PRAMANA
— journal of
physics

Vol. 86, No. 5
May 2016
pp. 985–996

Application of quasiexactly solvable potential method to the *N*-body problem of anharmonic oscillators

H PANAHI* and M MOHAMMADKAZEMI GAVABAR
Department of Physics, University of Guilan, Rasht 51335-1914, Iran
*Corresponding author. E-mail: t-panahi@guilan.ac.ir

MS received 16 July 2014; accepted 7 July 2015
DOI: 10.1007/s12043-015-1175-5; ***e*Publication:** 3 March 2016

Abstract. The quasiexactly solvable potential method is used to determine the energies and the corresponding exact eigenfunctions for a system of N particles with equal mass interacting via an anharmonic potential. For systems with five and seven particles, we compute the ground state and the first excited state only, and compare the spectrums with the results obtained by Ritz approximation method.

Keywords. Quasiexactly solvable method; Ritz approximation method; anharmonic oscillator.

PACS Nos 03.65.−w; 03.65.Ge

1. Introduction

The importance of exactly solvable potentials in quantum mechanics is well known. But even in a one-dimensional case, the number of such potentials are limited [1]. The quasiexactly solvable (QES) potentials are intermediate to exactly solvable and non-solvable ones and the Schrödinger equations of these potentials are generally solved numerically or by approximating the studied potentials by solvable ones [2,3]. For these systems, a finite number of energy levels and the corresponding wave functions are known. The first examples of QES potentials have been studied in refs [4–11]. Anharmonic potentials, some of which have a double-well structure, are the first examples of one-dimensional QES problems. A potential with a centrifugal barrier is also a QES potential that appears in the treatment of ring laser, where the Fokker–Planck equation is converted to an equivalent Schrödinger eigenvalue problem [12]. In recent years much attention has been given to QES potentials and many new ideas and concepts have been discovered and developed [13–21]. In ref. [13], a general approach has been introduced to construct multidimensional QES potentials with explicitly known eigenfunctions for two energy levels and some examples of new QES potentials are presented. In refs [15,17], by combining the QES method with the perturbative method, an approximating method that

can solve the problem without imposing any constraint on the potential parameters has been proposed. Based on this method, the radial Schrödinger equation for the QES central potentials have been solved in ref. [15]. Also, three families of QES potentials which play important roles in the description of interactions occurring between three particles of equal mass have been studied in ref. [17].

In this paper, we introduce an N-body quantum model interacting via an anharmonic potential and try to solve the problem by QES method given in [17]. We solve the problem without imposing any condition on the potential parameters. Hence in §2, we give the general formalism to solve the problem and in §3 and 4 we obtain the spectrum of system for $N = 5$ and $N = 7$ respectively. We also compare the spectrums of the system with the results obtained by Ritz approximation method in each section. The paper ends with a brief conclusion in §5.

2. Application of QES method to N-particle quantum model interacting via an anharmonic potential

In this section, we first introduce the anharmonic potential of N particles with the same mass as

$$V(x_1, x_2, ..., x_N) = \sum_{j=1}^{N} \left(A_{2j} \left(\sum_{i=1}^{N-1} (N-2i)\, x_i \right)^{2i} + B_{2i} \left(\sum_{i=1}^{N-1} x_i - (N-1)\, x_N \right)^{2i} \right), \tag{2.1}$$

where A_{2j} and B_{2j} are the constant coupling coefficients of the interactions. Potential (2.1) for $N = 3$ has been studied in ref. [17]. This potential can be interpreted as the interaction of N similar particles in which the first type of interaction corresponding to the interaction between the first $(N - 1)$ bodies in the absence of the last one and the second type corresponding to the interaction between the last body and the other $(N - 1)$ bodies.

For obtaining the spectrum of this potential, we use the QES method which has been used in ref. [17] for solving the 3-body system.

The Schrödinger equation for N similar particles with potential (2.1) has the following form (for $\hbar = m = 1$):

$$-\frac{1}{2} \sum_{i=1}^{\mathrm{N}} \frac{\partial^2 \psi}{\partial x_{\mathrm{i}}^2} + V\,(x_1, x_2, \ldots, x_N)\psi = E\psi, \tag{2.2}$$

where the eigenfunction ψ is a function of the coordinates x_1, x_2, ..., x_N and E is its energy eigenvalue. For solving this system, it is convenient to work with the following Jacobi-like coordinates:

$$x = \frac{\sum_{i=1}^{N-1}(N-2i)\, x_i}{\sqrt{\sum_{i=1}^{N-1}(N-2i)^2}}, \qquad y = \frac{\left[\sum_{i=1}^{N-1} x_i\right] - (N-1)\, x_N}{\sqrt{(N-1)^2 + N - 1}}, \qquad R = \frac{\sum_{i=1}^{N} x_i}{N}. \tag{2.3}$$

The coordinate x can be interpreted as the sum of the relative distance of $N - 1$ bodies, in the absence of the last body and the coordinate y means the sum of the relative distance

of the last body with the other $(N-1)$ bodies, and R is the centre-of-mass coordinate of the bodies. Now, if we choose the centre of mass R as the origin of the coordinates, then the terms corresponding to the central mass will be omitted as a constant. So after applying transformation (2.3) into (2.1), the potential function will be transformed to a two-dimensional potential which has the following general form:

$$V(x, y) = \sum_{j=1}^{N} a_{2j} x^{2j} + \sum_{j=1}^{N} b_{2j} y^{2j}, \tag{2.4}$$

where

$$a_{2j} = A_{2j} \left(\sum_{i=1}^{N-1} (N-2i) \right)^{i}, \quad b_{2j} = B_{2i} \left((N-1)^2 + N - 1 \right)^{i}. \tag{2.5}$$

Also by using this transformation and performing the tedious calculation, the kinetic term of the Hamiltonian will change to a simple and convenient two-dimensional form, that is, in the centre of-mass system of reference, the Schrödinger equation (2.2) will be transformed as

$$-\frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} \right) + V(x, y)\psi = E\psi. \tag{2.6}$$

Therefore, the Hamiltonian in Jacobi coordinates yields to a two-dimensional problem, and/or the N-body quantum system is transformed to a two-dimensional system.

It is obvious that the two-dimensional eigenvalue equation (2.6) does not always admit exact solutions. However, we try to solve it by the quasiexactly solvable method which yields the exact wave function and the corresponding energy for restricted values of the parameters a_{2j} and b_{2j}. We shall also use the Ritz approximation method to test the accuracy of our results.

Now for studying and solving the eigenvalue problem (2.6) by QES method, we introduce the trial wave function related to (x, y) as [15,17]

$$\psi = C \exp(f(x) + g(y)), \tag{2.7}$$

where for odd number of bodies, we consider

$$\begin{aligned} f(x) &= n_x \ln x + \sum_{i=0}^{(N-1)/2} \alpha_i x^{N+1-2i}, \\ g(y) &= n_y \ln y + \sum_{i=0}^{(N-1)/2} \alpha'_i y^{N+1-2i} \end{aligned} \tag{2.8}$$

and for even number of bodies

$$f(x) = n_x \ln x + \sum_{i=0}^{(N-1)/2} \alpha_i x^{i+2}, \quad g(y) = n_y \ln y + \sum_{i=0}^{(N-1)/2} \alpha'_i y^{i+2} \tag{2.9}$$

and C being a normalization constant. The real parameters of the trial wave function are calculated from the parameters of the potential.

If we substitute the trial wave function (2.7) into (2.6), it can be easily seen that

$$-\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=-\frac{1}{2}\left[\left(f'\right)^2+f''+\left(g'\right)^2+g''\right]\psi$$
$$=\left[E-V\left(x,y\right)\right]\psi. \tag{2.10}$$

By inserting f', f'', g', g'' and $V(x, y)$ into (2.10) and comparing two sides, we can arrive at many algebraic constraints between the parameters and the energy spectrum can be calculated from them. In the next sections, we try to study the quantum system with only odd number of N ($N = 5$ in §3 and $N = 7$ in §4) and show how the QES method works. We also obtain the spectrum by Ritz approximation method for comparing the results in the two different approaches.

3. The quantum model of 5-body anharmonic potential

The 3-body model with potential function (2.1) is studied in ref. [17] and in this section, we consider the same problem for 5-body system with the same mass. According to (2.1) we have

$$V\left(x_1,x_2,x_3,x_4,x_5\right)=\sum_{i=1}^{5}\left[A_{2i}\left(3x_1+x_2-x_3-3x_4\right)^{2i}\right.$$
$$\left.+B_{2i}\left(x_1+x_2+x_3+x_4-4x_5\right)^{2i}\right], \tag{3.1}$$

where A_{2i} and B_{2i} are the constant coefficients. As we have mentioned, this potential can be interpreted as the interaction of five similar particles such that the first type of interaction is the interaction of four particles with each other and the second one is the interaction of the fifth particle with the other four bodies. On the other hand, the first type of the interaction can be considered as the sum of the relative distance of four particles with each other when the the last particle is absent, that is

$$3x_1+x_2-x_3-3x_4=\left(x_1-x_2\right)+\left(x_1-x_3\right)+\left(x_1-x_4\right)+\left(x_2-x_3\right)$$
$$+\left(x_2-x_4\right)+\left(x_3-x_4\right). \tag{3.2}$$

Also, the second type of interaction can be interpreted as the sum of the relative distance of the last particle with the other four bodies

$$x_1+x_2+x_3+x_4-4x_5=\left(x_1-x_5\right)+\left(x_2-x_5\right)+\left(x_3-x_5\right)+\left(x_4-x_5\right). \tag{3.3}$$

In §3.1, we first use the QES method to obtain the spectrum of the potential and then in §3.2, we use the Ritz approximation method to show both results are almost equal.

3.1 *The 5-body model and QES method*

For $N = 5$, the Jacobi-like coordinates (2.3) are as follows:

$$x=\frac{3x_1+x_2-x_3-3x_4}{\sqrt{20}},\quad y=\frac{x_1+x_2+x_3+x_4-4x_5}{\sqrt{20}},$$
$$R=\frac{x_1+x_2+x_3+x_4+x_5}{5}. \tag{3.4}$$

Using these coordinates and assuming that the centre of mass is the origin, the potential (3.1) will change to the following two-dimensional potential:

$$V(x, y) = \sum_{i=1}^{5} \left(a_{2i}x^{2i} + b_{2i}y^{2i}\right) = a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + b_2y^2 + b_4y^4 + b_6y^6 + b_8y^8 + b_{10}y^{10}, \tag{3.5}$$

where

$$a_{2i} = 20^i A_{2i} \quad \text{and} \quad b_{2i} = 20^i B_{2i}.$$

The Schrödinger equation of this potential is also the same as (2.6) and the eigenfunction $\psi(x_1, x_2, x_3, x_4, x_5)$ will be a function of the two coordinates (x, y). In other words, we must solve the two-dimensional quantum system (2.6) with potential (3.5). Now we consider the eigenfunction of the system as in (2.7) and so according to (2.8), we have

$$\begin{aligned} f(x) &= -\alpha_1x^6 + \beta_1x^4 + \gamma_1x^2 + n_x \ln x, \\ g(y) &= -\alpha_2y^6 + \beta_2y^4 + \gamma_2y^2 + n_y \ln y, \end{aligned} \tag{3.6}$$

where $n_x, n_y, \alpha_i, \beta_i, \gamma_i$ are the real parameters. It is necessary to notice that due to the integrability of the wave function, the coefficients of the highest power in $f(x)$ and $g(y)$ should be minus.

If we use (3.5) and (3.6) and substitute f', f'', g', g'', $V(x, y)$ into (2.10), then we get the following algebraic equation:

$$\begin{aligned} &\left(-18\alpha_1^2 + a_{10}\right)x^{10} + (24\alpha_1\beta_1 + a_8)x^8 + \left(-8\beta_1^2 + 12\alpha_1\gamma_1 + a_6\right)x^6 \\ &+ (6\alpha_1n_x - 8\beta_1\gamma_1 + 15\alpha_1 + a_4)\,x^4 + \left(-6\beta_1 - 2\gamma_1^2 - 4\beta_1n_x + a_2\right)x^2 \\ &+ \left(-18\alpha_2^2 + b_{10}\right)y^{10} + (24\alpha_2\beta_2 + b_8)y^8 + \left(-8\beta_2^2 + 12\alpha_2\gamma_2 + b_6\right)y^6 \\ &+ (6\alpha_2n_y - 8\beta_2\gamma_2 + 15\alpha_2 + b_4)y^4 + \left(-6\beta_2 - 2\gamma_2^2 - 4\beta_2n_y + b_2\right)y^2 \\ &- \frac{n_x(n_x-1)}{2x^2} - \frac{n_y(n_y-1)}{2y^2} - \gamma_1(2n_x+1) - \gamma_2(2n_y+1) = E. \end{aligned} \tag{3.7}$$

As the right-hand side of (3.7) is a constant, it can be easily concluded that the coefficients of each variable term should be zero. Also, the parameters n_x, n_y should be zero or one. Therefore, the following set of constraint equations between the parameters of the potential and the wave function should exist:

$$\begin{aligned} &-18\alpha_1^2 + a_{10} = 0, \quad 24\alpha_1\beta_1 + a_8 = 0, \quad -8\beta_1^2 + 12\alpha_1\gamma_1 + a_6 = 0, \\ &-18\alpha_2^2 + b_{10} = 0, \quad 24\alpha_2\beta_2 + b_8 = 0, \quad -8\beta_2^2 + 12\alpha_2\gamma_2 + b_6 = 0, \\ &6\alpha_1n_x - 8\beta_1\gamma_1 + 15\alpha_1 + a_4 = 0, \quad -6\beta_1 - 2\gamma_1^2 - 4\beta_1n_x + a_2 = 0, \\ &6\alpha_2n_y - 8\beta_2\gamma_2 + 15\alpha_2 + b_4 = 0, \quad -6\beta_2 - 2\gamma_2^2 - 4\beta_2n_y + b_2 = 0. \end{aligned} \tag{3.8}$$

Now, from (3.8), it is seen that the parameters $b_6, b_8, b_{10}, a_6, a_8$ and a_{10} are free, while the remaining parameters b_2, b_4, a_2 and a_4 are not free and can be expressed in terms of the free ones and the quantum number n_x and n_y. The parameters α_i, β_i and γ_i, appearing in the expression of the wave function are also obtained from the free parameters. So,

Table 1. The initial values of the parameters according to QES method.

$a_2 = b_2$	$a_4 = b_4$	a_6 (QES) = b_6 (QES)	$a_8 = b_8$	$a_{10} = b_{10}$	$n_x = n_y$
0.284559194	0.027144684	0.015	0.001	0.0001	0
0.213848518	0.013002552	0.015	0.001	0.0001	1

we can give some initial values for the free parameters of the potential (3.5) and then calculate the other parameters from (3.8). We can also deduce the spectrum energy of the quantum system by constant terms of (3.7), which is called the quasiexactly solvable (QES) energy as

$$E_{n_x,n_y}\,(\mathrm{QES}) = -\gamma_1(2n_x + 1) - \gamma_2\big(2n_y + 1\big)\,. \tag{3.9}$$

This form is due to the symmetry of the potential with respect to x and y and it is very simple to calculate the energy spectrum of 5-body system in terms of $\big(n_x, n_y\big)$ and the parameters γ_1, γ_2. These parameters are calculated from (3.8), and the wave function of the system can be easily calculated from (2.7) and (3.6).

3.2 *The 5-body model and Ritz method*

For solving the quantum system with Hamiltonian (2.6) by Ritz method, we should calculate the following equations for x and y parts of the Hamiltonian:

$$E_{n_x} = \frac{\langle\psi_x|H_x|\psi_x\rangle}{\langle\psi_x|\psi_x\rangle}, \quad E_{n_y} = \frac{\langle\psi_y|H_y|\psi_y\rangle}{\langle\psi_y|\psi_y\rangle}, \tag{3.10}$$

where

$$H_x = -\frac{\partial^2}{2\,\partial x^2} + V(x), \quad H_y = -\frac{\partial^2}{2\partial y^2} + V(y) \tag{3.11}$$

and ψ_x, ψ_y are the trial wave functions which should be written in terms of some variational parameters such that they minimize the eigenvalue energy. In this work, we choose the trial wave function for the ground state and the first exited state respectively as

$$\psi_{0x} = \exp\big(-\xi_1 x^2\big)\,, \quad \psi_{0y} = \exp\big(-\xi_2 y^2\big)\,, \tag{3.12}$$

$$\psi_{1x} = x\exp\big(-\eta_1 x^2\big)\,, \quad \psi_{1y} = y\exp\big(-\mu_2 y^2\big)\,, \tag{3.13}$$

which satisfy the physical conditions of the problem. After substituting (3.12) and (3.13) into (3.10) and performing the numerical calculation, we have computed the minimum energy by Ritz method. In table 1, we have given the initial values and the calculated parameters from (3.8) and in table 2, we have written the values of energy from QES and Ritz methods.

It is necessary to explain that in table 1, first we choose some values for the free parameters, then the values of the wave function's parameters are calculated from (3.8) and so by applying the condition where (n_x, n_y) should be zero or one, we calculate the numerical values of b_2, b_4, a_2 and a_4. Of course, for simplicity, we have assumed that the initial

Table 2. The values of energy by using QES and Ritz methods for the parameters of table 1.

E_{n_x,n_y} (QES)	E (Ritz)	(n_x, n_y)
0.883883712	0.894756360	(0, 0)
1.767767424	1.790753134	(1, 0)
1.767767424	1.790753134	(0, 1)
2.651651136	2.684858704	(1, 1)

values of the free parameters a_{10}, a_8 and a_6 are equal to the initial values of the free parameters b_{10}, b_8 and b_6 respectively. In table 2, we see that the energy eigenvalues are almost equal using two methods and also for the two states $(n_x = 0, n_y = 1)$ and $(n_x = 1, n_y = 0)$, the values are exactly equal. This is because the free parameters of the potential that correspond to the variables x and y are equal.

Now, we assume that the initial values of the parameters for potential (3.5) have different values from what is mentioned above, that is, the initial values of the parameters do not satisfy the constraint equations (3.8). Then it is possible to decompose the potential (3.5) as the sum of a QES potential and a perturbation term as

$$\begin{aligned} V(x, y) = {} & a_2x^2 + a_4x^4 + a_{6\,\mathrm{QES}}x^6 + a_8x^8 + a_{10}x^{10} \\ & + b_2y^2 + b_4y^4 + b_{6\,\mathrm{QES}}y^6 + b_8y^8 + b_{10}y^{10} \\ & + (a_6 - a_{6\mathrm{QES}})x^6 + (b_6 - b_{6\mathrm{QES}})y^6, \end{aligned} \tag{3.14}$$

in which we have assumed that the two parameters (b_6, a_6) do not satisfy eq. (3.8), that is, the term $(a_6 - a_{6\mathrm{QES}})x^6 + (b_6 - b_{6\mathrm{QES}})y^6$ is considered as a perturbation term to the QES potential. Hence, according to (3.9), we easily know the value of QES energy and we should only solve the energy of the perturbation term. This perturbation energy is obtained from the following equation:

$$\Delta E_{n_x,n_y} = \frac{(a_6 - a_{6\mathrm{QES}})\displaystyle\int_{-\infty}^{+\infty} x^6\psi_x^*\psi_x \mathrm{d}x}{\displaystyle\int_{-\infty}^{+\infty} \psi_x^*\psi_x \mathrm{d}x} + \frac{(b_6 - b_{6\mathrm{QES}})\displaystyle\int_{-\infty}^{+\infty} y^6\psi_y^*\psi_y \mathrm{d}y}{\displaystyle\int_{-\infty}^{+\infty} \psi_y^*\psi_y \mathrm{d}y} \tag{3.15}$$

and the energy of the system becomes

$$E_{n_x,n_y} = E_{n_x,n_y}(\mathrm{QES}) + \Delta E_{n_x,n_y}. \tag{3.16}$$

For example, if we take the value of perturbation parameters, which are (b_6, a_6), equal to 0.02 and the other free parameters are those given in table 1, then we observe the values of energy given in table 3 are in good agreement with those obtained by Ritz method.

At the end of this section, we must notice that if the other free parameters of (3.5) do not satisfy eqs (3.8) then eq. (3.14) must be written in an appropriate form according to the parameters and $\Delta E_{n_x,n_y}$, as the contribution of the perturbation terms should be calculated for those free parameters.

Table 3. The values of energy for the perturbed system using two different methods.

E_{n_x,n_y}	E (Ritz)	(n_x, n_y)
0.89766973175	0.897669731806	(0, 0)
1.812789373075	1.812789373253	(1, 0)
1.812789373075	1.812789373253	(0, 1)
2.7279090144	2.7279090147	(1, 1)

4. The quantum model of 7-body anharmonic potential

In this section, we write (2.1) for $N = 7$ and obtain the spectrum of the system by the same methods as in the previous section. The potential is

$$V(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \sum_{i=1}^{7} \left[A_{2i}(5x_1 + 3x_2 + x_3 - x_4 - 3x_5 - 5x_6)^{2i} + B_{2i}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 6x_7)^{2i} \right], \tag{4.1}$$

where A_{2i} and B_{2i} are the constant coefficients. As mentioned in the previous section, this potential can be interpreted as the interaction of seven similar particles with each other in the absence of the last body, and the second kind is the interaction between the seventh body with the other six particles. On the other hand, similar to relations (3.2) and (3.3), respectively, the first type of interaction can be considered as the sum of the relative distance of six particles with each other in the absence of the last one and the second type of interaction can be interpreted as the sum of the relative distance of the last particle with the other six bodies.

In the following subsection, we first use the QES method to obtain the spectrum of the potential (4.1) and then compare the results with Ritz approximation method.

4.1 *The 7-body model and QES method*

We use the suitable Jacobi-like coordinates (2.3) for solving the system as

$$\begin{aligned} x &= \frac{5x_1 + 3x_2 + x_3 - x_4 - 3x_5 - 5x_6}{\sqrt{70}}, \\ y &= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 6x_7}{\sqrt{42}}, \\ R &= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7}{7}, \end{aligned} \tag{4.2}$$

where x is the sum of the relative distance of the bodies with each other, in the absence of the seventh one, y is considered as the sum of the relative distance of the seventh body with the other six bodies and R is the coordinate of the central mass. By doing the same

calculation as in the previous section, in centre-of-mass system of reference, we get the following two-dimensional potential via (4.1)

$$V(x, y) = \sum_{i=1}^{7} \left(a_{2i}x^{2i} + b_{2i}y^{2i}\right) = a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12} + a_{14}x^{14} + b_2y^2 + b_4y^4 + b_6y^6 + b_8y^8 + b_{10}y^{10} + b_{12}y^{12} + b_{14}y^{14}, \tag{4.3}$$

where

$$a_{2i} = 70^i A_{2i} \quad \text{and} \quad b_{2i} = 42^i B_{2i}.$$

The Schrödinger equation for this potential is also the same as eq. (2.6) and the eigenfunction $\psi(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ will be a function of two coordinate (x, y). Hence, we must solve the two-dimensional quantum system (2.6) for potential (4.3) where its eigenfunction is given as (2.7) and the functions (3.6) are given as

$$f(x) = -\alpha_1x^8 + \beta_1x^6 + \gamma_1x^4 + \eta_1x^2 + n_x \ln x,$$
$$g(x) = -\alpha_2y^8 + \beta_2y^6 + \gamma_2y^4 + \eta_2y^2 + n_y \ln y, \tag{4.4}$$

where α_i, β_i , γ_i and η_i are the real parameters. Similar to (3.6) and due to the integrability of the wave function, the coefficients of the highest power in $f(x)$ and $g(y)$ should be minus. Now substituting (4.3) and (4.4) into (2.10) gives the following algebraic equation:

$$\begin{aligned}
&\left(-32\alpha_1^2 + a_{14}\right)x^{14} + (48\alpha_1\beta_1 + a_{12})x^{12} + \left(-18\beta_1^2 + 32\alpha_1\gamma_1 + a_{10}\right)x^{10} \\
&+ (16\alpha_1\eta_1 - 24\beta_1\gamma_1 + a_8)x^8 + \left(28\alpha_1 - 8\gamma_1^2 - 12\beta_1\eta_1 + 8\alpha_1n_x + a_6\right)x^6 \\
&+ (-6\beta_1n_x - 8\eta_1\gamma_1 - 15\beta_1 + a_4)x^4 + \left(-2\eta_1^2 - 4n_x\gamma_1 - 6\gamma_1 + a_2\right)x^2 \\
&+ \left(-32\alpha_2^2 + b_{14}\right)y^{14} + (48\alpha_2\beta_2 + b_{12})y^{12} + \left(-18\beta_2^2 + 32\alpha_2\gamma_2 + b_{10}\right)y^{10} \\
&+ (16\alpha_2\eta_2 - 24\beta_2\gamma_2 + b_8)y^8 + \left(28\alpha_2 - 8\gamma_2^2 - 12\beta_2\eta_2 + 8\alpha_2n_y + b_6\right)y^6 \\
&+ \left(-6\beta_2n_y - 8\eta_2\gamma_2 - 15\beta_2 + b_4\right)y^4 + \left(-2\eta_2^2 - 4n_y\gamma_2 - 6\gamma_2 + b_2\right)y^2 \\
&- \frac{n_x(n_x - 1)}{2x^2} - \eta_1(2n_x + 1) - \frac{n_y(n_y - 1)}{2y^2} - \eta_2(2n_y + 1) = E.
\end{aligned} \tag{4.5}$$

The right-hand side of (4.5) is a constant, and thus the coefficients of each variable term on the left-hand side should be zero, and also the parameters n_x, n_y should be zero or one. Therefore, the following set of constraint equations between the parameters of the potential and the wave function should exist:

$$\begin{aligned}
&-32\alpha_1^2 + a_{14} = 0, \quad 48\alpha_1\beta_1 + a_{12} = 0, \quad -18\beta_1^2 + 32\alpha_1\gamma_1 + a_{10} = 0, \\
&16\alpha_1\eta_1 - 24\beta_1\gamma_1 + a_8 = 0, \quad 28\alpha_1 - 8\gamma_1^2 - 12\beta_1\eta_1 + 8\alpha_1n_x + a_6 = 0, \\
&-6\beta_1n_x - 8\eta_1\gamma_1 - 15\beta_1 + a_4 = 0, \quad -2\eta_1^2 - 4n_x\gamma_1 - 6\gamma_1 + a_2 = 0, \\
&-32\alpha_2^2 + b_{14} = 0, \quad 48\alpha_2\beta_2 + b_{12} = 0, \quad -18\beta_2^2 + 32\alpha_2\gamma_2 + b_{10} = 0, \\
&16\alpha_2\eta_2 - 24\beta_2\gamma_2 + b_8 = 0, \quad 28\alpha_2 - 8\gamma_2^2 - 12\beta_2\eta_2 + 8\alpha_2n_y + b_6 = 0,
\end{aligned}$$

Table 4. The initial values of the parameters according to QES method.

$a_2 = b_2$	$a_4 = b_4$	$a_6 = b_6$	a_8 (QES) = b_8 (QES)	$a_{10} = b_{10}$	$a_{12} = b_{12}$	$a_{14} = b_{14}$	(n_x, n_y)
10.90625	10.6875	2.08625	0.3	0.04	0.002	0.0002	(0, 0)
15.15625	10.5875	2.06625	0.3	0.04	0.002	0.0002	(1, 1)

$$-6\beta_2 n_y - 8\eta_2\gamma_2 - 15\beta_2 + b_4 = 0, \quad -2\eta_2^2 - 4n_y\gamma_2 - 6\gamma_2 + b_2 = 0. \tag{4.6}$$

Now, from (4.6), it is seen that the parameters $a_8, a_{10}, a_{12}, a_{14}, b_8, b_{10}, b_{12}$ and b_{14} are free, while the remaining parameters a_2, a_4, a_6, b_2, b_4 and b_6 are not free and can be expressed in terms of the free ones and the quantum number n_x and n_y. The wave function parameters are also obtained from the free parameters. The QES energy of the 7-body quantum system is also deduced as

$$E_{n_x,n_y}(\text{QES}) = -\eta_1(2n_x + 1) - \eta_2(2n_y + 1), \tag{4.7}$$

where its eigenfunction can be obtained from (2.7) and (4.4), where the required parameters of (4.4) are calculated from (4.6).

4.2 *The 7-body system and Ritz method*

For this model too, the trial wave functions (3.12) and (3.13) taken as the ground and excited states respectively, satisfy the physical conditions of the problem. Substituting them into (3.10) and performing some numerical calculations, we obtain the minimized energy eigenvalue by Ritz method. In table 4, we have given the initial values of the free parameters and the remaining parameters are obtained from (4.6). In table 5, we have also obtained the values of energy from QES and Ritz methods.

Similar to the previous section, in table 4, we first choose some values for free parameters, then the values of the wave function's parameters are calculated from (4.6) and so by applying the condition where (n_x, n_y) should be zero or one, we calculate the numerical values of a_2, a_4, a_6, b_2, b_4 and b_6. We have also assumed that the initial values of the free parameters a_8, a_{10}, a_{12} and a_{14} are equal to the initial values of the free parameters b_8, b_{10}, b_{12} and b_{14}, respectively. According to the values of table 5, in this model too, we see that the energy eigenvalues are almost equal using two methods and also for the two states $(n_x = 0, n_y = 1)$ and $(n_x = 1, n_y = 0)$, they are exactly equal.

Table 5. The values of energy using QES and Ritz methods for the parameters of table 4.

E_{n_x,n_y} (QES)	E(Ritz)	(n_x, n_y)
6.25	5.300555306	(0, 0)
12.5	12.04504551	(1, 0)
12.5	12.04504551	(0, 1)
18.75	18.78953571	(1, 1)

Table 6. The values of energy for the perturbed system by two different methods.

E_{n_x,n_y}	E(Ritz)	(n_x, n_y)
8.25	5.301669062	(0, 0)
13.50217312	12.04814637	(1, 0)
13.50217312	12.04814637	(0, 1)
18.75434623	18.79462368	(1, 1)

Now, we assume that the initial values of parameters of the potential (4.3) have different values from what is mentioned in table 4, i.e., the initial values of the parameters do not satisfy eqs (4.6). Then, we can write the potential (4.3) as the sum of QES potential and a perturbation term:

$$\begin{aligned} V(x, y) = {} & a_2x^2 + a_4x^4 + a_6x^6 + a_{8\text{QES}}x^8 \\ & + a_{10}x^{10} + a_{12}x^{12} + a_{14}x^{14} \\ & + b_2y^2 + b_4y^4 + b_6y^6 + b_{8\text{QES}}y^8 \\ & + b_{10}y^{10} + b_{12}y^{12} + b_{14}y^{14} \\ & + (a_8 - a_{8\text{QES}})x^8 + (b_8 - b_{8\text{QES}})y^8, \end{aligned} \tag{4.8}$$

in which we have assumed that the two parameters (b_6, a_6) do not satisfy eq. (4.6). So in (4.8), the term $(a_8 - a_{8\text{QES}})x^8 + (b_8 - b_{8\text{QES}})y^8$ is considered as a perturbation term to the QES potential. Hence, the perturbation energy is obtained from the following equation:

$$\begin{aligned} \Delta E_{n_x,n_y} = {} & \frac{(a_8 - a_{8\text{QES}}) \int_{\infty}^{+\infty} \int x^8 \psi_x^* \psi_x \mathrm{d}x}{\int_{-\infty}^{+\infty} \psi_x^* \psi_x \mathrm{d}x} \\ & + \frac{(b_8 - b_{8\text{QES}}) \int_{-\infty}^{+\infty} y^8 \psi_y^* \psi_y \mathrm{d}y}{\int_{-\infty}^{+\infty} \psi_y^* \psi_y \mathrm{d}y} \end{aligned} \tag{4.9}$$

and the energy of the system becomes

$$E_{n_x,n_y} = E_{n_x,n_y}(\text{QES}) + \Delta E_{n_x,n_y}. \tag{4.10}$$

For example, if we take the value of perturbation parameters, which are (b_8, a_8), equal to 0.4 and the other free parameters are those given in table 4, then the values of energy given in table 6, except for the (0,0) state, do not have a major difference with those obtained from the Ritz method.

We must notice that if the other independent parameters of (4.3) do not satisfy eqs (4.6) then (4.8) must be written in an appropriate form according to the parameters and $\Delta E_{n_x,n_y}$, as the contribution of the perturbation terms, should be calculated for those free parameters.

5. Conclusion

We have calculated the spectrum of a quantum system with N particles with equal mass interacting via an anharmonic potential by QES method. For $N = 5$ and 7, we have obtained the eigenenergies for the ground state and the first excited state only, and we have shown that the results are in good agreement with those obtained by Ritz method. Also, we have shown that the method and the obtained eigenvalues are not limited to a given range of magnitude of the potential parameters. This procedure is very convenient for other many-body potentials which can be transformed into QES ones.

References

[1] A V Turbiner and A G Ushveridze, *Phys. Lett. A* **126**, 181 (1987)
[2] K Datta and A Rampal, *Phys. Rev. D* **23**, 2875 (1981)
[3] S Coleman, *Aspects of symmetry*, Selected Erice Lectures (Cambridge University Press, Cambridge, 1988) p. 234
[4] V Singh, S N Biswas and K Dutta, *Phys. Rev. D* **18**, 1901 (1978)
[5] G P Flessas, *Phys. Lett. A* **72**, 289 (1979)
[6] M Razavy, *Am. J. Phys.* **48**, 285 (1980)
[7] M Razavy, *Phys. Lett. A* **82**, 7 (1981)
[8] A Khare, *Phys. Lett. A* **83**, 237 (1981)
[9] M Znojil, *Phys. Lett. A* **13**, 1445 (1983)
[10] N Kamran and P J Olver, *J. Math. Ann. Appl.* **145**, 342 (1990)
[11] A González-López, N Kamran and P J Olver, *J. Phys. A* **24**, 3995 (1991)
[12] L Mandel and E Wolf, *Optical coherence and quantum optics* (Cambridge University Press, Cambridge, 1995) Chapter 19
[13] V M Tkachuk and T Fityo, *Phys. Lett. A* **309**, 351 (2003)
[14] R Atre and P K Panigrahi, *Phys. Lett. A* **317**, 46 (2003)
[15] A Chouchaoui and A Diaf, *Ann. Phys.* **306**, 232 (2003)
[16] T Tanaka, *Ann. Phys.* **309**, 239 (2004)
[17] F Chafa, A Chouchaoui, M Hachemane and F Z Ighezou, *Ann. Phys.* **322**, 1034 (2007)
[18] A Chouchaoui, *Ann. Phys.* **312**, 431 (2004)
[19] H Panahi and M Baradaran, *Eur. Phys. J. Plus* **128**, 39 (2013)
[20] H Panahi and M Baradaran, *Mod. Phys. Lett. A* **27**, 1250176 (2012)
[21] D Agboola and Y Z Zhang, *Ann. Phys.* **330**, 246 (2013)