PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 86, No. 6
June 2016
pp. 1173–1182

Average weighted receiving time in recursive weighted Koch networks

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MS received 8 May 2015; accepted 30 July 2015
DOI: 10.1007/s12043-016-1196-8; ***e*Publication:** 23 March 2016

Abstract. Motivated by the empirical observation in airport networks and metabolic networks, we introduce the model of the recursive weighted Koch networks created by the recursive division method. As a fundamental dynamical process, random walks have received considerable interest in the scientific community. Then, we study the recursive weighted Koch networks on random walk i.e., the walker, at each step, starting from its current node, moves uniformly to any of its neighbours. In order to study the model more conveniently, we use recursive division method again to calculate the sum of the mean weighted first-passing times for all nodes to absorption at the trap located in the merging node. It is showed that in a large network, the average weighted receiving time grows sublinearly with the network order.

Keywords. Weighted Koch network; recursive division method; average weighted receiving time.

PACS Nos 05.40.Fb; 05.45.Df; 89.75.Hc

1. Introduction

In nature and society, many complex systems can be represented as networks or graphs, where the nodes represent the elements of the systems and the edges stand for the interactions between the nodes [1–3]. Typical examples include the scientific collaboration networks [4], the worldwide airport networks [5] and metabolic networks [6].

The properties of a graph can be expressed via its adjacency matrix a_{ij}, whose elements take the value 1 if an edge connects the node i to the node j, and 0 otherwise. In weighted networks, the weight ω_{ij} can be represented as $\omega_{ij} = g(i, j)a_{ij}$, where $g(i, j)$ is a function of i and j. Recently, several studies of networks with weights on the links, such as the worldwide airport network and metabolic networks, show that the weights are correlated with the network topology [6,7].

For example, take the worldwide airport networks, where the number of scheduled flights between two airports increases with the number of flights at each airport.

In such cases, the average link weight scales with the degrees of the nodes on the two ends of a link as

$$\langle \omega_{ij} \rangle = \omega_{ji} \sim (k_i k_j)^{\theta}, \tag{1}$$

where k_i and k_j are the degrees of the two nodes and θ is a parameter that controls the strength of correlation between the topology [7–9]. The introduced controllable parameter conveniently determines the level of link heterogeneity in a weighted network. When $\theta > 0$, it is a weighted network where links have different weights. The larger the θ is, the wider will be the difference between links.

Recently, fractals have also attracted an increasing attention in physics and other scientific fields, owing to the striking beauty intrinsic in their structures and the significant impact of the idea of fractals [10–12]. These structures have been a focus of research and many underlying properties have been found. So it makes sense to combine weighted networks with fractals, which are then called weighted fractal networks. Daudert and Lapidus [13] studied weighted graphs and random walks on the Koch snowflake. Carletti and Righi [14] defined a class of weighted complex networks whose topology can be completely analytically characterized in terms of the involved parameters and the fractal dimension.

Lately, particular attention is paid to three kinds of random walks, i.e., random walk, weight-dependent walk and strength-dependent walk on weighted networks. For these three kinds of random walks, Li *et al* obtained the analytical expressions of the MFPT and these expressions increase with network parameters [15]. Zhang *et al* proposed a mapping technique converting Koch fractals into a family of deterministic networks called Koch networks, which integrate the observed properties of real works in a single framework [16]. Sun *et al* studied the novel evolving small-world scale-free Koch networks [17]. In 2012, Dai *et al* presented weighted Koch networks on weight-dependent walk with one weight factor [18], and developed a multilayered division method to determine the average receiving time (ART). Based on that work, they developed the non-homogeneous weighted Koch networks [19] and the weighted tetrahedron Koch networks [20]. The average weighted receiving time (AWRT) was defined for the first time in [19]. They introduced a family of deterministic non-homogeneous weighted Koch networks on random walk with three scaling factors (i.e., $r_1, r_2, r_3 \in (0, 1)$), and showed that in a large network, the AWRT grows as power-law function of the network order with the exponent, represented by $\log_4(1 + r_1 + r_2 + r_3)$. In those networks, the weight is determined by the weight factor and the weighting mechanism for edges.

Inspired by airport networks and metabolic networks, we introduce the recursive weighted Koch networks in which each edge weight in every triangle is related to the triangle's topology and is dependent on the scaling factor $0 < r < 1$. We discuss the average weighted receiving time on random walk in the recursive weighted Koch networks. Our results show that in a large network, the AWRT grows as power-law function of the network order with the exponent, represented by $\log_4(2r + 2)$, $0 < r < 1$.

2. Weighted Koch networks created by recursive division method

This section aims at constructing the recursive weighted Koch networks. Intuited by weight-degree correlated networks e.g., airport networks and metabolic networks, and

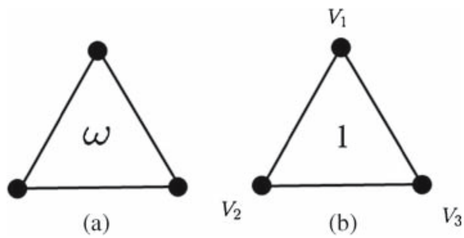

Figure 1. (**a**) w represents three edges with weight w in the triangle and (**b**) K_0 (i.e., G_0).

Koch networks, we build the recursive weighted Koch networks. In these networks, the same topology of every triangle is the same weight of three sides. In order to describe the model more conveniently, we use recursive division method to define the recursive weighted Koch networks.

Let us fix a positive real number $0 < r < 1$.

(1) Starting with a given initial network K_0 (i.e., G_0), K_0 consists of three nodes, called the merging nodes V_1, V_2, V_3 hereafter, and three edges with unit weight forming a triangle (as shown in figures 1a and 1b).

(2) Let $G_0^{(1)}$, $G_0^{(2)}$, $G_0^{(3)}$ (i.e., $K_0^{(1)}$, $K_0^{(2)}$, $K_0^{(3)}$) be three copies of G_0, whose weighted edges have been scaled by a factor r. Let us denote by $V_i^i (i = 1, 2, 3)$, the node in $K_0^{(i)}$ image of the labelled node $V_i \in K_0$. K_1 is obtained through amalgamating three copies $K_0^{(1)}$, $K_0^{(2)}$, $K_0^{(3)}$ and K_0 by merging, respectively, three pairs of V_i^i and V_i into a single new node, which is then the merging nodes V_1, V_2, V_3 of K_1 (as shown in figure 2a).

For convenience of construction of K_2, K_1 can also be regarded as merging H_1^i and G_1^i, respectively, as follows (as shown in figure 2b).

From the self-similarity of K_1, K_1 can be regarded as four groups, sequentially denoted by K_1^0, K_1^1, K_1^2 and K_1^3. One group is $K_1^0 = G_0 - \{V_1, V_2, V_3\}$, then $K_1 - K_1^0$ consists of the same three disjoint groups $K_1^i (i = 1, 2, 3)$, whose merging

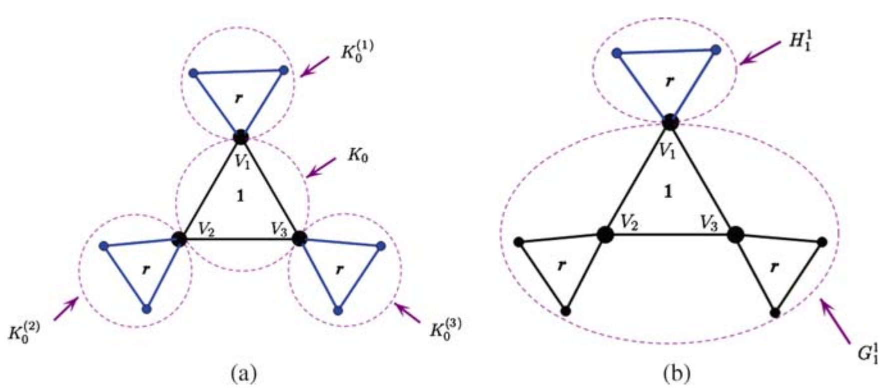

Figure 2. (**a**) The construction of K_1 and (**b**) K_1 is regarded as merging H_1^1 and G_1^1.

node is linked to the corresponding merging node V_i of G_0. Let $H_1^i = K_1^i$, whose merging node is $V_i (i = 1, 2, 3)$ and $G_1^i = (K_1 - H_1^i) \cup \{V_i\}$, then $H_1^i \cap G_1^i = \{V_i\}$.

(3) Let $G_1^{(i)}$ be the copy of $G_1^i (i = 1, 2, 3)$, whose weighted edges have been scaled by a factor r, and let $K_1^{(i)} = G_1^{(i)} \cup H_1^i$, where $G_1^{(i)} \cap H_1^i = \{V_i^i\}$, where V_i^i is the image in $K_1^{(i)}$ of the labelled node $V_i (i = 1, 2, 3)$. K_2 is obtained through amalgamating three groups $K_1^{(1)}$, $K_1^{(2)}$, $K_1^{(3)}$ and K_1 by merging, respectively, three pairs of $V_i^i \in K_1^{(i)}$ and $V_i \in K_1$ into a single new node, which is then the merging nodes V_1, V_2, V_3 of K_2 (as shown in figure 3a).

For convenience of the construction of K_3, K_2 can also be regarded as merging H_2^i and G_2^i as follows: From the self-similarity of K_2, K_2 can be regarded as four groups, sequentially denoted by K_2^0, K_2^1, K_2^2, K_2^3. One group is $K_2^0 = G_0 - \{V_1, V_2, V_3\}$, then $K_2 - K_2^0$ consists of the same three disjoint groups $K_2^i (i = 1, 2, 3)$, whose merging node is linked to the corresponding merging node V_i of G_0. Let $H_2^i = K_2^i$, whose merging node is $V_i (i = 1, 2, 3)$ and $G_2^i = (K_2 - H_2^i) \cup \{V_i\}$. Then $H_2^i \cap G_2^i = \{V_i\}$ (as shown in figure 3b).

⋮

(4) Given the generation $n + 1$, K_{n+1} may be obtained as follows:

First, for convenience of construction of K_{n+1}, K_n can also be regarded as merging H_n^i and G_n^i as follows:

From the self-similarity of K_n, K_n can be regarded as four groups, sequentially denoted by $K_n^0, K_n^1, K_n^2, K_n^3$. One group is $K_n^0 = G_0 - \{V_1, V_2, V_3\}$, then $K_n - K_n^0$ consists of the same three disjoint groups $K_n^i (i = 1, 2, 3)$, whose merging node is linked to the corresponding merging node V_i of G_0. Let $H_n^i = K_n^i$, whose merging node is $V_i (i = 1, 2, 3)$ and $G_n^i = (K_n - H_n^i) \cup \{V_i\}$. Then $H_n^i \cap G_n^i = \{V_i\}$ (as shown in figure 4a).

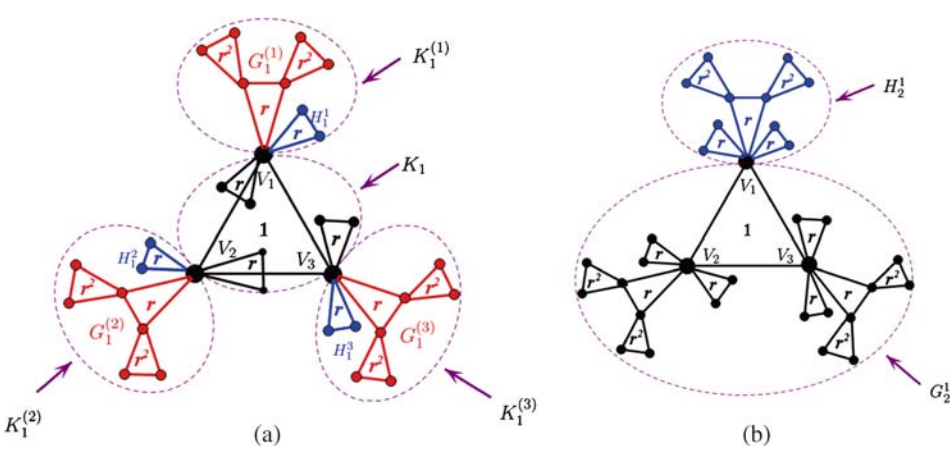

Figure 3. (**a**) The construction of K_2 and (**b**) K_1 is regarded as merging H_2^1 and G_2^1.

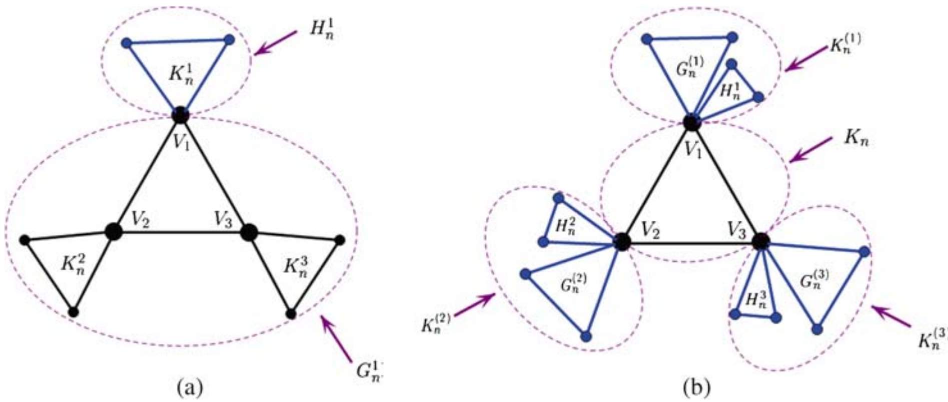

Figure 4. (**a**) K_n is regarded as merging H_n^1 and G_n^1 and (**b**) the construction method of K_{n+1}.

Then, let $G_n^{(i)}$ be the copy of $G_n^i (i = 1, 2, 3)$, whose weighted edges have been scaled by a factor r and let $K_n^{(i)} = G_n^{(i)} \bigcup H_n^i$, $G_n^{(i)} \bigcup H_n^i = \{V_i^i\}$, where V_i^i is the image in $K_n^{(i)}$ of the labelled node $V_i (i = 1, 2, 3)$.

Finally, K_{n+1} is obtained through amalgamating three groups $K_n^{(1)}$, $K_n^{(2)}$, $K_n^{(3)}$ and K_n by merging, respectively, three pairs of $V_i^i \in K_n^{(i)}$ and $V_i \in K_n$ into a single new node, which is then the merging nodes V_1, V_2, V_3 of K_{n+1} (as shown in figure 4b).

The recursive weighted Koch networks created by recursive division method is set up. By the construction, we can easily obtain some basic quantities, which are very useful for computing some quantities which we are concerned in this paper. The number of triangles $L_\triangle(n)$ present at iteration n is $L_\triangle(n) = 4^n$, and the number of nodes generated at iteration n is $L_v(n) = 6L_\triangle(n-1) = 6 \times 4^{n-1}$. Then, the numbers of edges and nodes in G_t are

$$E_n = 3L_\triangle(n) = 3 \times 4^n$$

and

$$N_n = \sum_{i=0}^{n} L_v(i) = 2 \times 4^n + 1,$$

respectively.

3. Average weighted receiving time

The purpose of this section is to determine explicitly the average weighted receiving time (AWRT) $\langle T \rangle_n$ and to show how $\langle T \rangle_n$ scales with network order. We aim at a particular case on K_n with the trap placed on one of its merging nodes V_1, V_2, V_3. The process is the random walk, i.e., the walker, at each step, starting from its current node, moves uniformly to any of its neighbours.

For convenience, let us denote by 1, 2, 3, the three merging nodes V_1, V_2, V_3 in K_n, and by $4, 5, \ldots, N_n - 1$ and N_n all other nodes except for the three merging nodes.

In a weighted network, the length of the edge can be introduced as some function of the weight and the weighted shortest path length d_{ij} is defined as the smallest sum of the edge lengths throughout all the possible paths in the graph from i to j. In the particular case of unweighted graphs, the weighted shortest path length d_{ij} reduces to the minimum number of edges traversed to get from i to j [14,21]. Inspired by the definition of the weighted shortest path length, for two adjacency nodes i and j in weighted networks, the weighted time is defined as the corresponding edge weight w_{ij}, which is equivalent to 1 of unweighted networks. The mean weighted first-passing time (MWFPT) is the expected first arriving weighted time for the walk starting from a source node to a given target node. Let $F_{ij}(n)$ be the mean weighted first-passage time (MWFPT) for a walker starting from node i to node j in K_n. Let $F_i(n)$ be the MWFPT from node i to the trap. $\langle T\rangle_n$ is the average weighted receiving time (AWRT), which is defined as the average of $F_i(n)$ over all starting nodes other than the trap in K_n. $\langle T\rangle_n$ is the key question concerned in this paper.

By definition, $\langle T\rangle_n$ is given by

$$\langle T\rangle_n = \frac{1}{N_n - 1}\sum_{i=2}^{N_n} F_i(n).$$

Here we denote by $T_t(n)$, the sum of MWFPTs for all nodes to absorption at the trap located the merging node V_1 in K_n, i.e.,

$$T_t(n) = \sum_{i=2}^{N_n} F_i(n).$$

Thus, the problem of determining $\langle T\rangle_n$ is reduced to finding $T_t(n)$.

By the construction of K_n, K_n can also be regarded as merging H_n^1 and G_n^1 (G_n in short), whose merging node is V_1. We denote by $T(n)$, the sum of MWFPTs for all nodes to absorption at the trap located at the merging node V_1 in G_n.

3.1 *Recursive formulas for $T(n)$ and $T_t(n)$*

Using the recursive division method, the derivation process of the recurrence formula for $T(n)$ in G_n is as follows.

From the construction of K_n, $K_n^i (i = 1, 2, 3)$ can be regarded as merging three groups $G_{n-1}^{(i)}$, H_{n-1}^i and K_{n-1}^i with the merging node V_i, then $K_n^i = G_{n-1}^{(i)} \cup H_{n-1}^i \cup K_{n-1}^i$. Note that $H_{n-1}^i = K_{n-1}^i$, then

$$K_n^i = G_{n-1}^{(i)} \cup 2K_{n-1}^i,$$

where

$$2K_{n-1}^i = K_{n-1}^i \cup K_{n-1}^i.$$

We can solve the equation inductively to yield

$$K_n^i = G_{n-1}^{(i)} \cup 2G_{n-2}^{(i)} \cup \cdots \cup 2^{n-2}G_1^{(i)} \cup 2^{n-1}G_0^{(i)}, \tag{2}$$

where

$$2^m G^{(i)}_{n-m-1} = \underbrace{G^{(i)}_{n-m-1} \cup \cdots \cup G^{(i)}_{n-m-1}}_{2^m},$$
$$m = 1, \ldots, n-1 \text{ and } i = 1, 2, 3.$$

We have

$$\begin{aligned} T(n) &= 2[rT(n-1) + 2rT(n-2) + \cdots \\ &\quad +2^{n-2}rT(1) + 2^{n-1}rT(0) + \frac{1}{3}N_n F_2(n)] \\ &= 2rT(n-1) + 2^2 rT(n-2) + \cdots \\ &\quad +2^{n-1}rT(1) + 2^n rT(0) + \frac{2}{3}N_n F_2(n). \end{aligned} \tag{3}$$

Then, eq. (3) is written for $n-1$ and doubled

$$\begin{aligned} 2T(n-1) &= 2^2 rT(n-2) + 2^3 rT(n-3) + \cdots \\ &\quad +2^{n-1}rT(1) + 2^n rT(0) \\ &\quad +\frac{4}{3}N_{n-1}F_2(n-1). \end{aligned} \tag{4}$$

Subtracting eq. (4) from eq. (3), one can obtain

$$\begin{aligned} T(n) &= 2(r+1)T(n-1) + \frac{2}{3}N_n F_2(n) \\ &\quad -\frac{4}{3}N_{n-1}F_2(n-1). \end{aligned} \tag{5}$$

From eq. (2) we also have

$$T_t(n) = T(n) + rT(n-1) + 2rT(n-2) + \cdots + 2^{n-1}rT(0) \tag{6}$$

and

$$\begin{aligned} T(n-1) &= 2rT(n-2) + 2^2 rT(n-3) + \cdots + 2^{n-2}rT(1) + 2^{n-1}rT(0) \\ &\quad +\frac{2}{3}N_{n-1}F_2(n-1). \end{aligned} \tag{7}$$

Subtracting eq. (7) from eq. (6), one can obtain

$$T_t(n) = T(n) + (r+1)T(n-1) - \frac{2}{3}N_{n-1}F_2(n-1). \tag{8}$$

Substituting eq. (5) into eq. (8), we obtain eq. (9) as follows

$$T_t(n) = 3(r+1)T(n-1) + \frac{2}{3}N_n F_2(n) - 2N_{n-1}F_2(n-1). \tag{9}$$

Equation (5) is multiplied by $3/2$ to get

$$\frac{3}{2}T(n) = 3(r+1)T(n-1) + N_n F_2(n) - 2N_{n-1}F_2(n-1). \tag{10}$$

From eqs (9) and (10), one can obtain

$$T_t(n) = \frac{3}{2}T(n) - \frac{1}{3}N_n F_2(n). \tag{11}$$

From eqs (5) and (11), one can obtain

$$\begin{aligned} T_t(n) &= 2(r+1)T_t(n-1) + \frac{2}{3}N_n F_2(n) \\ &\quad + \frac{2}{3}(r-2)N_{n-1}F_2(n-1). \end{aligned} \tag{12}$$

Thus, the problem of determining $T_t(n)$ is reduced to finding $F_2(n)$.

Let $P(n-1) = N_n F_2(n) + (r-2)N_{n-1}F_2(n-1)$, then

$$T_t(n) = 2(r+1)T_t(n-1) + \frac{2}{3}P(n-1). \tag{13}$$

3.2 *The formula of $F_2(n)$*

Let $R(n)$ be the weighted expected time for a walker in network G_n originally from node V_1 to return to the starting node V_1 for the first time, named mean weighted return time (MWRT).

Using the construction of K_{n-1} and K_n, we have

$$\begin{aligned} F_2(n-1) &= \frac{1}{2^n}[1 + (1 + F_3(n-1)) + 2^{n-1}(rR(0) \\ &\quad + F_2(n-1)) + \cdots + 2(rR(n-2) + F_2(n-1))]. \end{aligned}$$

i.e.,

$$F_2(n-1) = 2 + 2^{n-1}rR(0) + 2^{n-2}rR(1) + \cdots + 2rR(n-2)$$

and

$$\begin{aligned} F_2(n) &= 2 + 2^n rR(0) + 2^{n-1}rR(1) + \cdots + 2rR(n-1) \\ &= 2 + 2[2^{n-1}rR(0) + 2^{n-2}rR(1) \\ &\quad + \cdots + 2rR(n-2)] + 2rR(n-1) \\ &= 2 + 2[F_2(n-1) - 2] + 2rR(n-1) \\ &= 2F_2(n-1) - 2 + 2rR(n-1). \end{aligned} \tag{14}$$

By the definition of $R(n)$, we have

$$R(n-1) = \frac{1}{2}[1 + F_2(n-1) + 1 + F_3(n-1)] = 1 + F_2(n-1). \tag{15}$$

Plugging eq. (15) into eq. (14) leads to

$$F_2(n) = 2(1+r)F_2(n-1) + 2(r-1).$$

Considering the initial condition $F_2(0) = 2$, we could finally get

$$F_2(n) = \frac{6r}{2r+1}(2r+2)^n + \frac{2(1-r)}{2r+1}. \tag{16}$$

3.3 *Formulas of $T_t(n)$ and $\langle T \rangle_n$*

Using eq. (16), we could work out explicitly the expression of $P(n-1)$

$$\begin{aligned} P(n-1) &= (2\times 4^n + 1)\left[\frac{6r(2r+2)^n}{2r+1} + \frac{2(1-r)}{2r+1}\right] \\ &\quad +(r-2)(2\times 4^{n-1}+1)\left[\frac{6r(2r+2)^{n-1}}{2r+2} + \frac{2(1-r)}{2r+1}\right] \\ &= \frac{36r(3r+2)}{2r+1}(8r+8)^{n-1} + \frac{4(1-r)(r+2)}{2r+1}4^{n-1} \\ &\quad +\frac{18r^2}{2r+1}(2r+2)^{n-1} - \frac{2(1-r)^2}{2r+1}. \end{aligned}$$

Then

$$\begin{aligned} P(n) &= \frac{36r(3r+2)}{2r+1}(8r+8)^n + \frac{4(1-r)(r+2)}{2r+1}4^n \\ &\quad +\frac{18r^2}{2r+1}(2r+2)^n - \frac{2(1-r)^2}{2r+1}. \end{aligned} \tag{17}$$

Using eqs (13) and (17) and also the initial condition $T_t(0) = 4$, $T_t(n)$ can be obtained

$$\begin{aligned} T_t(n) &= \frac{4r(3r+2)}{(r+1)(2r+1)}(8r+8)^n + \frac{8(r+2)}{3(2r+1)}4^n + \frac{12r^2}{2r+1}n(2r+2)^{n-1} \\ &\quad -\frac{4r(3r^2-r+1)}{(r+1)(2r+1)^2}(2r+2)^n + \frac{4(1-r)^2}{3(2r+1)^2} \\ &\approx \frac{4r(3r+2)}{(r+1)(2r+1)}(8r+8)^n. \end{aligned} \tag{18}$$

Recalling $N_n = 2\times 4^n + 1$ and eq. (18), $\langle T \rangle_n$ can be obtained as

$$\begin{aligned} \langle T \rangle_n &\approx \frac{2r(3r+2)}{(r+1)(2r+1)}(2r+2)^n \\ &= \frac{\sqrt{2}r(3r+2)}{(2r+1)(r+1)^{3/2}}(N_n-1)^{\log_4(2r+2)}, \quad 0 < r < 1. \end{aligned}$$

4. Conclusions

In summary, intuited by airport networks and metabolic networks, we have first built the recursive weighted Koch networks. In these networks, links with similar topological structures have the same weights, which can model the systems better in real-world networks. Then, we have investigated random walks on the weighted Koch networks created by recursive division method. Previous studies focussing on unweighted or homogeneous weighted Koch networks are simple compared to the present model. Noticeably, comparatively few investigations have been reported in the literature relative to random walks on the recursive weighted Koch networks created by recursive division method. Finally, we have discussed the impact of key parameter $0 < r < 1$ on AWRT. Our analysis indicates that in a large network, the AWRT grows as power-law function of the network order with the exponent, represented by $\theta(r) = \log_2(2r + 2)$. When r grows from 0 to 1, the exponent increases from 0 and approaches 1, indicating that the AWRT grows sublinerly with network order.

Acknowledgements

This research is supported by the Humanistic and Social Science Foundation from the Ministry of Education of China (Grants 14YJAZH012).

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