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Significance of power average of sinusoidal and non-sinusoidal periodic excitations in nonlinear non-autonomous system

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Abstract. Additional sinusoidal and different non-sinusoidal periodic perturbations applied to the periodically forced nonlinear oscillators decide the maintainance or inhibitance of chaos. It is observed that the weak amplitude of the sinusoidal force without phase is sufficient to inhibit chaos rather than the other non-sinusoidal forces and sinusoidal force with phase. Apart from sinusoidal force without phase, i.e., from various non-sinusoidal forces and sinusoidal force with phase, square force seems to be an effective weak perturbation to suppress chaos. The effectiveness of weak perturbation for suppressing chaos is understood with the total power average of the external forces applied to the system. In any chaotic system, the total power average of the external forces is constant and is different for different nonlinear systems. This total power average decides the nature of the force to suppress chaos in the sense of weak perturbation. This has been a universal phenomenon for all the chaotic non-autonomous systems. The results are confirmed by Melnikov method and numerical analysis. With the help of the total power average technique, one can say whether the chaos in that nonlinear system is to be supppressed or not.

Keywords. Chaos; controlling chaos; Melnikov; power average.

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1. Introduction

Chaos is ubiquitous, widespread and observed in numerous physical, chemical and biological systems. The presence of chaos both in nature and in man-made devices is very common and has been extensively demonstrated in recent decades [1]. Even though chaos is useful, as in mixing processes, in heat transfer, or in secure communication [2–4], often it is unwanted or undesirable. For example, increasing drag in flow systems, erratic fibrillations of heart beating, extreme weather predictions, and complicated circuit oscillations [5–7] are some situations where chaos is harmful. Thus, one may wish to avoid or control chaos with minimal efforts and without altering the underlying system significantly.

Recently, investigations on suppressing chaos in nonlinear dynamical systems have became increasingly popular. Various algorithms or methods have been proposed and implemented successfully to avoid the harmful effects of chaos [8–12]. Various control techniques, such as feedback and non-feedback methods [13–15], sliding mode control [16–20], minimum entropy control technique [21] and linearization technique [22] are often implemented for controlling chaos. Feedback and non-feedback methods make use of a small perturbating external force such as a small driving force, a small noise term, a small constant bias or a weak modulation to some system parameters to modify the underlying chaotic dynamical system weakly, so that stable orbit appears [13–15]. (i) Parametric excitation of adjustable parameter [23,24] and (ii) external periodic excitation [23,25–27] are the two basic techniques admitted in non-feedback chaos control [14,15,27–31].

As the success of these chaos control schemes depends strongly on the chance response of a chaotic attractor to the weak perturbation, it is considerably interesting and important to find how ‘weak’ is the perturbation. Here, the term ‘weak’ or ‘small’ essentially means that one would like to effect changes as minimal as possible to the original system so that it is not grossly deformed. For instance, one can usually expect that a small second periodic perturbation should be able to bring the system out of the chaotic region. However, in certain systems a large value of second periodic perturbation is required to inhibit chaos for the system [24,27,32–36]. The reason for having such a large value of second periodic perturbation is still unknown.

So far, most of the methods treat the second harmonic perturbation as sinusoidal force with/without phase effect [27]. For practical purposes, it could be hard to implement an exact phase difference between two periodic forces. Thus, it prompts us to look into other periodic signals which produce the same dynamical effect as by the second periodic force with phase. It is observed in [37,38] that pulsed signals such as square, sawtooth and triangle waves come in handy to handle the situation effectively. Moreover, they can be easily generated and employed successfully in any real systems. In some cases, it is found that the square wave force is the most suitable signal for suppressing chaos in the sense of weak periodic perturbation, whereas in other situations, particularly at resonant perturbation it is not so. The reason why the square wave force cannot suppress chaos in the sense of weak perturbation is still unknown.

On employing second harmonic perturbation to a periodically-driven nonlinear system, certain underlying phenomena like strange non-chaotic attractors (SNA) and vibrational resonance (VR) can also be observed. If the frequency of second harmonic force and driving force are incommensurate, then their ratio will be irrational and one can find the occurence of SNA [39–42]. On the other hand, an optimal amplitude of the high-frequency second harmonic force enhances the response of a nonlinear non-autonomous system to a low-frequency first harmonic signal. Such a resonant behaviour is known as vibrational resonance [43–45]. These phenomena have been realized in various theoretical models [39,46–59] and experimental systems [60–66].

A typical non-feedback method can be generally modelled as

$$\ddot{x} = F(x, f_1(t), f_2(t)), \tag{1}$$

where the first force $f_1(t)$ drives the system into the chaotic state, while the second one $f_2(t)$ is a weak periodic force which suppresses the chaotic behaviour without altering the underlying system significantly.

In the present paper, we explain the facts behind the unanswered question using the power average concept and also whether the chaos in any nonlinear non-autonomous system can be suppressed or not. Power is defined as the rate of doing work by the external forces or amount of energy required to do certain work by the external forces. If the force $f(t)$ is time-dependent, then power obtained should be instantaneous. Power at any instant is defined as the square of the time-dependent force (i.e., $|f(t)|^2$). In general, power average for one complete cycle is defined as the mean of the sum of the instantaneous values of power taken during one complete cycle T and is given as

$$P_{\mathrm{av}} = \frac{1}{T}\int_0^T |f(t)|^2 \mathrm{d}t. \tag{2}$$

For the sum of two different external forces of different time periods say T_1, T_2, the total power average is evaluated by taking a common time period $T = n_1 T_1 = n_2 T_2$, where n_1 and n_2 are integers and their ratios $T_1/T_2 = n_2/n_1$ should be a rational number.

$$P_{\mathrm{av}} = \frac{1}{T}\int_0^T |f_1(t)|^2 \mathrm{d}t + \frac{1}{T}\int_0^T |f_2(t)|^2 \mathrm{d}t. \tag{3}$$

The total power average value of different periodic sinusoidal and non-sinusoidal forces which can be considered as the net external driven force suppresses or induces chaos in a system. If the total power average value of any two external forces namely, driven force and second periodic control force, in any system for suppressing chaos is identified, then it will be the same for all the other external forces in the same system. It is found in our analytical study that the small-amplitude value of square wave force is sufficient to inhibit chaotic behaviour in the system at subharmonic or superharmonic resonance condition. In contrast, small-amplitude value of sinusoidal force is sufficient to inhibit chaos at main resonance condition. This fact is also confirmed by analytically investigating the effect of different periodic perturbation of pulsed signals such as cosine, square, sawtooth and triangular wave on non-autonomous chaotic systems using Melnikov's method. Melnikov's method represents one of the tools in which global information on specific systems can be obtained analytically. It is applied to homoclinic orbits passing through a hyperbolic saddle point. It defines an integral function which measures the first variation of separation between the perturbed stable and unstable manifolds of the hyperbolic saddle point. The integral function, usually called

Melnikov's integral, is a formal way for evaluating the distance, provided that an explicit expression of the unperturbed periodic trajectory is known. From the Melnikov function one can get sufficient condition to inhibit chaos.

The results are also confirmed numerically by evaluating the intersection of the trajectory of the dynamical system on a surface of the section, usually known as Poincaré map. Poincaré map describes exactly the dynamical system with the evolution of time, but is subject to the limitations of numerical work, mainly the accuracy of the numerical integration namely Runge–Kutta fourth (RK IV) order method. Based on the number of intersections of the trajectory on a surface for long time intervals, the periodic or chaotic behaviour is identified and is plotted in a phase plot diagram.

Another tool, namely bifurcation diagram, is different from phase plot diagram in the sense that the evolution of trajectory is measured at every instant of time rather than after specific time period as in the Poincaré map. The purpose of the bifurcation diagram is to display qualitative information about equilibria, across the dyanamical system, obtained by varying the control parameter. In other words, it represents the sudden appearance of qualitatively different solution for a nonlinear system as some parameter is varied. Period doubling, period-halfing and other phenomena that accompany the onset of chaos have been clearly understood from the bifurcation diagram.

2. Analytical treatment for the suppression of chaos through weak periodic excitation

To look at the foregoing ideas in a concrete model, we consider a double well Duffing oscillator

$$\ddot{x} + \alpha\dot{x} - \omega_0^2 x + \beta x^3 = F_1 \cos\omega_1 t + f_2(\omega_2 t), \tag{4}$$

where α, β and ω_0^2 are the damping coefficient, coefficient of stiffness and natural frequency of the system respectively. $f_1(\omega_1 t) = F_1 \cos\omega_1 t$ is the driving sinusoidal force and $f_2(\omega_2 t)$ is the weak periodic force (controlling/suppressing force) which may be either sinusodial or non-sinusoidal. It is a stable model for oscillatory processes in not only physics, but also biology, sociology, chemistry, engineering and even economics [67]. Also it can be easily implemented using electronic circuit as well as mathematical modelling and can be used as a model for more complicated and modified systems in various fields like radio and telecommunications, neurology, human cardiosystem [68–72] etc.

The Melnikov method yields the Melnikov function for the system (4)

$$M^{\pm}(t_0) = A \pm BF_1 \sin\omega_1 t_0 \pm CF_2 \mathrm{har}(\omega_2 t_0), \tag{5}$$

where $\mathrm{har}(\omega_2 t_0)$ means indistinctly $\sin(\omega_2 k t_0)$ or $\cos(\omega_2 k t_0)$, F_1 and F_2 are the amplitudes of the driving and controlling forces, respectively. The Melnikov function $M^{\pm}(t_0)$ will be used here to illustrate the approach to the enhancement or suppression of chaos.

It was mentioned [34] that a sufficient condition for $F_2 = F_{2\,\mathrm{min}}$ to inhibit chaos is,

$$F_2 \geq F_{2\,\mathrm{min}} = (F_1 + A/B) * (B/C) \tag{6}$$

or

$$F_2 \geq F_{2\,\mathrm{min}} = (F_1 + A/B)(\omega_1/\omega_2)R, \tag{7}$$

where $R = \cosh(\pi\omega_2/2\omega_0)/\cosh(\pi\omega_1/2\omega_0)$. For main resonance, $\omega_1 = \omega_2$ and hence $R = 1$. In order to suppress chaos at sub/superharmonic condition in the sense of weak perturbation, $F_{2\,\mathrm{min}}$ can further be modified by changing B to $-B$ and the value R as small as possible. As the value of R can take a very large value, if $\omega_1 \ll \omega_2$, it is impossible to suppress chaos at weaker amplitude in subharmonic resonance condition, whereas, the value of R can have small value if $\omega_1 \gg \omega_2$. Hence it is possible for us to suppress the chaos at weaker amplitudes of second harmonic excitation in superharmonic resonance condition alone.

If the external forces are of the cosine form i.e., $f_1(\omega_1 t) = F_1 \cos(\omega_1 t)$ and $f_2(\omega_2 t) = F_{2\mathrm{ph}} \cos(\omega_2 t + \psi)$, the total power average value of external forces is

$$P_{\mathrm{av}} = \frac{1}{T}\int_0^T |F_1 \cos(\omega_1 t)|^2 \mathrm{d}t + \frac{1}{T}\int_0^T |F_{2\mathrm{ph}} \cos(\omega_2 t + \psi)|^2 \mathrm{d}t, \tag{8}$$

where $\omega_2 = (n_2/n_1)\omega_1$. For primary resonance, $n_2/n_1 = 1$.

Substituting the above condition in eq. (8) and evaluating the integral, one can get

$$P_{\mathrm{av}} = \frac{F_1^2 + F_{2\mathrm{ph}}^2 + 2F_1 F_{2\mathrm{ph}} \cos\psi}{2} \tag{9}$$

or

$$F_{2\mathrm{ph}} = -F_1 \cos\psi + \sqrt{F_1^2 \cos^2\psi + 2P_{\mathrm{av}} - F_1^2}. \tag{10}$$

For sub/superharmonic resonance, $n_2/n_1 \neq 1$ and plugging this condition in eq. (8) and solving the integral, we get

$$P_{\rm av} = \frac{F_1^2 + F_{\rm 2ph}^2}{2} \tag{11}$$

or

$$F_{\rm 2ph}^2 = 2P_{\rm av} - F_1^2. \tag{12}$$

Here F_1 is the strength of the first periodic force which induces chaotic behaviour in the system. $F_{\rm 2ph}$ is the strength of the second periodic perturbation with phase. If $F_{\rm 2ph}^2$ is positive, then $F_{\rm 2ph}$ exists and can lead to the suppression of chaos. On the other hand, if $F_{\rm 2ph}^2$ is negative, then $F_{\rm 2ph}$ becomes imaginary and does not exist in reality. Hence it is impossible to suppress chaos by second periodic perturbation with phase. $P_{\rm av}$ is the total power average value of the external forces requiring to bring back system (4) into a stable orbit. On the other hand, if the phase ψ is zero, then from eq. (9)

$$F_{2\ \text{without phase}} = \sqrt{2P_{\rm av}} - F_1. \tag{13}$$

At main resonance, from eqs (12) and (13), for a fixed value of $P_{\rm av}$ and F_1, it is clearly found that $F_{2\ \text{without phase}}$ is always less than $F_{\rm 2ph}$. Thus, weak amplitude of the sinusoidal force without phase is sufficient to inhibit chaos rather than sinusoidal force with phase.

2.1 *Effect of square force as the second harmonic perturbation*

System (4) can be rewritten as

$$\dot{x} = y,$$
$$\dot{y} = \omega_0^2 x - \beta x^3 + F_1 \cos\omega_1 t + f_2(\omega_2 t) - \alpha\dot{x}. \tag{14}$$

If the second harmonic perturbation is of the square force of the form represented as

$$f_2(\omega_2 t) = \begin{cases} F_2, & 0 \le t < \pi/\omega_2 \\ -F_2, & \pi/\omega_2 \le t < 2\pi/\omega_2, \end{cases} \tag{15}$$

using Fourier series, the above square force can be rewritten as

$$f_2(\omega_2 t) = \frac{4}{\pi}\sum_{k=1,3,5}^{N} \frac{\sin(k\omega_2 t)}{k}. \tag{16}$$

The Melnikov function becomes

$$M(t_0) = \int_{-\infty}^{\infty} y_h[-\alpha y_h + F_1\cos\omega_1(\tau + t_0) + f_2(\omega_2(\tau + t_0))]\mathrm{d}\tau. \tag{17}$$

Using the explicit form of unperturbed system solutions given by

$$[x_h(\tau), y_h(\tau)] = \left[\pm\sqrt{\frac{2\omega_0^2}{\beta}}\,\mathrm{sech}\left(\sqrt{\omega_0^2}\tau\right) \mp\sqrt{\frac{2}{\beta}}\omega_0^2\,\mathrm{sech}\left(\sqrt{\omega_0^2}\tau\right)\tan\left(\sqrt{\omega_0^2}\tau\right)\right], \tag{18}$$

where $\tau = t - t_0$, the above integral is worked out to be

$$M^{\pm}(t_0) = A \pm BF_1\sin\omega_1 t_0 \pm \sum_{k=1,3,5}^{N} C_k F_2 \cos\omega_2 k t_0, \tag{19}$$

where

$$A = \frac{4\alpha(\omega_0^2)^{3/2}}{3\beta}, \qquad B = \sqrt{2/\beta}\pi\omega_1\,\mathrm{sech}\left(\frac{\pi\omega_1}{2\sqrt{\omega_0^2}}\right)$$

and

$$C_k = 4\sqrt{2/\beta}\omega_2\,\mathrm{sech}\left(\frac{k\pi\omega_2}{2\sqrt{\omega_0^2}}\right).$$

The total power average of the external force along with square wave force either at primary, subharmonic or at superharmonic condition gives

$$P_{\rm av} = \frac{F_1^2}{2} + F_{\rm 2sq}^2 \tag{20}$$

or

$$F_{\rm 2sq}^2 = P_{\rm av} - \frac{F_1^2}{2}. \tag{21}$$

If we assume that the effective values of power average of both sinusoidal force with phase and square wave force are the same, it is found from eqs (12) and (21), that $F_{\rm 2sq}^2 = F_{\rm 2ph}^2/2$. It means that the amplitude of the square wave force is $1/\sqrt{2}$ times less than the amplitude of second periodic sinusoidal force with phase. Here for all positive values for $F_{\rm 2sq}^2$, the chaos is suppressed and for negative values it is not so.

2.2 *Effect of sawtooth force as the second harmonic perturbation*

For system (4) if the second harmonic perturbation is the sawtooth force of the form

$$f_2(\omega_2 t) = \begin{cases} \dfrac{2F_2 t}{T}, & 0 < t < \pi/\omega_2, \\ \dfrac{2F_2 t}{T} - 2F_2, & \pi/\omega_2 < t < 2\pi/\omega_2. \end{cases} \quad (22)$$

Then from Fourier series, it can be expressed as

$$f_2(\omega_2 t) = \frac{2}{\pi} \sum_{k=1,2,3}^{N} (-1)^{(k+1)} \frac{\sin(k\omega_2 t)}{k}, \quad (23)$$

and solving the Melnikov integral represented by eq. (17) using eqs (22) and (18), the Melnikov integral is worked out to be

$$M^{\pm}(t_0) = A \pm BF_1 \sin\omega_1 t_0 \pm \sum_{k=1,2,3}^{N} C_k F_2 \cos\omega_2 k t_0, \quad (24)$$

where

$$A = \frac{4\alpha(\omega_0^2)^{3/2}}{3\beta}, \qquad B = \sqrt{2/\beta}\,\pi\omega_1 \,\mathrm{sech}\left(\frac{\pi\omega_1}{2\sqrt{\omega_0^2}}\right)$$

and

$$C_k = 2\sqrt{2/\beta}(-1)^{k+1}\omega_2 \,\mathrm{sech}\left(\frac{k\pi\omega_2}{2\sqrt{\omega_0^2}}\right).$$

The total power average of the external force along with sawtooth wave force either at the main, the subharmonic or at the superharmonic condition yields,

$$P_{\mathrm{av}} = \frac{F_1^2}{2} + \frac{F_{2\mathrm{saw}}^2}{3} \quad (25)$$

or

$$F_{2\mathrm{saw}}^2 = 3\left[P_{\mathrm{av}} - \frac{F_1^2}{2}\right]. \quad (26)$$

Like the sinusoidal force with phase and square force, only positive values for $F_{2\mathrm{saw}}^2$ can suppress chaos. Again we assume that the power average values of both sinusoidal force with phase and sawtooth wave force are the same, and it is noted from eqs (12) and (26), that $F_{2\mathrm{saw}}^2 = 3(F_{2\mathrm{ph}}^2/2)$, the amplitude of the sawtooth wave force is $\sqrt{3/2}$ times greater than the amplitude of the second periodic sinusoidal force with phase.

2.3 *Effect of triangular force as the second harmonic perturbation*

If the second harmonic perturbation in system (4) is the triangular force of the form

$$f_2(\omega_2 t) = \begin{cases} \dfrac{4F_2 t}{T}, & 0 < t < \pi/2\omega_2, \\ -\dfrac{4F_2 t}{T} + 2F_2, & \pi/2\omega_2 < t < 3\pi/2\omega_2, \\ \dfrac{4F_2 t}{T} - 4F_2, & 3\pi/2\omega_2 < t < 2\pi/\omega_2, \end{cases} \quad (27)$$

and its equivalent Fourier form is

$$f_2(\omega_2 t) = \frac{8}{\pi^2} \sum_{k=1,3,5}^{N} (-1)^{((k-1)/2)} \frac{\sin(k\omega_2 t)}{k^2}. \quad (28)$$

By solving the Melnikov integral represented by eq. (17) using eqs (27) and (18), the Melnikov integral is worked out to be

$$M^{\pm}(t_0) = A \pm BF_1 \sin\omega_1 t_0 \pm \sum_{k=1,3,5}^{N} C_k F_2 \cos\omega_2 k t_0, \quad (29)$$

where

$$A = \frac{4\alpha(\omega_0^2)^{3/2}}{3\beta}, \qquad B = \sqrt{2/\beta}\,\pi\omega_1 \,\mathrm{sech}\left(\frac{\pi\omega_1}{2\sqrt{\omega_0^2}}\right)$$

and

$$C_k = 4\sqrt{2/\beta}\frac{\omega_2}{\pi k} \,\mathrm{sech}\left(\frac{k\pi\omega_2}{2\sqrt{\omega_0^2}}\right).$$

The total power average of the external force along with the triangular wave force either at the main, at the subharmonic or at the superharmonic condition provides

$$P_{\mathrm{av}} = \frac{F_1^2}{2} + \frac{F_{2\mathrm{tri}}^2}{3} \quad (30)$$

or

$$F_{2\mathrm{tri}}^2 = 3\left[P_{\mathrm{av}} - \frac{F_1^2}{2}\right]. \quad (31)$$

Similar to the sinusodial force with phase, square force and sawtooth force, positive values for $F_{2\mathrm{tri}}^2$ alone inhibit chaos. From eqs (12) and (31), it is inferred that $F_{2\mathrm{tri}}^2 = 3F_{2\mathrm{ph}}^2/2$, that is, the amplitude of triangular wave force is $\sqrt{3/2}$ times greater than the amplitude of the second periodic sinusoidal force with phase (figure 1).

3. Analytical result

Sufficient condition for inhibiting/suppressing chaos is obtained by substituting the corresponding value of A, B and C_k in eq. (6). Figure 2 explains the minimum value of F_2 (eq. (6)) required to suppress chaos for various values of F_1 for four cases, namely cosine force with phase (solid line), square force (dashed line), sawtooth force (dotted line), triangular force (dash with dotted line) respectively as second harmonic excitations. From figure 2, the region below the line is periodic, while the one above is chaotic. To suppress chaos for system (4), the amplitude of F_1 and F_2 leads to the power average $P_{\mathrm{av}} = 0.3411$ W. Fixing $P_{\mathrm{av}} = 0.3411$, the minimum/threshold value of second periodic force F_2 required to suppress chaos for various values of F_1 for four cases, namely cosine force (eq. (12)), square force (eq. (21)), sawtooth force (eq. (26)), triangular force (eq. (31)) respectively as second harmonic excitations are plotted in figure 1. The region below the threshold curve is periodic, whereas the region above is chaotic in nature.

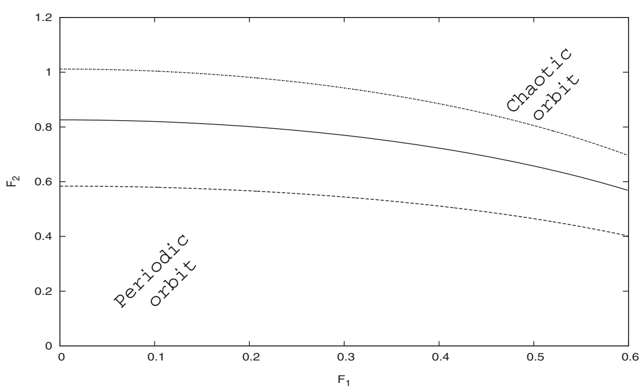

Figure 1. Minimum value of F_2 required to suppress chaos for various amplitudes F_1 using power average method for four cases, namely cosine force with phase (solid line), square force (short dashed line), sawtooth force (long dashed line), triangular force (long dashed line) as second harmonic excitations.

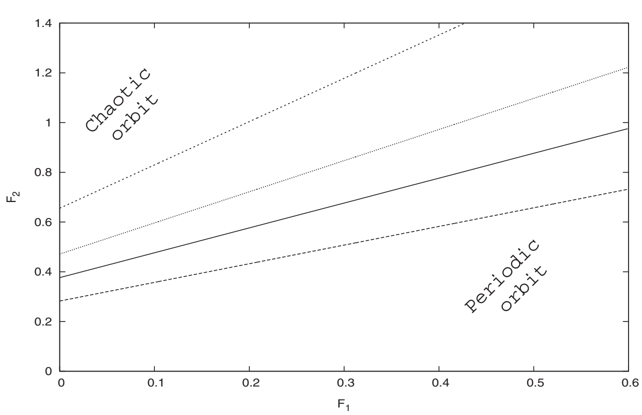

Figure 2. Threshold value of F_2 required to suppress chaos for various amplitudes F_1 using Melnikov method for four cases, namely cosine force with phase (solid line), square force (dashed line), sawtooth force (dotted line), triangular force (dash with dotted line) as second harmonic excitations.

4. Numerical result

Numerical simulation of system (4) is performed using Runge–Kutta fourth-order algorithm by fixing the parameters $\alpha = 0.5$, $\beta = 1.0$, $\omega_0^2 = 1.0$ and $\omega_1 = \omega_2 = 1$. The response of the system to various second periodic forces with frequency $\omega_1 = \omega_2 = 1.0$ (i.e., at main resonance) is analysed by varying the amplitudes F_1 and F_2. The phase plot displays the overall dynamics on the $F_1 - F_2$ plane.

Figures 3a and 3b show the resulting phase plot diagram in the F_1–F_2 plane for eq. (4) with cosine force without phase $\psi = 0$ and cosine force with phase $\psi = 1.5844$ respectively as second periodic forces, and figure 4 for various non-sinusoidal forces namely square force (from eq. (15) and equivalent square force from Fourier series eq. (16)), sawtooth force (from eq. (22) and equivalent sawtooth force from Fourier series eq. (23)) and triangular force (from eq. (27) and equivalent triangular force from Fourier series eq. (28)).

With $F_2 = 0$, as F_1 increases, period T to chaos is observed. In all the five cases (cosine force without phase, cosine force with phase, square force, sawtooth force, triangular force respectively as second harmonic excitations), the chaos is seen in the range $0.35 < F_1 < 0.6$ at lower values of F_2. As F_2 increases, the chaos reaches periodic motion. It is very clear from figures 3 and 4, that the chaos (white region) is spread over in the $F_1 - F_2$ plane. With proper selection of F_2 values, one can eliminate or suppress the chaotic nature. It is found that the chaos can be suppressed earlier for cosine force without phase rather than square force or cosine force with phase or triangular force or sawtooth force. This has been confirmed by evaluating the second harmonic non-sinusoidal excitations by eqs (15), (22), (27) and their corresponding Fourier series eqs (16), (23), (28) respectively.

To elucidate this concept, in the absence of the second force, i.e., $f_2(\omega_2 t) = 0$, it was shown that system (4) exhibits the familiar period-doubling route to chaos, then periodic windows and so on when the amplitude F_1 is varied from zero [34] (see figure 5a).

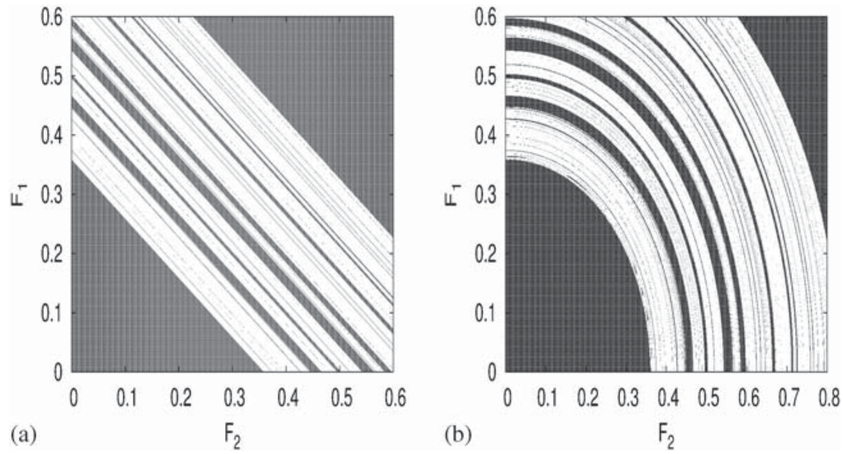

Figure 3. Phase diagram of eq. (4) for the cosine periodic force at the main harmonic resonance (black dot represents periodic region and white space represents chaotic region). (**a**) Cosine force without phase ($\psi = 0$) and (**b**) cosine force with phase ($\psi = 1.5844$) as second harmonic excitations.

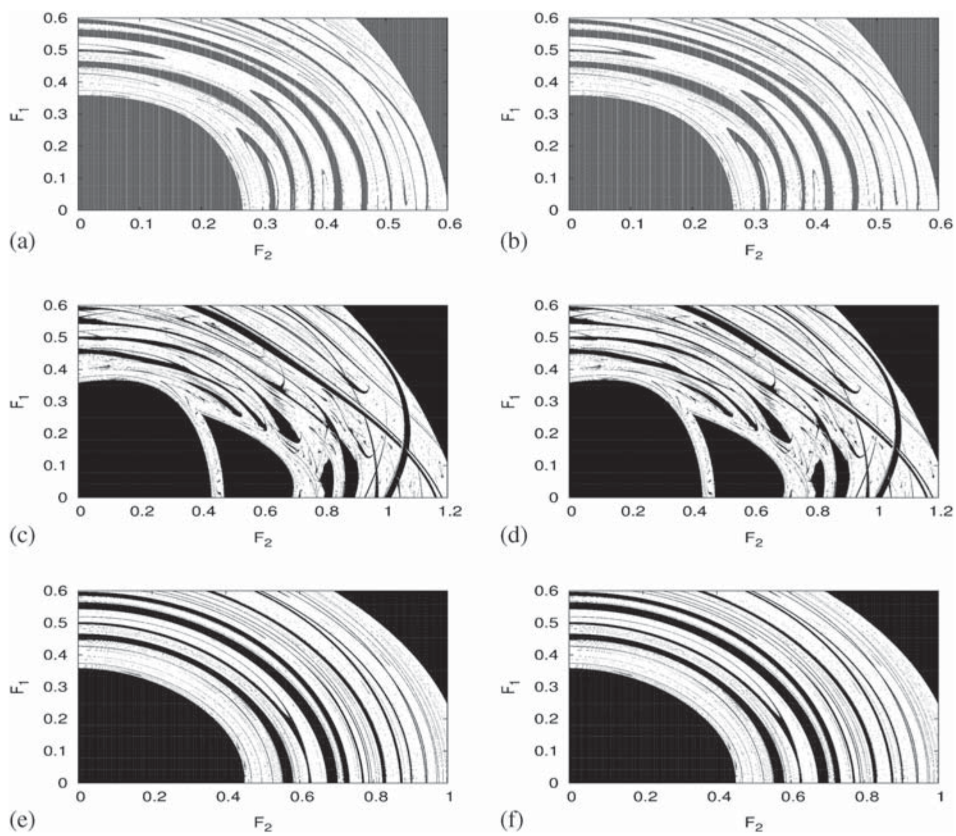

Figure 4. Phase diagram of eq. (4) for various periodic forces at main harmonic resonance (black dot represents periodic region and white space represents chaotic region). (**a**) Square force of the form eq. (15), (**b**) equivalent square force from Fourier series eq. (16), (**c**) sawtooth force of the form eq. (22), (**d**) equivalent sawtooth force using Fourier series eq. (23), (**e**) triangular force of the form eq. (27) and (**f**) equivalent triangular force obtained from Fourier series eq. (28) as second harmonic excitations.

Now, we include periodic force $f_2(\omega_2 t)$ as $f_2(\omega_2 t) = F_2 \cos(\omega_2 t + \psi)$ in eq. (4) with $\omega_2 = \omega_1$. When the parameters are fixed as $F_1 = 0.4$, $\omega_2 = \omega_1 = 1$ and $\psi = 0$, the system exhibits chaotic oscillations at $F_2 = 0$. As one varies the value of F_2 further, it is found that system (4) undergoes inverse period-doubling and finally ends up with the period T orbit [34] (see figure 5b).

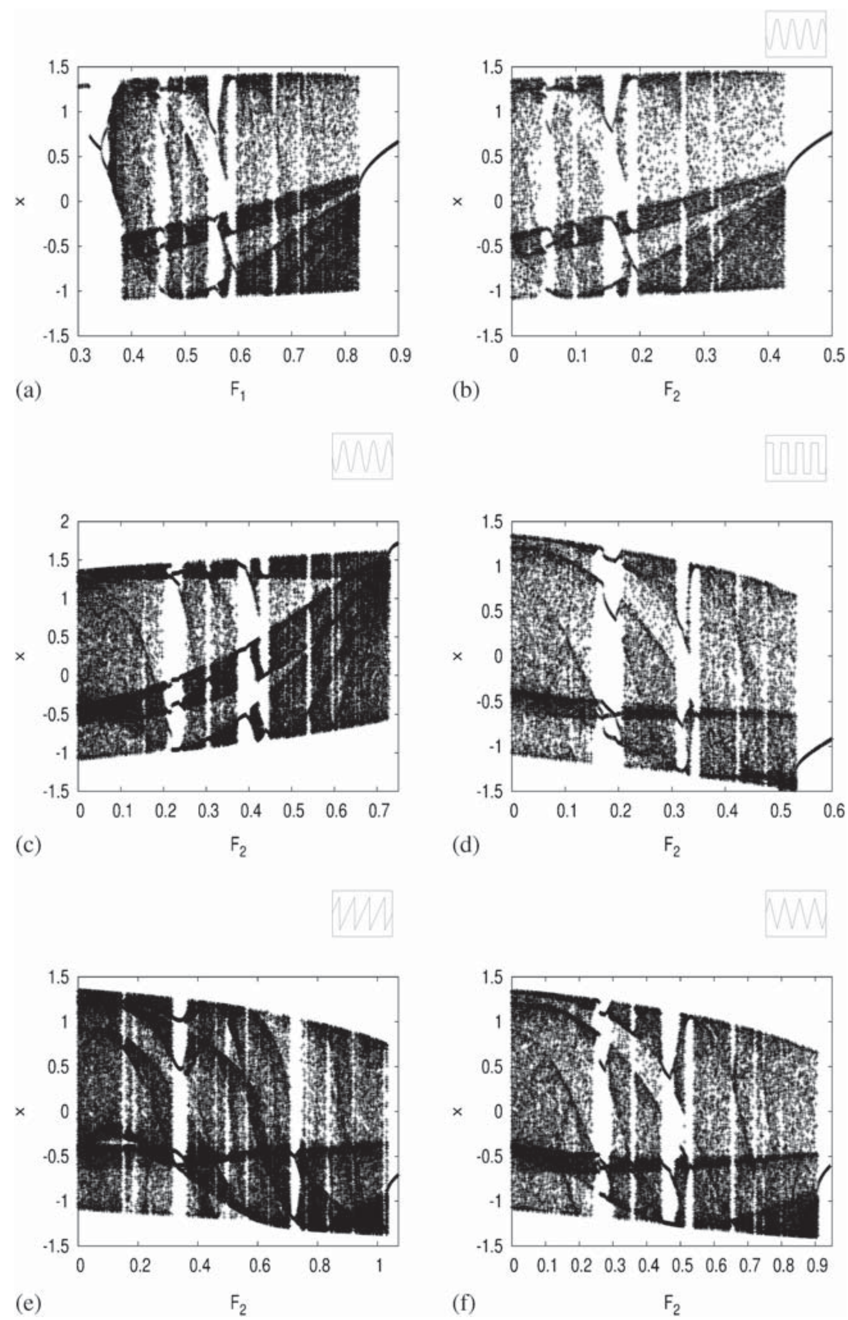

Figure 5. Bifurcation diagram of eq. (4) for various periodic forces as second harmonic at the main resonance. (**a**) Bifurcation diagram in the absence of second harmonic excitations, (**b**) cosine force without phase, (**c**) cosine force with phase, (**d**) square force, (**e**) sawtooth force and (**f**) triangular force as second harmonic excitations.

It is inferred from figure 5b that in order to reach stable periodic orbit, one has to vary the amplitude of the second periodic force without phase to an extent upto $F_2 = 0.426$. With $F_1 = 0.4$, $F_2 = 0.426$ and $\psi = 0$, using eq. (9), it is observed that the power average of the external forces to suppress chaos is found to be $P_{\mathrm{av}} = 0.3411$ W and it remains constant for whatever force is employed as second perturbation. From eq. (13), using power average value $P_{\mathrm{av}} = 0.3411$, it is confirmed analytically that the minimum value for F_2 to suppress chaos is 0.4259.

In the controlling aspects, it would be expected that the second periodic force must be weak in such a way that the amplitude of the second periodic force should be small in comparison with the original driving force. To check for weak perturbation, consider the effect of phase difference between the second periodic force with its original force. It is observed that by fixing

Table 1. Amplitude of different second harmonic excitation forces to suppress chaos in chaotic double well Duffing oscillator system (4) using Melnikov method, power average method and numerical RK IV order method.

Method	Different forms of suppression force (F_2)				
	Cosine $\psi = 0$	Cosine $\psi = 1.5844$	Square	Triangular	Sawtooth
Melnikov	–	0.7765	0.5824	0.9722	1.3522
Power average	0.4259	0.7226	0.5110	0.8850	0.8850
Numerical	0.4260	0.7290	0.5330	0.9070	1.0995

$\psi = 1.5844$ and varying the value of F_2, the system reaches periodic orbit for an amplitude of $F_2 = 0.729$ (see figure 5c). This is in agreement with the theoretically predicted value which is evaluated from eq. (10) as $F_2 = 0.7226$.

Let us now introduce the square wave force of the form represented by eq. (15) as a second periodic perturbation. From eq. (21), with the constant power average value $P_{\text{av}} = 0.3411$, it is obtained theoretically that the amplitude of the square wave force for suppressing chaos is found to be $F_2 = 0.5109$. This is in agreement with the numerically predicted value which is observed from figure 5d that the system undergoes familiar period-halving route and ends up with period T-orbit at $F_2 = 0.533$. It is also interesting to note that the system approaches periodic with larger amplitude than periodic forced sinusoidal signal without phase effect. This is confirmed by the Melnikov analytical method which is calculated from eqs (19) and (6) as $F_2 = 0.5824$.

If the square wave force is replaced by the sawtooth wave represented by eq. (22) as a second periodic perturbation, then from eq. (26) with the constant power average value $P_{\text{av}} = 0.3411$, the amplitude of the sawtooth wave force for suppressing chaos is found to be $F_2 = 0.885$. This is in close agreement with the numerically predicted value which is observed from figure 5e reverse period doubling ends with period T orbit at $F_2 = 1.0995$. It is obvious that the chaotic behaviour is suppressed at $F_2 = 1.0995$, which is larger than the sinusoidal force without/with phase and the square wave. From eqs (24) and (6), it is found analytically by Melnikov method that the value of F_2 is 1.3522.

Alternatively, if the triangular wave represented by eq. (27) is employed as a second periodic perturbation, then from eq. (31) with the constant power average value $P_{\text{av}} = 0.3411$, the amplitude of the triangular wave force for suppressing chaos is found to be $F_2 = 0.885$. This is in close agreement with the numerically predicted value which is observed from figure 5f reverse period doubling ends with period T orbit at $F_2 = 0.907$. It is evident that the chaotic behaviour is suppressed at $F_2 = 0.9722$ from analytical calculation, which is larger than sinusoidal force without/with phase, square wave and smaller than sawtooth force. The above results are listed in table 1.

It is found that from eqs (12), (21), (26) and (31), the values of the second harmonic sinusoidal force with phase and non-sinusoidal forces become imaginary when $P_{\text{av}} < F_1^2/2$. As a result, the sinusoidal force with phase and non-sinusoidal forces are unable to suppress chaos.

5. Conclusion

Finally, we have considered the chaotic double well Duffing oscillator system is subjected to various periodic non-sinusoidal forces as second harmonic excitations. We have studied the role played by the shape of different second harmonic excitation forces on chaotic system. In particular, it is observed that the sinusoidal wave periodic signal is the most suitable perturbation rather than any other non-sinusoidal excitations for suppressing chaos, whereas in non-sinusoidal excitations, square force is more effective rather than others. These results have been confirmed by our analytical expression for power average of different signals, and are also in close agreement with both numerical simulation and Melnikov method.

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