r(\phi_{r,x_1x_1} + \phi_{r,x_2x_2} + \phi_{r,x_3x_3} + 2\phi_{r,x_1}g_{r,x_1} \\ &+2\phi_{r,x_2}g_{r,x_2} + 2\phi_{r,x_3}g_{r,x_3} - 2\phi_{i,x_1}g_{i,x_1} \\ &-2\phi_{i,x_2}g_{i,x_2} - 2\phi_{i,x_3}g_{i,x_3}) + \phi_i(\phi_{i,x_1x_1} + \phi_{i,x_2x_2} \\ &+\phi_{i,x_3x_3} + 2\phi_{r,x_1}g_{i,x_1} + 2\phi_{r,x_2}g_{i,x_2} + 2\phi_{r,x_3}g_{i,x_3} \\ &+2\phi_{i,x_1}g_{r,x_1} + 2\phi_{i,x_2}g_{r,x_2} + 2\phi_{i,x_3}g_{r,x_3})] \\ &+2(E_r - V_r) = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} &g_{i,x_1x_1} + g_{i,x_2x_2} + g_{i,x_3x_3} + 2g_{r,x_1}g_{i,x_1} + 2g_{r,x_2}g_{i,x_2} \\ &+2g_{r,x_3}g_{i,x_3} + \frac{1}{(\phi_r^2 + \phi_i^2)}[\phi_r(\phi_{i,x_1x_1} + \phi_{i,x_2x_2} \\ &+\phi_{i,x_3x_3} + 2\phi_{r,x_1}g_{i,x_1} + 2\phi_{r,x_2}g_{i,x_2} + 2\phi_{r,x_3}g_{i,x_3} \\ &+2\phi_{i,x_1}g_{i,x_1} + 2\phi_{i,x_2}g_{i,x_2} + 2\phi_{i,x_3}g_{i,x_3}) \\ &+\phi_i(-\phi_{r,x_1x_1} - \phi_{r,x_2x_2} - \phi_{r,x_3x_3} + 2\phi_{i,x_1}g_{i,x_1} \\ &+2\phi_{i,x_2}g_{i,x_2} + 2\phi_{i,x_3}g_{i,x_3} - 2\phi_{r,x_1}g_{i,x_1} \\ &-2\phi_{r,x_2}g_{i,x_2} - 2\phi_{r,x_3}g_{i,x_3})] + 2(E_i - V_i) = 0. \end{aligned} \quad (16)$$

The ground state solutions can be obtained by selecting $\phi(x, y, z)$ as constant. Then, under such a choice eqs (15) and (16) reduce to

$$\begin{aligned} &g_{r,x_1x_1} + g_{r,x_2x_2} + g_{r,x_3x_3} + (g_{r,x_1})^2 + (g_{r,x_2})^2 \\ &+(g_{r,x_3})^2 - (g_{i,x_1})^2 - (g_{i,x_2})^2 - (g_{i,x_3})^2 \\ &+2(E_r - V_r) = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} &g_{i,x_1x_1} + g_{i,x_2x_2} + g_{i,x_3x_3} + 2g_{r,x_1}g_{i,x_1} + 2g_{r,x_2}g_{i,x_2} \\ &+2g_{r,x_3}g_{i,x_3} + 2(E_i - V_i) = 0. \end{aligned} \quad (18)$$

For the given functional form of $\phi(x, y, z)$ and $g(x, y, z)$, eqs (15) and (16) yield excited state solutions, whereas for the constant value of $\phi(x, y, z)$, eqs (17) and (18) give rise to ground state solutions of a given complex potential.

3. Illustrative examples

In this section, we find the eigenspectra of a coupled shifted harmonic potential and its variant including $\mathcal{PT}$-symmetric version as

3.1 *Coupled shifted harmonic potential*

Consider a coupled shifted harmonic potential in three dimensions as

$$\begin{aligned} V(x, y, z) = &a_{10}x + a_{20}y + a_{30}z + a_{11}x^2 + a_{22}y^2 \\ &+a_{33}z^2 + a_{12}xy + a_{23}yz + a_{31}zx, \end{aligned} \quad (19)$$

where, the parameters a_{ij}'s are complex constants.

Under transformation (2), the real and imaginary parts of the potential (19) are written as

$$\begin{aligned} V_r = &a_{10r}x_1 - a_{10i}p_4 + a_{20r}x_2 - a_{20i}p_5 + a_{30r}x_3 \\ &-a_{30i}p_6 + a_{11r}(x_1^2 - p_4^2) - 2a_{11i}x_1p_4 \\ &+a_{22r}(x_2^2 - p_5^2) - 2a_{22i}x_2p_5 + a_{33r}(x_3^2 - p_6^2) \\ &-2a_{33i}x_3p_6 + a_{12r}(x_1x_2 - p_4p_5) \\ &-a_{12i}(x_1p_5 + x_2p_4) + a_{23r}(x_2x_3 - p_5p_6) \\ &-a_{23i}(x_2p_6 + x_3p_5) + a_{31r}(x_1x_3 - p_4p_6) \\ &-a_{31i}(x_1p_6 + x_3p_4), \end{aligned} \quad (20)$$

$$\begin{aligned} V_i = &a_{10i}x_1 + a_{10r}p_4 + a_{20i}x_2 + a_{20r}p_5 + a_{30i}x_3 \\ &+a_{30r}p_6 + a_{11i}(x_1^2 - p_4^2) + 2a_{11r}x_1p_4 \\ &+a_{22i}(x_2^2 - p_5^2) + 2a_{22r}x_2p_5 + a_{33i}(x_3^2 - p_6^2) \\ &+2a_{33r}x_3p_6 + a_{12i}(x_1x_2 - p_4p_5) \\ &+a_{12r}(x_1p_5 + x_2p_4) + a_{23i}(x_2x_3 - p_5p_6) \\ &+a_{23r}(x_2p_6 + x_3p_5) + a_{31i}(x_1x_3 - p_4p_6) \\ &+a_{31r}(x_1p_6 + x_3p_4). \end{aligned} \quad (21)$$

The functional forms of g_r and g_i in consonance with the conditions (14) are considered as

$$\begin{aligned} g_r = &\alpha_{10}x_1 - \beta_{10}p_4 + \alpha_{20}x_2 - \beta_{20}p_5 + \alpha_{30}x_3 \\ &-\beta_{30}p_6 - \frac{1}{2}\alpha_1(x_1^2 - p_4^2) + \frac{1}{2}\alpha_2(x_2^2 - p_5^2) \\ &+\frac{1}{2}\alpha_3(x_3^2 - p_6^2) + \beta_1x_1p_4 + \beta_2x_2p_5 + \beta_3x_3p_6 \\ &+\gamma_{12}(x_1x_2 - p_4p_5) - \gamma_{21}(x_1p_5 + x_2p_4) \\ &+\gamma_{23}(x_2x_3 - p_5p_6) - \gamma_{32}(x_2p_6 + x_3p_5) \\ &+\gamma_{31}(x_1x_3 - p_4p_6) - \gamma_{13}(x_1p_6 + x_3p_4), \end{aligned} \quad (22)$$

$$g_i = \beta_{10}x_1 + \alpha_{10}p_4 + \beta_{20}x_2 + \alpha_{20}p_5 + \beta_{30}x_3 + \alpha_{30}p_6 - \frac{1}{2}\beta_1(x_1^2 - p_4^2) - \frac{1}{2}\beta_2(x_2^2 - p_5^2) - \frac{1}{2}\beta_3(x_3^2 - p_6^2) + \alpha_1 x_1 p_4 + \alpha_2 x_2 p_5 + \alpha_3 x_3 p_6 + \gamma_{21}(x_1x_2 - p_4p_5) + \gamma_{12}(x_1p_5 + x_2p_4) + \gamma_{32}(x_2x_3 - p_5p_6) + \gamma_{23}(x_2p_6 + x_3p_5) + \gamma_{13}(x_1x_3 - p_4p_6) + \gamma_{31}(x_1p_6 + x_3p_4). \quad (23)$$

Now, by selecting a suitable form of the function $\phi(x, y, z)$ and substituting eqs (20)–(23) in eqs (15)–(17), the ground state as well as the excited state solutions for the given potential are obtained.

Case 1. *Ground state solutions*. For the ground state solutions, the functional form of $\phi(x, y, z)$ is taken as constant, i.e., $\phi(x, y, z) = 1$. Now, substituting eqs (20)–(23) in eqs (17) and (18) and equating the coefficients of x_1, x_2, x_3, p_4, p_5, p_6 and their various products to zero, we obtain a set of 20 non-repeating equations as

$$E_r = -\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_{10}^2 + \alpha_{20}^2 + \alpha_{30}^2 - \beta_{10}^2 - \beta_{20}^2 - \beta_{30}^2), \quad (24)$$

$$E_i = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3 - \alpha_{10}\beta_{10} - \alpha_{20}\beta_{20} - \alpha_{30}\beta_{30}), \quad (25)$$

$$\alpha_{10}\alpha_1 + \beta_{10}\beta_1 + \alpha_{20}\gamma_{12} - \beta_{20}\gamma_{21} - \beta_{30}\gamma_{13} + \alpha_{30}\gamma_{31} = a_{10r}, \quad (26)$$

$$\alpha_{10}\beta_1 - \beta_{10}\alpha_1 - \alpha_{20}\gamma_{21} - \beta_{20}\gamma_{12} - \beta_{30}\gamma_{31} - \alpha_{30}\gamma_{13} = -a_{10i}, \quad (27)$$

$$\alpha_{20}\alpha_2 + \beta_{20}\beta_2 + \alpha_{10}\gamma_{12} - \beta_{10}\gamma_{21} - \beta_{30}\gamma_{32} + \alpha_{30}\gamma_{23} = a_{20r}, \quad (28)$$

$$\alpha_{20}\beta_2 - \beta_{20}\alpha_2 - \alpha_{10}\gamma_{21} - \beta_{10}\gamma_{12} - \beta_{30}\gamma_{23} - \alpha_{30}\gamma_{32} = -a_{20i}, \quad (29)$$

$$\alpha_{30}\alpha_3 + \beta_{30}\beta_3 + \alpha_{10}\gamma_{31} - \beta_{10}\gamma_{13} - \beta_{20}\gamma_{32} + \alpha_{20}\gamma_{23} = a_{30r}, \quad (30)$$

$$\alpha_{30}\beta_3 - \beta_{30}\alpha_3 - \alpha_{10}\gamma_{13} - \beta_{10}\gamma_{31} - \beta_{20}\gamma_{23} - \alpha_{20}\gamma_{32} = -a_{30i}, \quad (31)$$

$$\alpha_1^2 - \beta_1^2 + \gamma_{12}^2 - \gamma_{21}^2 + \gamma_{31}^2 - \gamma_{13}^2 = 2a_{11r}, \quad (32)$$

$$\alpha_1\beta_1 - \gamma_{12}\gamma_{21} - \gamma_{13}\gamma_{31} = -a_{11i}, \quad (33)$$

$$\alpha_2^2 - \beta_2^2 + \gamma_{12}^2 - \gamma_{21}^2 + \gamma_{23}^2 - \gamma_{32}^2 = 2a_{22r}, \quad (34)$$

$$\alpha_2\beta_2 - \gamma_{12}\gamma_{21} - \gamma_{23}\gamma_{32} = -a_{22i}, \quad (35)$$

$$\alpha_3^2 - \beta_3^2 + \gamma_{31}^2 - \gamma_{13}^2 + \gamma_{23}^2 - \gamma_{32}^2 = 2a_{33r}, \quad (36)$$

$$\alpha_3\beta_3 - \gamma_{31}\gamma_{13} - \gamma_{23}\gamma_{32} = -a_{33i}, \quad (37)$$

$$(\alpha_1 + \alpha_2)\gamma_{12} + (\beta_1 + \beta_2)\gamma_{21} + \gamma_{23}\gamma_{31} - \gamma_{13}\gamma_{32} = a_{12r}, \quad (38)$$

$$-(\alpha_1 + \alpha_2)\gamma_{21} + (\beta_1 + \beta_2)\gamma_{12} - \gamma_{23}\gamma_{13} - \gamma_{31}\gamma_{32} = a_{12i}, \quad (39)$$

$$(\alpha_2 + \alpha_3)\gamma_{23} + (\beta_2 + \beta_3)\gamma_{32} + \gamma_{12}\gamma_{31} - \gamma_{13}\gamma_{21} = a_{23r}, \quad (40)$$

$$-(\alpha_2 + \alpha_3)\gamma_{32} + (\beta_2 + \beta_3)\gamma_{23} - \gamma_{12}\gamma_{13} - \gamma_{31}\gamma_{21} = a_{23i}, \quad (41)$$

$$(\alpha_1 + \alpha_3)\gamma_{31} + (\beta_1 + \beta_3)\gamma_{13} + \gamma_{12}\gamma_{23} - \gamma_{32}\gamma_{21} = a_{31r}, \quad (42)$$

$$-(\alpha_1 + \alpha_3)\gamma_{13} + (\beta_1 + \beta_3)\gamma_{31} - \gamma_{12}\gamma_{32} - \gamma_{23}\gamma_{21} = a_{31i}. \quad (43)$$

As such, the solutions of various α's, β's and γ_{ij}'s in these equations seem to be difficult. For this purpose, one can make a number of choices among the ansatz parameters to obtain the eigenvalue and eigenfunction for a given potential. Therefore, one should make some plausible choices among these potential parameters so that there is no conflict between the general solutions and the well-established results. In this view, we choose

$$\alpha_{10} = \alpha_{20} = -\frac{1}{2}\alpha_{30}, \quad \beta_{10} = \beta_{20} = -\frac{1}{2}\beta_{30},$$

$$\gamma_{12} = \gamma_{21} = \gamma_{23} = \gamma_{32} = \gamma_{31} = \gamma_{13},$$

$$\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = -\gamma_{12}\gamma_{21} = -\gamma_{23}\gamma_{32} = -\gamma_{31}\gamma_{13}.$$

Under the above choices, eqs (26)–(37) immediately lead to

$$\alpha_{10} = \alpha_{20} = -\frac{1}{2}\alpha_{30} = -\frac{a_{10r}u + a_{10i}v}{u^2 + v^2}, \quad (44)$$

$$\beta_{10} = \beta_{20} = -\frac{1}{2}\beta_{30} = \frac{a_{10r}v - a_{10i}u}{u^2+v^2}, \quad (45)$$

$$\alpha_1 = -a_+,\ \ \alpha_2 = -b_+,\ \ \alpha_3 = -c_+, \quad (46)$$

$$\beta_1 = a_-,\quad \beta_2 = b_-,\ \ \beta_3 = c_-, \quad (47)$$

$$\gamma_{12} = \gamma_{21} = \gamma_{23} = \gamma_{32} = \gamma_{31} = \gamma_{13} = \sqrt{\frac{a_{11i}}{3}}, \quad (48)$$

where

$$u = \left(a_+ + \sqrt{\frac{a_{11i}}{3}}\right), \quad v = \left(a_- + 2\sqrt{\frac{a_{11i}}{3}}\right)$$

and

$$a_{11i} = a_{22i} = a_{33i}.$$

$$a_\pm = \sqrt{\pm a_{11r} + \sqrt{a_{11r}^2 + a_{11i}^2/9}},$$

$$b_\pm = \sqrt{\pm a_{22r} + \sqrt{a_{22r}^2 + a_{22i}^2/9}},$$

$$c_\pm = \sqrt{\pm a_{33r} + \sqrt{a_{33r}^2 + a_{33i}^2/9}}.$$

Further, eqs (38) and (43) give the following constraining relations:

$$\sqrt{a_{11i}/3}(-a_+ - b_+ + a_+ + b_-) = a_{12r}, \quad (49)$$

$$\sqrt{a_{11i}/3}(a_+ - b_+ + a_+ + b_-) = a_{12i}, \quad (50)$$

$$\sqrt{a_{11i}/3}(-b_+ - c_+ + b_+ + c_-) = a_{23r}, \quad (51)$$

$$\sqrt{a_{11i}/3}(b_+ - c_+ + b_+ + c_-) = a_{23i}, \quad (52)$$

$$\sqrt{a_{11i}/3}(-c_+ - a_+ + c_+ + a_-) = a_{31r}, \quad (53)$$

$$\sqrt{a_{11i}/3}(c_+ - a_+ + c_+ + a_-) = a_{31i}. \quad (54)$$

Thus, after substituting the values of various ansatz parameters [(46)–(48)] in eqs (24) and (25), the real and imaginary components of eigenvalue are written as

$$E_r^{(0)} = \frac{1}{2}(a_+ + b_+ + c_+) - 3\left[\frac{(a_{10r}^2 - a_{10i}^2)(u^2-v^2) - 4a_{10r}a_{10i}uv}{(u^2+v^2)^2}\right], \quad (55)$$

$$E_i^{(0)} = \frac{1}{2}(a_- + b_- + c_-) + 3\left[\frac{(a_{10r}^2 - a_{10i}^2)uv - 4a_{10r}a_{10i}(u^2-v^2)}{(u^2+v^2)^2}\right] \quad (56)$$

and the eigenfunction is given by

$$\psi^{(0)}(x,y,z) = N\exp\Bigg[-a_{10}\frac{u-iv}{u^2+v^2}(x+y-2z) - \frac{1}{2}(a_+ + ia_-)x^2 - \frac{1}{2}(b_+ + ib_-)y^2 - \frac{1}{2}(c_+ + ic_-)z^2 + (1+i)\sqrt{\frac{a_{11i}}{3}}(xy+yz+zx)\Bigg]. \quad (57)$$

$\mathcal{PT}$-symmetric case. After imposing $\mathcal{PT}$-symmetric condition (1) on potential (19), one finds $a_{11i} = a_{22i} = a_{33i} = a_{12i} = a_{23i} = a_{31i} = 0$, and the potential (19) turns out to be

$$V(x,y,z) = a_{10i}x + a_{20i}y + a_{30i}z + a_{11r}x^2 + a_{22r}y^2 + a_{33r}z^2 + a_{12r}xy + a_{23r}yz + a_{31r}zx. \quad (58)$$

By following the same prescription as above, we find that

$$\gamma_{21} = \gamma_{32} = \gamma_{13} = \beta_1 = \beta_2 = \beta_3 = \alpha_{10} = \alpha_{20} = \alpha_{30} = 0.$$

Then, rationalization of eqs (17) and (18) provide a set of 11 equations as

$$E_r = -\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 - \beta_{10}^2 - \beta_{20}^2 - \beta_{30}^2), \quad (59)$$

$$E_i = 0, \quad (60)$$

$$\beta_{10}\alpha_1 + \beta_{20}\gamma_{12} + \beta_{30}\gamma_{31} = a_{10i}, \quad (61)$$

$$\beta_{20}\alpha_2 + \beta_{10}\gamma_{12} + \beta_{30}\gamma_{23} = a_{20i}, \quad (62)$$

$$\beta_{30}\alpha_3 + \beta_{10}\gamma_{31} + \beta_{20}\gamma_{23} = a_{30i}, \quad (63)$$

$$\alpha_1^2 + \gamma_{12}^2 + \gamma_{31}^2 = 2a_{11r}, \quad (64)$$

$$\alpha_2^2 + \gamma_{12}^2 + \gamma_{23}^2 = 2a_{22r}, \quad (65)$$

$$\alpha_3^2 + \gamma_{31}^2 + \gamma_{23}^2 = 2a_{33r}, \quad (66)$$

$$(\alpha_1 + \alpha_2)\gamma_{12} + \gamma_{23}\gamma_{31} = a_{12r}, \quad (67)$$

$$(\alpha_2 + \alpha_3)\gamma_{23} + \gamma_{12}\gamma_{31} = a_{23r}, \quad (68)$$

$$(\alpha_1 + \alpha_3)\gamma_{31} + \gamma_{12}\gamma_{23} = a_{31r}. \quad (69)$$

On choosing $\gamma_{12} = \gamma_{23} = \gamma_{31} = 2\alpha_1$ and $\beta_{10} = \beta_{20} = -\frac{1}{2}\beta_{30}$, eqs (61)–(66) lead to

$$\beta_{10} = \beta_{20} = -\frac{1}{2}\beta_{30} = 3\frac{a_{10i}}{2a_{11r}}, \quad \alpha_1 = -\frac{1}{3}\sqrt{2a_{11r}},$$
$$\alpha_2 = -\frac{1}{3}\sqrt{18a_{22r} - 16a_{11r}},$$
$$\alpha_3 = -\sqrt{18a_{33r} - 16a_{11r}},$$
$$\gamma_{12} = \gamma_{23} = \gamma_{31} = -\frac{2}{3}\sqrt{2a_{11r}}. \tag{70}$$

Inserting eq. (70) in eqs (59) and (60), the eigenvalues are given by

$$E(r)^{(0)}_{\mathcal{PT}} = \frac{1}{6}\left[\sqrt{2a_{11r}} + \sqrt{18a_{22r} - 16a_{11r}} + \sqrt{18a_{33r} - 16a_{11r}} + \frac{27a_{10i}^2}{2a_{11r}}\right], \tag{71}$$

$$E(i)^{(0)}_{\mathcal{PT}} = 0, \tag{72}$$

and the corresponding eigenfunction becomes

$$\psi^{(0)}(x, y, z)_{\mathcal{PT}} = N \exp\left[\frac{3ia_{10i}}{\sqrt{2a_{11r}}}(x + y - 2z) - \frac{1}{3}\sqrt{2a_{11r}}\, x^2 - \frac{1}{3}\sqrt{18a_{22r} - 16a_{11r}}\, y^2 - \frac{1}{3}\sqrt{18a_{33r} - 16a_{11r}} z^2 - \frac{2}{3}\sqrt{2a_{11r}}(xy + yz + zx)\right], \tag{73}$$

where, the $\mathcal{PT}$-symmetric form of the potential is given by

$$V(x, y, z)_{\mathcal{PT}} = a_{10i}x + a_{20i}y + a_{30i}z + a_{11r}x^2 + a_{22r}y^2 + a_{33r}z^2 + a_{12r}xy + a_{23r}yz + a_{31r}zx.$$

Special case. If $a_{11} = a_{22} = a_{33}$, then under the same prescription as above, the eigenspectra are given by

$$E(r)^{(0)}_{\mathcal{PT}} = \frac{1}{2}\left[\sqrt{2a_{11r}} + 27\frac{a_{10i}^2}{2a_{11r}}\right], \quad E(i)^{(0)}_{\mathcal{PT}} = 0, \tag{74}$$

$$\psi^{(0)}(x, y, z)_{\mathcal{PT}} = N \exp\left[\frac{3ia_{10i}}{\sqrt{2a_{11r}}}(x + y - 2z) - \frac{1}{3}\sqrt{2a_{11r}}(x^2 + y^2 + z^2) - \frac{2}{3}\sqrt{2a_{11r}}(xy + yz + zx)\right], \tag{75}$$

and the $\mathcal{PT}$-symmetric version of the potential becomes

$$V(x, y, z)_{\mathcal{PT}} = a_{10i}x + a_{20i}y + a_{30i}z + a_{11r}(x^2 + y^2 + z^2) + a_{12r}xy + a_{23r}yz + a_{31r}zx.$$

Case 2. *Excited state solutions*. In this case, we are concerned with the excited state solutions of the potential (19). The functional form of $\phi(x, y, z)$ for the first excited state is assumed to be [15,16]

$$\phi(x, y, z) = \alpha x + \beta y + \gamma z + \sigma. \tag{76}$$

Substituting transformation (1) on eq. (76), the real and imaginary parts of $\phi(x, y, z)$ are given by

$$\phi_r = \alpha x_1 + \beta x_2 + \gamma x_3 + \sigma, \tag{77}$$

$$\phi_i(x_1, p_4, x_2, p_5, x_3, p_6) = \alpha p_4 + \beta p_5 + \gamma p_6. \tag{78}$$

Here, α, β, γ and σ are considered to be real constants. On substituting eqs (20)–(23) and eq. (78) in eqs (15) and (16), a complicated expression is obtained, which is very tedious to solve without imposing some restrictions on the ansatz parameters. To avoid this problem, we make the following restrictions on the ansatz parameters:

$$\alpha_{10} = \alpha_{20} = -\frac{1}{2}\alpha_{30}, \quad \beta_{10} = \beta_{20} = \frac{1}{2}\beta_{30},$$
$$\alpha_1 = \alpha_2 = \alpha_3,$$
$$\beta_1 = \beta_2 = \beta_3, \quad \gamma_{12} = \gamma_{21}, \quad \gamma_{23} = \gamma_{32}, \quad \gamma_{31} = \gamma_{13}$$
$$\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = -\gamma_{12}\gamma_{21} = -\gamma_{23}\gamma_{32} = -\gamma_{31}\gamma_{13}.$$

Under these restrictions, rationalization of the resultant expression yields the following set of non-repeating equations:

$$E_r = -\frac{5}{2}\alpha_1 - 2\gamma_{12} - 3(\alpha_{10}^2 - \beta_{10}^2), \tag{79}$$

$$E_i = \frac{5}{2}\beta_1 + 2\gamma_{21} - 3\alpha_{10}\beta_{10}, \tag{80}$$

$$\alpha_1^2 - \beta_1^2 = 2a_{11r}, \tag{81}$$

$$\alpha_1\beta_1 - 2\gamma_{12}\gamma_{21} = -a_{11i}, \tag{82}$$

$$2\alpha_1\gamma_{12} + 2\beta_1\gamma_{21} = a_{12r}, \tag{83}$$

$$-2\alpha_1\gamma_{21} + 2\beta_1\gamma_{12} - 2\gamma_{23}\gamma_{13} = a_{12i}. \tag{84}$$

Now, with these restrictions, eqs (81)–(82) immediately lead to

$$\alpha_1=\alpha_2=\alpha_3=-a_+,\quad \beta_1=\beta_2=\beta_3=a_-, \tag{85}$$

$$\gamma_{12}=\gamma_{21}=\gamma_{23}=\gamma_{32}=\gamma_{31}=\gamma_{13}=\sqrt{\frac{a_{11i}}{3}}. \tag{86}$$

In addition to this, eqs (83)–(84) provide the following constraining relation with the condition $a_{11}=a_{22}=a_{33}$ and $a_{12}=a_{23}=a_{31}$. The constraints obtained from eqs (83)–(84) are

$$a_{11}=a_{22}=a_{33},\quad a_{12}=a_{23}=a_{31}, \tag{87}$$

$$2(-a_+ + a_-)\sqrt{\frac{a_{11i}}{3}}=a_{12r},$$

$$2(a_+ + a_-)\sqrt{\frac{a_{11i}}{3}}-\frac{2a_{11i}}{3}=a_{12i}. \tag{88}$$

Employing the values of the ansatz parameters (eqs (85) and (86)) in eqs (79) and (80), the first excited state energy eigenvalues are given by

$$E_r^{(1)}=\frac{5}{2}a_+-2\sqrt{\frac{a_{11i}}{3}}-3\left[\frac{(a_{10r}^2-a_{10i}^2)(u^2-v^2)-4a_{10r}a_{10i}uv}{(u^2+v^2)^2}\right], \tag{89}$$

$$E_i^{(1)}=\frac{5}{2}a_+2\sqrt{\frac{a_{11i}}{3}}+3\left[\frac{(a_{10r}^2-a_{10i}^2)uv-4a_{10r}a_{10i}(u^2-v^2)}{(u^2+v^2)^2}\right], \tag{90}$$

and the corresponding eigenfunction turns out to be

$$\psi^{(1)}(x,y,z)=\alpha(x+y+z)\,\exp\left[-a_{10}\frac{u-iv}{u^2+v^2}\times(x+y-2z)-\frac{1}{2}(a_++ia_-)\,x^2-\frac{1}{2}(b_++ib_-)\,y^2-\frac{1}{2}(c_++ic_-)\,z^2+(1+i)\sqrt{\frac{a_{11i}}{3}}\,(xy+yz+zx)\right]. \tag{91}$$

$\mathcal{PT}$-symmetric case. By imposing $\mathcal{PT}$-symmetric condition (1) on potential (19), one finds $a_{11i}=a_{22i}=a_{33i}=a_{12i}=a_{23i}=a_{31i}=0$, then under the same prescription as above, the $\mathcal{PT}$ eigenspectra for the first excited state becomes

$$E(r)^{(1)}_{(\mathcal{PT})}=\frac{5}{3}\sqrt{\frac{a_{11r}}{2}}-2\sqrt{\frac{a_{11i}}{3}}+27\frac{a_{10i}^2}{2a_{11r}},$$
$$E(i)^{(1)}_{(\mathcal{PT})}=0, \tag{92}$$

$$\psi^{(1)}_{(\mathcal{PT})}=\alpha(x+y+z)\exp\left[\frac{3ia_{10i}}{\sqrt{2a_{11r}}}(x+y-2z)-\frac{1}{3}\sqrt{2a_{11r}}(x^2+y^2+z^2)-\frac{2}{3}\sqrt{2a_{11r}}(xy+yz+zx)\right], \tag{93}$$

where, the $\mathcal{PT}$-symmetric form of the potential is

$$V(x,y,z)_{\mathcal{PT}}=a_{10i}x+a_{20i}y+a_{30i}z+a_{11r}x^2+a_{22r}y^2+a_{33r}z^2+a_{12r}xy+a_{23r}yz+a_{31r}zx.$$

3.2 *Variant of coupled shifted harmonic potential*

The variant of a coupled shifted harmonic potential by including inverse harmonic and cross terms in the potential (19) is written as

$$V(x,y,z)=V_1+A_1\frac{x}{y}+A_2\frac{y}{x}+B_1\frac{y}{z}+B_2\frac{z}{y}+C_1\frac{z}{x}+C_2\frac{x}{z}+\frac{A_3}{x^2}+\frac{B_3}{y^2}+\frac{C_3}{z^2}+\frac{A_4}{x}+\frac{B_4}{y}+\frac{C_4}{z}, \tag{94}$$

where, the parameters A_i, B_i, C_i with $i=1,2,3,4$ are complex constants. After substituting transformation (1) on the potential (94), we have

$$V_r=V_{r1}+\frac{A_{3r}(x_1^2-p_4^2)}{(x_1^2+p_4^2)^2}+\frac{2A_{3i}x_1p_4}{(x_1^2+p_4^2)^2}+\frac{B_{3r}(x_2^2-p_5^2)}{(x_2^2+p_5^2)^2}+\frac{2B_{3i}\,x_2p_5}{(x_2^2+p_5^2)^2}+\frac{C_{3r}(x_3^2-p_6^2)}{(x_3^2+p_6^2)^2}+\frac{2C_{3i}x_3p_6}{(x_3^2+p_6^2)^2}+\frac{A_{1r}(x_1x_2+p_4p_5)}{(x_2^2+p_5^2)}-\frac{A_{1i}(x_2p_4-x_1p_5)}{(x_2^2+p_5^2)}$$

$$+\frac{A_{2r}(x_1x_2+p_4p_5)}{(x_1^2+p_4^2)}+\frac{A_{2i}(x_2p_4-x_1p_5)}{(x_1^2+p_4^2)}$$
$$+\frac{B_{1r}(x_2x_3+p_5p_6)}{(x_3^2+p_6^2)}-\frac{B_{1i}(x_3p_5-x_2p_6)}{(x_3^2+p_6^2)}$$
$$+\frac{B_{2r}(x_2x_3+p_4p_5)}{(x_2^2+p_5^2)}+\frac{B_{2i}(x_3p_5-x_2p_6)}{(x_2^2+p_5^2)}$$
$$+\frac{C_{1r}(x_1x_3+p_4p_6)}{(x_1^2+p_4^2)}-\frac{C_{1i}(x_1p_6-x_3p_4)}{(x_1^2+p_4^2)}$$
$$+\frac{C_{2r}(x_1x_3+p_4p_6)}{(x_3^2+p_6^2)}+\frac{C_{2i}(x_1p_6-x_3p_4)}{(x_3^2+p_6^2)}$$
$$+\frac{A_{4r}x_1}{x_1^2+p_4^2}+\frac{A_{4i}p_4}{x_1^2+p_4^2}+\frac{B_{4r}x_2}{x_2^2+p_5^2}+\frac{B_{4i}p_5}{x_2^2+p_5^2}$$
$$+\frac{C_{4r}x_3}{x_3^2+p_6^2}+\frac{C_{4i}p_6}{x_3^2+p_6^2}, \tag{95}$$

$$V_i = V_{i1}+\frac{A_{3i}(x_1^2-p_4^2)}{(x_1^2+p_4^2)^2}-\frac{2A_{3r}x_1p_4}{(x_1^2+p_4^2)^2}$$
$$+\frac{B_{3i}(x_2^2-p_5^2)}{(x_2^2+p_5^2)^2}-\frac{2B_{3r}x_2p_5}{(x_2^2+p_5^2)^2}$$
$$+\frac{C_{3i}(x_3^2-p_6^2)}{(x_3^2+p_6^2)^2}-\frac{2C_{3r}x_3p_6}{(x_3^2+p_6^2)^2}$$
$$+\frac{A_{1i}(x_1x_2+p_4p_5)}{(x_2^2+p_5^2)}+\frac{A_{1r}(x_2p_4-x_1p_5)}{(x_2^2+p_5^2)}$$
$$+\frac{A_{2i}(x_1x_2+p_4p_5)}{(x_1^2+p_4^2)}-\frac{A_{2r}(x_2p_4-x_1p_5)}{(x_1^2+p_4^2)}$$
$$+\frac{B_{1i}(x_2x_3+p_5p_6)}{(x_3^2+p_6^2)}+\frac{B_{1r}(x_3p_5-x_2p_6)}{(x_3^2+p_6^2)}$$
$$+\frac{B_{2i}(x_2x_3+p_4p_5)}{(x_2^2+p_5^2)}-\frac{B_{2r}(x_3p_5-x_2p_6)}{(x_2^2+p_5^2)}$$
$$+\frac{C_{1i}(x_1x_3+p_4p_6)}{(x_1^2+p_4^2)}+\frac{C_{1r}(x_1p_6-x_3p_4)}{(x_1^2+p_4^2)}$$
$$+\frac{C_{2i}(x_1x_3+p_4p_6)}{(x_3^2+p_6^2)}-\frac{C_{2r}(x_1p_6-x_3p_4)}{(x_3^2+p_6^2)}$$
$$+\frac{A_{4i}x_1}{x_1^2+p_4^2}-\frac{A_{4r}p_4}{x_1^2+p_4^2}+\frac{B_{4i}x_2}{x_2^2+p_5^2}-\frac{B_{4r}p_5}{x_2^2+p_5^2}$$
$$+\frac{C_{4i}x_3}{x_3^2+p_6^2}-\frac{C_{4r}p_6}{x_3^2+p_6^2}, \tag{96}$$

where, V_{r1} and V_{i1} are the same as given by eqs (20) and (21). The ansatz for the eigenfunction consistent with eqs (95) and (96), and satisfying (14) are written as

$$g_r = g_{r1}+\delta_1\tan^{-1}\frac{x_1}{p_4}+\delta_2\tan^{-1}\frac{x_2}{p_5}+\delta_3\tan^{-1}\frac{x_3}{p_6}$$
$$-\frac{\gamma_1}{2}\ln(x_1^2+p_4^2)-\frac{\gamma_2}{2}\ln(x_2^2+p_5^2)$$
$$-\frac{\gamma_3}{2}\ln(x_3^2+p_6^2), \tag{97}$$

$$g_i = g_{i1}+\gamma_1\tan^{-1}\frac{x_1}{p_4}+\gamma_2\tan^{-1}\frac{x_2}{p_5}+\gamma_3\tan^{-1}\frac{x_3}{p_6}$$
$$+\frac{\delta_1}{2}\ln(x_1^2+p_4^2)+\frac{\delta_2}{2}\ln(x_2^2+p_5^2)$$
$$+\frac{\delta_3}{2}\ln(x_3^2+p_6^2), \tag{98}$$

where, g_{r1} and g_{i1} are the same as given by eqs (22) and (23). When eqs (95)–(98) are employed in eqs (17) and (18) then, rationalization of the resultant expression yields the following 26 equations in addition to eqs (32)–(43):

$$E_r = -\frac{1}{2}[(\alpha_1+\alpha_2+\alpha_3+\alpha_{10}^2+\alpha_{20}^2+\alpha_{30}^2$$
$$-\beta_{10}^2-\beta_{20}^2-\beta_{30}^2)+2(\delta_1\beta_1+\delta_2\beta_2+\delta_3\beta_3$$
$$-\gamma_1\alpha_1-\gamma_2\alpha_2-\gamma_3\alpha_3)], \tag{99}$$

$$E_i = -\frac{1}{2}\left[(\beta_1+\beta_2+\beta_3-\alpha_{10}\beta_{10}-\alpha_{20}\beta_{20}\right.$$
$$-\alpha_{30}\beta_{30})+2(\delta_1\alpha_1+\alpha_2\delta_2+\delta_3\alpha_3-\gamma_1\beta_1$$
$$\left.-\gamma_2\beta_2-\gamma_3\beta_3)\right], \tag{100}$$

$$\gamma_1^2-\delta_1^2+\gamma_1 = 2A_{3r}, \tag{101}$$
$$2\gamma_1\delta_1+\delta_1 = -2A_{3i}, \tag{102}$$
$$\gamma_2^2-\delta_2^2+\gamma_2 = 2B_{3r}, \tag{103}$$
$$2\gamma_2\delta_2+\delta_2 = -2B_{3i}, \tag{104}$$
$$\gamma_3^2-\delta_1^2+\gamma_3 = 2C_{3r}, \tag{105}$$
$$2\gamma_3\delta_3+\delta_3 = -2C_{3i}, \tag{106}$$
$$\gamma_{12}\gamma_2+\gamma_{21}\delta_2 = -A_{1r}, \tag{107}$$
$$\gamma_{21}\gamma_2-\gamma_{12}\delta_2 = -A_{1i}, \tag{108}$$
$$\gamma_{12}\gamma_1+\gamma_{21}\delta_1 = -A_{2r}, \tag{109}$$
$$\gamma_{21}\gamma_1-\gamma_{12}\delta_1 = -A_{2i}, \tag{110}$$

$$\gamma_{23}\gamma_3 + \gamma_{32}\delta_3 = -B_{1r}, \quad (111)$$

$$\gamma_{32}\gamma_3 - \gamma_{23}\delta_3 = -B_{1i}, \quad (112)$$

$$\gamma_{23}\gamma_2 + \gamma_{32}\delta_2 = -B_{2r}, \quad (113)$$

$$\gamma_{32}\gamma_2 - \gamma_{23}\delta_2 = -B_{2i}, \quad (114)$$

$$\gamma_{31}\gamma_1 + \gamma_{13}\delta_1 = -C_{1r}, \quad (115)$$

$$\gamma_{13}\gamma_1 - \gamma_{31}\delta_1 = -C_{1i}, \quad (116)$$

$$\gamma_{31}\gamma_3 + \gamma_{13}\delta_3 = -C_{2r}, \quad (117)$$

$$\gamma_{13}\gamma_3 - \gamma_{31}\delta_3 = -C_{2i}, \quad (118)$$

$$\alpha_{10}\gamma_1 + \beta_{10}\delta_1 = -A_{4r}, \quad (119)$$

$$\alpha_{10}\delta_1 - \beta_{10}\gamma_1 = A_{4i}, \quad (120)$$

$$\alpha_{20}\gamma_2 + \beta_{20}\delta_2 = -B_{4r}, \quad (121)$$

$$\alpha_{20}\delta_2 - \beta_{20}\gamma_2 = B_{4i}, \quad (122)$$

$$\alpha_{30}\gamma_3 + \beta_{30}\delta_3 = -C_{4r}, \quad (123)$$

$$\alpha_{30}\delta_3 - \beta_{30}\gamma_3 = C_{4i}. \quad (124)$$

Now, all α_i, β_i and γ_{ij} are the same as represented by eq. (70), whereas eqs (101)–(106) are solved to obtain δ_is, γ_is as

$$\delta_1 = -\frac{4A_{3i}}{A_1}, \quad \delta_2 = -\frac{4B_{3i}}{B_1}, \quad \delta_3 = -\frac{4C_{3i}}{C_1}, \quad (125)$$

$$\gamma_1 = -\frac{1}{2} + \frac{k_1}{4}, \quad \gamma_2 = -\frac{1}{2} + \frac{k_2}{4},$$

$$\gamma_3 = -\frac{1}{2} + \frac{k_3}{4}, \quad (126)$$

where

$$k_1 = \sqrt{2 + 16A_{3r} + 2\sqrt{1 + 16(A_{3r} + 4|A_3|^2)}},$$

$$k_2 = \sqrt{2 + 16B_{3r} + 2\sqrt{1 + 16(B_{3r} + 4|B_3|^2)}},$$

$$k_3 = \sqrt{2 + 16C_{3r} + 2\sqrt{1 + 16(C_{3r} + 4|C_3|^2)}}.$$

Further, eqs (107)–(118) yield the following constraining relations:

$$\sqrt{\frac{a_i}{3}}\left(-\frac{4B_{3i}}{k_2} - \frac{1}{2} + \frac{k_2}{4}\right) = -A_{1r},$$

$$\sqrt{\frac{a_i}{3}}\left(\frac{4B_{3i}}{k_2} - \frac{1}{2} + \frac{k_2}{4}\right) = -A_{1i}, \quad (127)$$

$$\sqrt{\frac{a_i}{3}}\left(-\frac{4A_{3i}}{k_1} - \frac{1}{2} + \frac{k_1}{4}\right) = -A_{2r}, \quad (128)$$

$$\sqrt{\frac{a_i}{3}}\left(\frac{4A_{3i}}{k_1} - \frac{1}{2} + \frac{k_1}{4}\right) = -A_{2i}, \quad (129)$$

$$\sqrt{\frac{a_i}{3}}\left(-\frac{4C_{3i}}{k_3} - \frac{1}{2} + \frac{k_3}{4}\right) = -B_{1r}, \quad (130)$$

$$\sqrt{\frac{a_i}{3}}\left(\frac{4C_{3i}}{k_3} - \frac{1}{2} + \frac{k_3}{4}\right) = -B_{1i}, \quad (131)$$

$$16(a_{1r}v - a_{1i}u)A_{3i} + k_1(a_{1r}u + a_{1i}v)(k_1 - 2)$$
$$= 4k_1A_{4r}(u^2 + v^2), \quad (132)$$

$$16(a_{1r}u + a_{1i}v)A_{3i} - k_1(a_{1r}v - a_{1i}u)(k_1 - 2)$$
$$= 4k_1A_{4i}(u^2 + v^2), \quad (133)$$

$$16(a_{1r}v - a_{1i}u)B_{3i} + k_2(a_{1r}u + a_{1i}v)(k_2 - 2)$$
$$= 4k_2B_{4r}(u^2 + v^2), \quad (134)$$

$$16(a_{1r}u + a_{1i}v)A_{3i} - k_2(a_{1r}v - a_{1i}u)(k_2 - 2)$$
$$= 4k_2B_{4i}(u^2 + v^2), \quad (135)$$

$$16(a_{1r}v - a_{1i}u)C_{3i} + k_3(a_{1r}u + a_{1i}v)(k_3 - 2)$$
$$= -2k_3C_{4r}(u^2 + v^2), \quad (136)$$

$$16(a_{1r}u + a_{1i}v)C_{3i} - k_3(a_{1r}v - a_{1i}u)(k_3 - 2)$$
$$= -2k_3C_{4i}(u^2 + v^2). \quad (137)$$

Now, inserting the values of various ansatz parameters in eqs (99) and (100), the eigenvalues are written as

$$E_r^{(0)} = a_+\left(1 - \frac{k_1}{4}\right) + b_+\left(1 - \frac{k_2}{4}\right) + c_+\left(1 - \frac{k_3}{4}\right)$$
$$+ \frac{4A_{3i}}{k_1}a_- + \frac{4B_{3i}}{k_2}b_- + \frac{4c_{3i}}{k_3}c_- - 3$$
$$\times \left[\frac{(a_{10r}^2 - a_{10i}^2)(u^2 - v^2) - 4a_{10r}a_{10i}uv}{(u^2 + v^2)^2}\right], \quad (138)$$

$$E_i^{(0)} = a_-\left(1-\frac{k_1}{4}\right)+b_-\left(1-\frac{k_2}{4}\right)+c_-\left(1-\frac{k_3}{4}\right) -\frac{4A_{3i}}{k_1}a_- -\frac{4B_{3i}}{k_2}b_- -\frac{4c_{3i}}{k_3}c_- +3 \times\left[\frac{(a_{10r}^2-a_{10i}^2)uv-4a_{10r}a_{10i}(u^2-v^2)}{(u^2+v^2)^2}\right], \quad (139)$$

and the corresponding eigenfunction turns out to be

$$\psi^{(0)}(x,y,z) = (x_1^2+p_4^2)^{\frac{i}{2}(\delta_1+i\gamma_1)}(x_2^2+p_5^2)^{\frac{i}{2}(\delta_2+i\gamma_2)} \times(x_3^2+p_6^2)^{\frac{i}{2}(\delta_3+i\gamma_3)} \times\exp\left[-a_{10}\frac{u-iv}{u^2+v^2}(x+y-2z) -\frac{1}{2}(a_++ia_-)x^2-\frac{1}{2}(b_++ib_-)y^2 -\frac{1}{2}(c_++ic_-)z^2+(1+i)\sqrt{\frac{a_{11i}}{3}} \times(xy+yz+zx)+(\gamma_1+i\delta_1) \times\tan^{-1}\frac{x_1}{p_4}+(\gamma_2+i\delta_2)\tan^{-1}\frac{x_2}{p_5} +(\gamma_3+i\delta_3)\tan^{-1}\frac{x_3}{p_6}\right]. \quad (140)$$

$\mathcal{PT}$-symmetric case. Under the $\mathcal{PT}$-symmetric condition (1), potential (94) yields $a_{11i}=a_{22i}=a_{33i}=a_{12i}=a_{23i}=a_{31i}=A_{ni}=B_{ni}=C_{ni}=\delta_n=\beta_n=0$ where $n=1,2,3,4$, and hence the potential eq. (94) becomes

$$V(x,y,z) = V_1+A_{1r}\frac{x}{y}+A_{2r}\frac{y}{x}+B_{1r}\frac{y}{z}+B_{2r}\frac{z}{y} +C_{1r}\frac{z}{x}+C_{2r}\frac{x}{z}+\frac{A_{3r}}{x^2}+\frac{B_{3r}}{y^2}+\frac{C_{3r}}{z^2} +\frac{A_{4i}}{x}+\frac{B_{4i}}{y}+\frac{C_{4i}}{z}, \quad (141)$$

where, V_{r1} is the same as given by eq. (58). After adopting the same procedure as above, we find the following set of equations in addition to eqs (64)–(69), i.e.,

$$E_r = \left[\alpha_1\left(\gamma_1-\frac{1}{2}\right)+\alpha_2\left(\gamma_2-\frac{1}{2}\right) +\alpha_3\left(\gamma_3-\frac{1}{2}\right)\right], \quad (142)$$

$$E_i = 0, \quad (143)$$

$$\gamma_1^2+\gamma_1 = 2A_{3r}, \quad (144)$$

$$\gamma_2^2+\gamma_2 = 2B_{3r}, \quad (145)$$

$$\gamma_3^2+\gamma_3 = 2C_{3r}, \quad (146)$$

$$\gamma_{12}\gamma_2 = -A_{1r}, \quad (147)$$

$$\gamma_{12}\gamma_1 = -A_{2r}, \quad (148)$$

$$\gamma_{23}\gamma_3 = -B_{1r}, \quad (149)$$

$$\gamma_{23}\gamma_2 = -B_{2r}, \quad (150)$$

$$\gamma_{31}\gamma_1 = -C_{1r}, \quad (151)$$

$$\gamma_{31}\gamma_3 = -C_{2r}. \quad (152)$$

$$\beta_{10}\gamma_1 = -A_{4i}, \quad (153)$$

$$\beta_{20}\gamma_2 = -B_{4i}, \quad (154)$$

$$\beta_{30}\gamma_3 = -C_{4i}. \quad (155)$$

From eqs (144)–(146), we have

$$\gamma_1 = \frac{-1+\sqrt{1+8A_{3r}}}{2}, \quad \gamma_2 = \frac{-1+\sqrt{1+8B_{3r}}}{2}, \quad \gamma_3 = \frac{-1+\sqrt{1+8C_{3r}}}{2}. \quad (156)$$

On employing the ansatz parameters represented by eqs (70) and (156) in eqs (142) and (143), the energy eigenvalues are written as

$$E(r)^{(0)}_{(\mathcal{PT})} = \frac{1}{6}\left[\sqrt{2a_{11r}}\left(2-\sqrt{1+8A_{3r}}\right) +\sqrt{18a_{22r}-16a_{11r}}\left(2-\sqrt{1+8B_{3r}}\right) +\sqrt{18a_{33r}-16a_{11r}}\left(2-\sqrt{1+8C_{3r}}\right)\right] +\frac{27a_{1i}^2}{2a_{11r}^2}, \quad (157)$$

$$E(i)^{(0)}_{(\mathcal{PT})} = 0 \quad (158)$$

and the eigenfunction is given by

$$\psi^{(0)}_{(\mathcal{PT})}(x,y,z) = (x_1^2+p_4^2)^{-\gamma_1/2}(x_2^2+p_5^2)^{-\gamma_2/2} \times (x_3^2+p_6^2)^{-\gamma_3/2} \times \exp\left[\frac{3ia_{10i}}{\sqrt{2a_{11r}}}(x+y-2z) - \frac{1}{3}\sqrt{2a_{11r}}(x^2+y^2+z^2) - \frac{2}{3}\sqrt{2a_{11r}}(xy+yz+zx) + i\gamma_1\tan^{-1}\frac{x_1}{p_4} + i\gamma_2\tan^{-1}\frac{x_2}{p_5} + i\gamma_3\tan^{-1}\frac{x_3}{p_6}\right], \tag{159}$$

where, the $\mathcal{PT}$-symmetric form of the potential is

$$V_{\mathcal{PT}} = V_1 + A_{1r}\frac{x}{y} + A_{2r}\frac{y}{x} + B_{1r}\frac{y}{z} + B_{2r}\frac{z}{y} + C_{1r}\frac{z}{x} + C_{2r}\frac{x}{z} + \frac{A_{3r}}{x^2} + \frac{B_{3r}}{y^2} + \frac{C_{3r}}{z^2} + \frac{A_{4i}}{x} + \frac{B_{4i}}{y} + \frac{C_{4i}}{z}.$$

4. Concluding remarks

In the present work, we have computed the ground state as well as the excited state solutions of the SE for a coupled shifted harmonic potential and its variant. For this purpose, the ansatz method is utilized and besides the complexity of the phase-space, complexity of the potential parameters is also taken into account [15,16]. It is found that the imaginary part of the eigenvalue always vanishes for the solvable cases of the SE as long as the potential parameters are real. In this respect, the results obtained in the present approach coincides with those derived from the invariance of a given Hamiltonian under $\mathcal{PT}$-operation. The important feature of the method is that it provides additional flexibility for obtaining real eigenspectra of a non-Hermitian Hamiltonian system. It is also emphasized that solutions of the SE obtained in this paper show the presence of some constraining relations among the potential parameters, which give rise to bound-state energies of a system. The interesting aspect of this method is an account of the complex coupling parameters in the potential in addition to the complex phase-space which leads to complex spectra. But for more involved complex systems, particularly in three dimensions, it is too tedious to obtain the eigenspectra of a coupled system in higher dimensions due to the expansion of algebra and problem in choosing appropriate forms of the functions ϕ, g_r and g_i. It is also observed that the real and imaginary parts of the eigenvalue follow just the opposite order for the discrete energy levels by retaining the conventional ordering for the magnitude of the eigenvalues.

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