



Bilinearization and new multisoliton solutions for the (4+1)-dimensional Fokas equation

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Abstract. The (4+1)-dimensional Fokas equation is derived in the process of extending the integrable Kadomtsev–Petviashvili and Davey–Stewartson equations to higher-dimensional nonlinear wave equations. This equation is under investigation in this paper. Hirota's bilinear method is, for the first time, used to solve such a higher-dimensional equation. In order to bilinearize the Fokas equation, some appropriate transformations are adopted. As a result, single-soliton solution, double-soliton solution and three-soliton solution are obtained. A new uniform formula of n -soliton solution is derived from this. It is shown that the transformations adopted in this work play a key role in converting the Fokas equation into Hirota's bilinear form.

Keywords. Bilinearization; multisoliton solution; Fokas equation; Hirota's bilinear method.

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1. Introduction

As pointed out by Drazin and Johnson [1], it is not easy to give a comprehensive and precise definition of a soliton. However, one can associate the term with any solution of nonlinear partial differential equations (PDEs) which (i) represents a wave of permanent form, (ii) is localized, so that it decays or approaches a constant at infinity, and (iii) can undergo a strong interaction with other solitons preserving its identity. As the soliton phenomena were first observed in 1834 [2] and the Korteweg–de Vries (KdV) equation was solved by the inverse scattering method [3], finding soliton solutions of nonlinear PDEs has become one of the most exciting and extremely active areas of research. In the past several decades, many methods have been proposed for solving nonlinear PDEs, such as Bäcklund transformation [4], Darboux transformation [5], Hirota's bilinear method [6], homogeneous balance method [7], function expansion methods [8–15] and some others [16–18]. Among them, Hirota's bilinear method [6] is a purely algebraic method for constructing multisoliton solutions [19–25]. The key step of the method is to convert the

given nonlinear PDE into the so-called bilinear form. For such bilinear forms, there is no general rule to follow and one often tries to take some transformations like rational transformation or logarithmic transformation [19].

Recently, Fokas [26] derived a new (4+1)-dimensional equation:

$$u_{tx_1} - \frac{1}{4}u_{x_1x_1x_1x_2} + \frac{1}{4}u_{x_2x_2x_2x_1} + \frac{3}{2}(u^2)_{x_1x_2} - \frac{3}{2}u_{y_1y_2} = 0 \tag{1}$$

in the process of extending the integrable Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations to some higher-dimensional nonlinear wave equations. Yang and Yan [27] constructed Jacobi elliptic double periodic solutions, hyperbolic function solutions and rational solutions of eq. (1) by investigating its symmetries. Lee *et al* [28] obtained some exact solutions of eq. (1) by using modified tanh–coth method, extended Jacobi elliptic function method and the exp-function method. Kim and Sakthivel [29] obtained hyperbolic function solutions, trigonometric function solutions and rational solution of eq. (1) by applying G'/G -expansion method. Zhang and Zhang [30] gave the space–time fractional derivative form of eq. (1) and obtained its generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution. More recently, He *et al* [31,32] obtained many new exact solutions of eq. (1) by using extended F-expansion method. To the best of our knowledge, eq. (1) has not been studied by Hirota’s bilinear method and multisoliton solutions of this equation have not been reported. Integrable nonlinear PDEs possess soliton solutions [33] and have important physical applications ranging from fluid mechanics and nonlinear optics to quantum gravity and field theories [26]. It is well known that both the celebrated KdV equation and the nonlinear Schrödinger (NLS) equation have (exist) n -soliton solutions [2]. As two extensions of the (1+1)-dimensional KdV and NLS equations into (2+1)-dimensional space, the integrable KP and DS equations also possess n -soliton solutions [2]. Does the (4+1)-dimensional Fokas equation (1) has multisoliton solutions? This paper will give a positive answer to this question.

The rest of this paper is organized as follows: In §2, following Hirota’s bilinear method, we first adopt some appropriate transformations to convert eq. (1) into the bilinear form and then construct its multisoliton solutions. In §3, we conclude this paper.

2. Multisoliton solutions

First, we take the following transformation:

$$x = k_1x_1 + k_2x_2, \tag{2}$$

where k_1 and k_2 are undetermined constants. Then eq. (1) becomes

$$\left[u_t + \frac{1}{4}k_2(k_2^2 - k_1^2)u_{xxx} + \frac{3}{2}k_2(u^2)_x \right]_{x_1} - \frac{3}{2}u_{y_1y_2} = 0. \tag{3}$$

Suppose $k_1 \neq k_2$ and let

$$u = (k_2^2 - k_1^2)(\ln f)_{xx}, \quad f = f(t, x_1, x_2, y_1, y_2). \tag{4}$$

Then substituting eq. (4) into eq. (3) and eliminating the common factor $k_2^2 - k_1^2$, we have

$$\left\{ (\ln f)_{xxt} + \frac{1}{4}k_2(k_2^2 - k_1^2)(\ln f)_{xxxx} + \frac{3}{2}k_2(k_2^2 - k_1^2) [(\ln f)_{xx}]_x^2 \right\}_{x_1} - \frac{3}{2}(\ln f)_{xxy_1y_2} = 0,$$

which can be written as follows (when $k_1 \neq 0$):

$$\left\{ (\ln f)_{xt} + \frac{1}{4}k_2(k_2^2 - k_1^2)(\ln f)_{xxx} + \frac{3}{2}k_2(k_2^2 - k_1^2) [(\ln f)_{xx}]^2 - \frac{3}{2k_1}(\ln f)_{y_1y_2} \right\}_{xx} = 0. \tag{5}$$

Integrating eq. (5) with respect to x twice and selecting the integration constants as zeros, we have

$$(\ln f)_{xt} + \frac{1}{4}k_2(k_2^2 - k_1^2)(\ln f)_{xxx} + \frac{3}{2}k_2(k_2^2 - k_1^2) [(\ln f)_{xx}]^2 - \frac{3}{2k_1}(\ln f)_{y_1y_2} = 0, \tag{6}$$

and hence obtain the bilinear form of eq. (1):

$$\left[D_x D_t + \frac{1}{4}k_2(k_2^2 - k_1^2)D_x^4 - \frac{3}{2k_1}D_{y_1}D_{y_2} \right] f \cdot f = 0, \tag{7}$$

where D_x, D_t, D_{y_1} and D_{y_2} are the Hirota's differential operators [19] defined by

$$D_t^m D_x^n f(t, x) \cdot g(t, x) = (\partial_t - \partial_t')^m (\partial_x - \partial_x')^n f(t, x)g(t', x') \Big|_{t'=t, x'=x}.$$

Next we employ eq. (7) to construct multisoliton solutions of eq. (1). For this purpose, it is necessary to consider the boundary condition of eq. (5). As \tanh^2 -type solution [27] of eq. (1) can be rewritten using the relationship $\text{sech}^2 + \tanh^2 = 1$, eq. (1) exists sech^2 -type solution with bell-shaped structure. Inspired by the structural features of such a sech^2 -type solution and eq. (4), in this paper we give an assumption of the boundary condition of eq. (5): $\forall t, (\ln f)_{xx} \rightarrow 0$ for $\eta \rightarrow \pm\infty$. Here η is a linear function of spatial variables x, y_1 and y_2 . In order to construct the single-soliton solution, we suppose

$$f = 1 + f^{(1)}\varepsilon + f^{(2)}\varepsilon^2 + f^{(3)}\varepsilon^3 + \dots, \tag{8}$$

substitute eq. (8) into eq. (7) and then collect the coefficients of the same order of ε . Then, this process yields a system of differential equations

$$2 \left[\partial_x \partial_t + \frac{1}{4}(k_2^2 - k_1^2)k_2 \partial_x^4 - \frac{3}{2k_1} \partial_{y_1} \partial_{y_2} \right] f^{(1)} = 0, \tag{9}$$

$$\begin{aligned} & 2 \left[\partial_x \partial_t + \frac{1}{4}k_2(k_2^2 - k_1^2) \partial_x^4 - \frac{3}{2k_1} \partial_{y_1} \partial_{y_2} \right] f^{(2)} \\ & = - \left[D_x D_t + \frac{1}{4}k_2(k_2^2 - k_1^2)D_x^4 - \frac{3}{2k_1}D_{y_1}D_{y_2} \right] f^{(1)} \cdot f^{(1)}, \end{aligned} \tag{10}$$

$$\begin{aligned}
 & 2 \left[\partial_x \partial_t + \frac{1}{4} k_2 (k_2^2 - k_1^2) \partial_x^4 - \frac{3}{2k_1} \partial_{y_1} \partial_{y_2} \right] f^{(3)} \\
 & = -2 \left[D_x D_t + \frac{1}{4} k_2 (k_2^2 - k_1^2) D_x^4 - \frac{3}{2k_1} D_{y_1} D_{y_2} \right] f^{(1)} \cdot f^{(2)}, \quad (11)
 \end{aligned}$$

and so forth. Let

$$f^{(1)} = e^{\xi_1}, \quad \xi_1 = r_1(\omega_1 t + x + p_1 y_1 + q_1 y_2) + \xi_1^{(0)}, \quad (12)$$

be a solution of eq. (9). Here r_1, ω_1, p_1 and q_1 are constants to be determined, and $\xi_1^{(0)}$ is an arbitrary constant. Insertion of eq. (12) into eq. (9) yields

$$\omega_1 = -\frac{1}{4} k_2 (k_2^2 - k_1^2) r_1^2 + \frac{3}{2k_1} p_1 q_1. \quad (13)$$

Substituting eq. (13) into eqs (10) and (11), we can verify that

$$f^{(n)} = 0, \quad n = 2, 3, \dots \quad (14)$$

In this case, we can write

$$f_1 = 1 + e^{\xi_1}, \quad (15)$$

and hence obtain the following single-soliton solution of eq. (1):

$$u = (k_2^2 - k_1^2) [\ln(1 + e^{\xi_1})]_{xx} = \frac{1}{4} r_1^2 (k_2^2 - k_1^2) \operatorname{sech}^2 \frac{1}{2} \xi_1, \quad (16)$$

where

$$\begin{aligned}
 \xi_1 = r_1 \left[-\frac{1}{4} k_2 r_1^2 (k_2^2 - k_1^2) t + \frac{3}{2k_1} p_1 q_1 t + k_1 x_1 + k_2 x_2 \right. \\
 \left. + p_1 y_1 + q_1 y_2 \right] + \xi_1^{(0)}.
 \end{aligned}$$

In figure 1, the single-soliton solution (16) is shown by selecting $r_1 = 1, k_1 = 1, k_2 = 3, p_1 = 3, q_1 = 2, \xi_1^{(0)} = 0, y_1 = 0, y_2 = 0$, from which we can see that the single-soliton propagating along the negative direction of x -axis possesses a bell-shaped structure.

For the double-soliton solution, we suppose

$$f^{(1)} = e^{\xi_1} + e^{\xi_2}, \quad \xi_2 = r_2(\omega_2 t + x + p_2 y_1 + q_2 y_2) + \xi_2^{(0)}, \quad (17)$$

and substitute eq. (17) into eq. (9). Here r_2, ω_2, p_2 and q_2 are constants to be determined, $\xi_2^{(0)}$ is an arbitrary constant. We then obtain

$$\omega_2 = -\frac{1}{4} k_2 r_2^2 (k_2^2 - k_1^2) + \frac{3}{2k_1} p_2 q_2. \quad (18)$$

In view of eq. (10), we further suppose

$$f^{(2)} = e^{\xi_1 + \xi_2 + A_{12}}, \quad (19)$$

where A_{12} is a constant to be determined. From eqs (10) and (19), we can obtain

$$e^{A_{12}} = \frac{k_1 k_2 (k_2^2 - k_1^2) (r_2 - r_1)^2 + 2(p_2 - p_1)(q_2 - q_1)}{k_1 k_2 (k_2^2 - k_1^2) (r_2 + r_1)^2 + 2(p_2 - p_1)(q_2 - q_1)}. \quad (20)$$

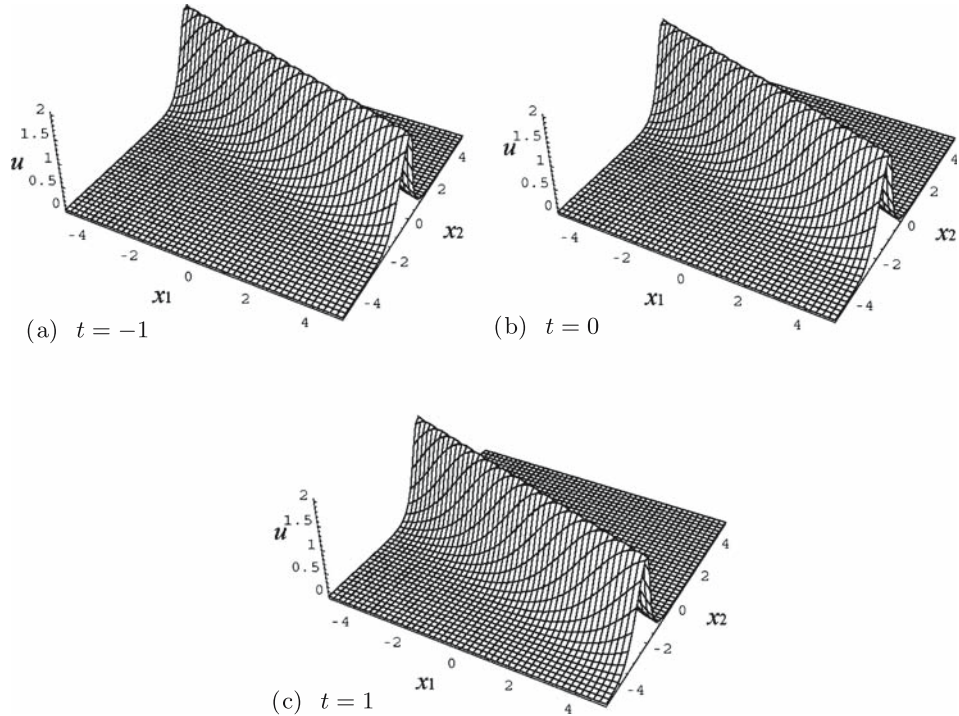


Figure 1. Evolutionary plots of the single-soliton solution (16).

Substituting eqs (17)–(20) into eq. (11), we can verify that

$$f^{(n)} = 0, \quad n = 3, 4, \dots \quad (21)$$

In this case, we can write

$$f_2 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}, \quad (22)$$

and hence obtain the following double-soliton solution of eq. (1):

$$u = (k_2^2 - k_1^2) \left[\ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}) \right]_{xx}, \quad (23)$$

where

$$\xi_i = r_i \left[-\frac{1}{4} k_2 (k_2^2 - k_1^2) r_i^2 t + \frac{3}{2k_1} p_i q_i t + k_1 x_1 + k_2 x_2 + p_1 y_1 + q_1 y_2 \right] + \xi_i^{(0)}, \quad i = 1, 2.$$

In figure 2, a double-soliton solution (23) is shown. Here, we select $r_1 = 1, r_2 = 2, k_1 = 1, k_2 = 2, p_1 = 3, p_2 = 2, q_1 = 2, q_2 = 1, \xi_1^{(0)} = 0, \xi_2^{(0)} = 0, x_1 = 0, x_2 = 0$. It is easy to see that collision between the pair of bell-shaped solitons has elastic characteristics. It must be noted that solitons do undergo elastic collision with colliding solitons coming out of collision without any change in shape. In case of solitary waves, this also can happen but is not general, for instance in [34].

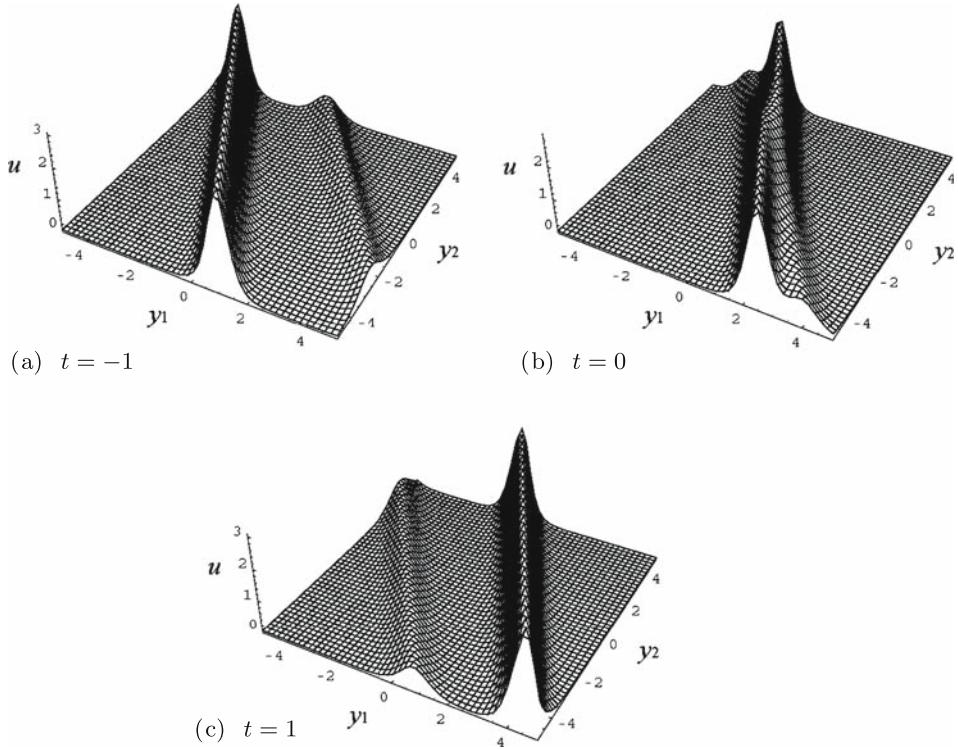


Figure 2. Evolutionary plots of the double-soliton solution (23).

We continue to construct the three-soliton solution. We suppose

$$f^{(1)} = e^{\xi_1} + e^{\xi_2} + e^{\xi_3}, \quad \xi_3 = r_3(\omega_3 t + x + p_3 y_1 + q_3 y_2) + \xi_3^{(0)}, \quad (24)$$

and substitute eq. (24) into eq. (9). Here r_3 , ω_3 , p_3 and q_3 are constants to be determined, $\xi_3^{(0)}$ is an arbitrary constant. We then obtain

$$\omega_3 = -\frac{1}{4}k_2 r_3^2 (k_2^2 - k_1^2) + \frac{3}{2k_1} p_3 q_3. \quad (25)$$

In view of eq. (10), we further suppose

$$f^{(2)} = e^{\xi_1 + \xi_2 + A_{12}} + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}}, \quad (26)$$

$$f^{(3)} = e^{\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}}, \quad (27)$$

where A_{13} and A_{23} are constants to be determined. From eqs (10), (26) and (27), we obtain

$$e^{A_{13}} = \frac{k_1 k_2 (k_2^2 - k_1^2) (r_3 - r_1)^2 + 2(p_3 - p_1)(q_3 - q_1)}{k_1 k_2 (k_2^2 - k_1^2) (r_3 + r_1)^2 + 2(p_3 - p_1)(q_3 - q_1)}, \quad (28)$$

$$e^{A_{23}} = \frac{k_1 k_2 (k_2^2 - k_1^2) (r_3 - r_2)^2 + 2(p_3 - p_2)(q_3 - q_2)}{k_1 k_2 (k_2^2 - k_1^2) (r_3 + r_2)^2 + 2(p_3 - p_2)(q_3 - q_2)}. \quad (29)$$

Substituting eqs (25)–(29) into eq. (11), we can verify that

$$f^{(n)} = 0, \quad n = 4, 5, \dots \quad (30)$$

In this case, we can write

$$f_3 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1+\xi_2+A_{12}} + e^{\xi_1+\xi_3+A_{13}} + e^{\xi_2+\xi_3+A_{23}} + e^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}}, \quad (31)$$

and hence obtain the following three-soliton solution of eq. (1):

$$u = (k_2^2 - k_1^2) \left[\ln \left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1+\xi_2+A_{12}} + e^{\xi_1+\xi_3+A_{13}} + e^{\xi_2+\xi_3+A_{23}} + e^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}} \right) \right]_{,xx}, \quad (32)$$

where

$$\xi_i = r_i \left[-\frac{1}{4} k_2 (k_2^2 - k_1^2) r_i^2 t + \frac{3}{2k_1} p_i q_i t + k_1 x_1 + k_2 x_2 + p_1 y_1 + q_1 y_2 \right] + \xi_i^{(0)}, \quad i = 1, 2, 3.$$

In figure 3, we select $r_1 = 1, r_2 = 2, r_3 = -1, k_1 = 1, k_2 = 2, k_3 = 3, p_1 = 1, p_2 = 2, p_3 = -1, q_1 = 2, q_2 = 1, q_3 = -1, \xi_1^{(0)} = 0, \xi_2^{(0)} = 0, \xi_3^{(0)} = 0, x_1 = 0, y_1 = 0$ and then show an elastic collision that happened among the three-soliton solution (32).

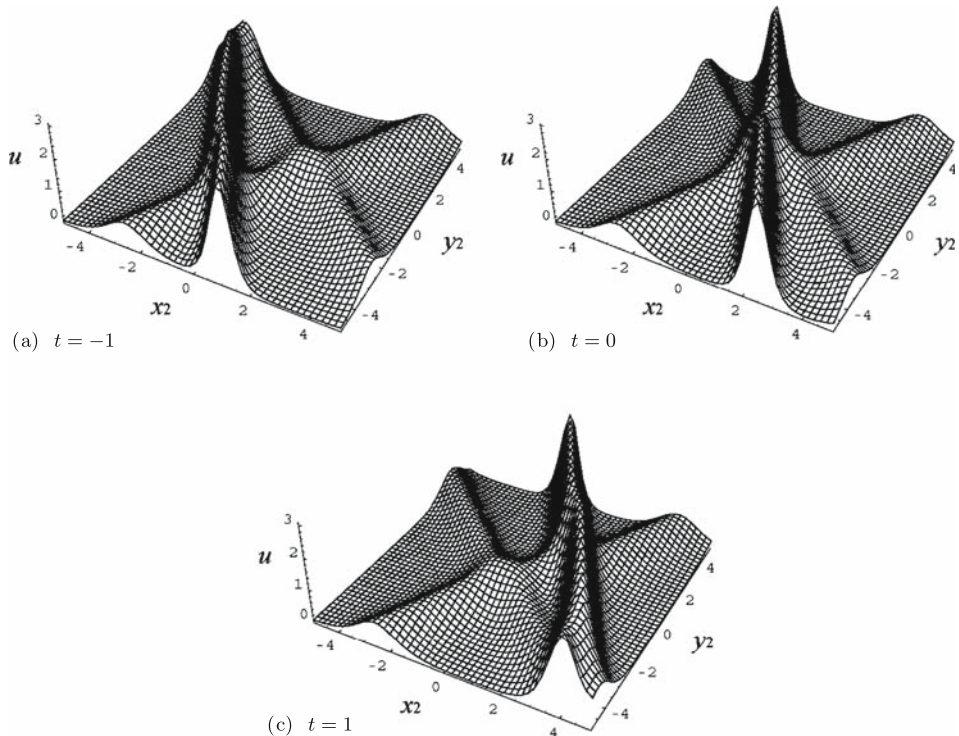


Figure 3. Evolutionary plots of the three-soliton solution (32).

Generally, by taking

$$f_n = \sum_{\mu=0,1} e^{\sum_{j=1}^n \mu_j \xi_j + A \sum_{1 \leq j < l} \mu_j \mu_l A_{jl}}, \quad (33)$$

$$e^{A_{jl}} = \frac{k_1 k_2 (k_2^2 - k_1^2) (r_l - r_j)^2 + 2(p_l - p_j)(q_l - q_j)}{k_1 k_2 (k_2^2 - k_1^2) (r_l + r_j)^2 + 2(p_l - p_j)(q_l - q_j)}, \quad (34)$$

we can obtain the uniform formula of n -soliton solution of eq. (1):

$$u = (k_2^2 - k_1^2) \left[\ln \left(\sum_{\mu=0,1} e^{\sum_{j=1}^n \mu_j \xi_j + A \sum_{1 \leq j < l} \mu_j \mu_l A_{jl}} \right) \right]_{,xx}, \quad (35)$$

where the summation $\sum_{\mu=0,1}$ refers to all possible combinations of each $\mu_i = 0, 1$ for $i = 1, 2, \dots, n$.

To the best of our knowledge, the obtained double-soliton solution (23), three-soliton solution (32) and n -soliton solution (35) are new.

Remark 1. As mentioned earlier, soliton solutions (16), (23), (32) and (35) have two constraints: $k_1 \neq k_2$ and $k_1 \neq 0$. Otherwise, the bilinear form (7) does not exist. For $k_1 = k_2$ or $k_1 = 0$, there are no soliton solutions as expected for eq. (1). If $k_1 = k_2$, eq. (3) is reduced to

$$k_1 u_{tx} + \frac{3}{2} k_1^2 (u^2)_{xx} - \frac{3}{2} u_{y_1 y_2} = 0. \quad (36)$$

According to the homogeneous balance method [7], we can easily see that eq. (36) has only trivial solutions. On the other hand, if $k_1 = 0$, eq. (3) becomes

$$-\frac{3}{2} u_{y_1 y_2} = 0, \quad (37)$$

which is a linear equation, and so has no soliton solution.

3. Conclusions

In summary, we have given bilinear form of the (4+1)-dimensional Fokas equation (1) and hence obtained its single-soliton solution, double-soliton solution, three-soliton solution and the new uniform formula of n -soliton solution through Hirota's bilinear method. In the process of using Hirota's bilinear method to solve eq. (1), one of the key steps is to convert eq. (1) into the bilinear form (7) by the appropriate transformations (2), (4)–(6) adopted in this work. Once a nonlinear PDE is converted into the bilinear form, then multisoliton solutions are usually obtained. In order to construct bilinear forms of nonlinear PDEs, some dependent variable transformations need to be taken. However, there is no general rule for the selection of such transformations. For the Fokas equation (1), not only the dependent variable transformation (4) but also the independent variable transformation (2) is taken. This paper shows that Hirota's bilinear method with some appropriate transformations may provide us with an effective mathematical tool for constructing multisoliton solutions of some other new higher-dimensional nonlinear PDEs.

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