PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 86, No. 5
May 2016
pp. 973–983

An entanglement concentration protocol for cluster states using CNOT gate operation

BINAYAK S CHOUDHURY and ARPAN DHARA*

Department of Mathematics, Indian Institute of Engineering Science and Technology,
Shibpur, B. Garden, Shibpur, Howrah 711 103, India
*Corresponding author. E-mail: arpanbesu88@gmail.com

MS received 11 July 2014; revised 15 April 2015; accepted 11 May 2015
DOI: 10.1007/s12043-015-1128-z; ***e*Publication:** 27 November 2015

Abstract. The purpose of this paper is a proposal on entanglement concentration protocol for cluster states. The protocol uses CNOT gate operation and is assisted with a single qubit. Moreover, the local and non-local operations are performed by a single party. We also make a comparative numerical study of the residual entanglement left out after the execution of each step of the protocol.

Keywords. CNOT gate; entanglement; entanglement concentration; cluster state.

PACS Nos 03.65.Ud; 03.67.Hk

1. Introduction

Entanglement is the most precious quantum resource in the theory of quantum information and communication [1,2]. Originally conceived in the famous work of Einstein, Podolsky and Rosen [3], its potential areas of applications began to be discovered only in the last decade of the 20th century [1].

The early literatures dealt with the two-party entanglement. Later, several multipartite entanglements were advanced through the categories of states like GHZ states [4], W-state [5], equitorial state [6], cluster state [7] etc., of which in the present paper we consider cluster states.

Maximally entangled states have many uses. Due to unavoidable interaction of the quantum system with the environment, and also after some use has been made of an entangled state, the amount of entanglement decreases. It becomes necessary to increase the amount of entanglement in order to make a practical use of it with maximum efficiency. Entanglement concentration protocols (ECP) are such protocols which are constructed for the purpose of converting less entangled states to states with more entanglement. In particular, partially entangled states are converted to the maximally entangled states using these protocols. It may be noted that these protocols are not the only approaches of generating maximally entangled states. There are several other protocols such as entanglement purification [8–12], quantum entanglement distillation [13,14], etc., to this effect.

The history of ECP starts in the work of Bennett *et al* [8] in 1996 in which he advanced a protocol which is also known as the Schmidt decomposition protocol. Since then, there have been several other interesting ECPs for photon systems and atomic systems. In 1999, Bose *et al* [15] proposed a protocol based on entanglement swapping. Shi *et al* [16] presented an ECP based on entanglement swapping and a collective unitary evolution on a qubit in a multiqubit system and an ancillary qubit. In 2001, Yamamoto *et al* [17] proposed an ECP based on polarizing beam splitters (PBSs) independently along with its completed experimental demonstration. In 2003, Zhao *et al* [18] proposed an experimental realization of entanglement concentration using two polarization entangled photon pairs produced by pulsed parametric down-conversion. In 2008, Sheng *et al* [19] represented an ECP based on cross-Kerr nonlinearities to distinguish the parity of two polarization photons with the help of Schmidt projection method. In 2009, the same researchers proposed another ECP for electrons based on their spins and their charges [20]. Subsequently, in 2010, they discussed an ECP based on a single photon with the help of cross-Kerr nonlinearity [21]. In 2012, the same researchers proposed two ECPs, the first one was implemented with linear optics and the second one was implemented with cross-Kerr nonlinearities [22]. In the same year, Deng [23] proposed an optimal non-local multipartite entanglement concentration based on projection measurement. In 2013, Choudhury *et al* [24] presented an ECP for cluster states based on single qubit and projection measurement. In the same year, Zhao *et al* [25] proposed a two-step entanglement concentration for arbitrary electronic cluster state.

The objective of our paper is to develop an ECP for even qubit cluster states. It has no natural extension to states like GHZ states involving odd numbers of qubits because the protocol uses the fact that an even number of qubits are involved. The protocol uses an ancillary qubit and a CNOT gate and is applicable to partially entangled pure cluster states. Cluster states have been used in many communication protocols like that given in [26]. A cluster state has some salient features like that it has better persistency than GHZ states and is maximally connected. A cluster state is harder to be destroyed by local operations than GHZ states [27]. ECPs for such cluster states have been proposed in [24,25]. We also make a comparative numerical study of the residual entanglement in the state left out after the execution of each iteration.

2. ECP for partially entangled four-qubit cluster states

In this section, we discuss an ECP for cluster states with four qubits. Then in the next section, we extend it to any even number of qubits (more than four). The schematic diagram of the ECP presented here is given in figure 1. Consider a four-qubit system composed of four qubits A, B, C, and D in the following partially entangled cluster state:

$$|\phi\rangle_{\mathrm{ABCD}} = a(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}}) \\ +b(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}}), \quad (1)$$

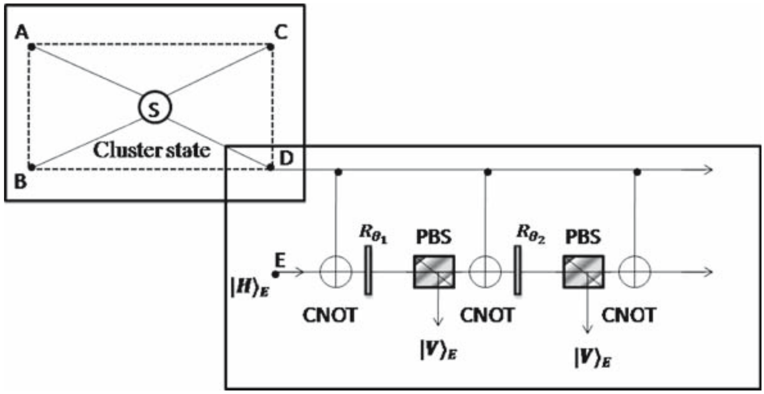

Figure 1. The schematic diagram of the present ECP for four-qubit cluster states assisted with an ancillary single qubit, CNOT gate and projection measurement. PBS stands for polarization beam splitter and R_θ denotes a unitary rotation operation through an angle θ acting on the ancillary single qubit E. S denotes the source of the non-maximally entangled cluster state.

where A, B, C and D represent the four qubits in the possession of four parties, say Alice, Bob, Carson and Dick, respectively and $|H\rangle$ and $|V\rangle$ stand for horizontal and vertical polarizations respectively and a and b are two known complex numbers, satisfying $2|a|^2 + 2|b|^2 = 1$.

Consider a single qubit E prepared by Dick whose initial state is given by $|H\rangle_{\mathrm{E}}$. Then he performs a CNOT gate on the two qubits D and E, where D is the control qubit and E is the target qubit. After applying the CNOT gate, the state of the composite system ABCDE is

$$\begin{aligned}|\Psi\rangle_{\mathrm{ABCDE}} &= a(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}})|H\rangle_{\mathrm{E}}\\ &\quad + b(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})|V\rangle_{\mathrm{E}}.\end{aligned}$$

Define a transformation, which is a rotation R_{θ_1}, given by

$$|H\rangle \rightarrow \frac{1}{\sqrt{|a|^2 + |b|^2}}(b|V\rangle - a|H\rangle),$$

$$|V\rangle \rightarrow \frac{1}{\sqrt{|a|^2 + |b|^2}}(a|V\rangle + b|H\rangle),$$

where a and b are complex numbers.

Applying this transformation R_{θ_1} on the single qubit E, the state of the system becomes

$$\begin{aligned}|\Psi'\rangle_{\mathrm{ABCDE}} = \frac{1}{\sqrt{(|a|^2 + |b|^2)}}\{&ab[|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}}\\ &+ |H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}}]|V\rangle_{\mathrm{E}}\\ &+ [-a^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}})\\ &+ b^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})]|H\rangle_{\mathrm{E}}\}.\end{aligned}$$

When Dick obtains the state $|V\rangle_{\mathrm{E}}$, the system ABCD is in the standard four-qubit cluster state

$$|\Psi_{\mathrm{CL}}\rangle_{\mathrm{ABCD}} = \frac{1}{2}(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}}).$$

The probability of obtaining the above cluster state is equal to

$$p_{s_1} = \frac{2|a|^2|b|^2}{(|a|^2+|b|^2)^2}.$$

When Dick obtains the state $|H\rangle_{\mathrm{E}}$, the system ABCD is the four-qubit cluster class state which is partially entangled, given by

$$|\phi'\rangle_{\mathrm{ABCD}} = \frac{1}{\sqrt{2(|a|^4+|b|^4)}}\{-a^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}}) + b^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})\}. \quad (2)$$

The probability of obtaining the above state is equal to

$$p_{f_1} = \frac{|a|^4+|b|^4}{(|a|^2+|b|^2)^2}.$$

When $|H\rangle_{\mathrm{E}}$ is obtained, i.e., when we have $|\phi'\rangle_{\mathrm{ABCD}}$, we perform the CNOT gate on the two qubits D and E, where D is the control qubit and E is the target qubit as used in the earlier case. Then the state of the composite system ABCDE is

$$|\Psi''\rangle_{\mathrm{ABCDE}} = \frac{1}{\sqrt{2(|a|^4+|b|^4)}}\{-a^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}})|H\rangle_{\mathrm{E}} + b^2(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})|V\rangle_{\mathrm{E}}\}.$$

Define another rotation operation R_{θ_2}, given by

$$|H\rangle \to \frac{1}{\sqrt{|a|^4+|b|^4}}(b^2|V\rangle - a^2|H\rangle),$$

$$|V\rangle \to \frac{1}{\sqrt{|a|^4+|b|^4}}(a^2|V\rangle + b^2|H\rangle).$$

We note that here a^2 and b^2 are complex numbers because a and b are assumed to be complex in R_{θ_1}.

Applying R_{θ_2} on the single qubit E, the state of the system becomes

$$|\Psi'''\rangle_{\mathrm{ABCDE}} = \frac{1}{\sqrt{2(|a|^4+|b|^4)(|a|^4+|b|^4)}} \times\{-a^2b^2[|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}}]|V\rangle_{\mathrm{E}} + [a^4(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}}) + b^4(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})]|H\rangle_{\mathrm{E}}\}.$$

When Dick obtains the state $|V\rangle_{\mathrm{E}}$, the system ABCD is in the standard four-qubit cluster state

$$|\Psi_{\mathrm{CL}}\rangle_{\mathrm{ABCD}} = \frac{1}{2}(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}}).$$

Probability of success of getting cluster state is equal to

$$p_{s_2} = p_{f_1}\frac{2|a|^4|b|^4}{(|a|^4+|b|^4)^2} = \frac{2|a|^4|b|^4}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)}.$$

When Dick obtains the state $|H\rangle_{\mathrm{E}}$, the system ABCD is in the four-qubit cluster class state with partial entanglement, given by

$$|\phi''\rangle_{\mathrm{ABCD}} = \frac{1}{\sqrt{2(|a|^8+|b|^8)}}\{a^4(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}} + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|H\rangle_{\mathrm{C}}|H\rangle_{\mathrm{D}}) + b^4(|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}} - |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}}|V\rangle_{\mathrm{C}}|V\rangle_{\mathrm{D}})\}. \tag{3}$$

The probability of obtaining this state is equal to

$$p_{f_2} = p_{f_1}\frac{|a|^8+|b|^8}{(|a|^4+|b|^4)^2} = \frac{|a|^8+|b|^8}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)}.$$

Then the probability of success of ECP after the execution of two steps is given by

$$P = \sum_{i=1}^{2} p_{s_i} = 2\left[\frac{|a|^2|b|^2}{(|a|^2+|b|^2)^2} + \frac{|a|^4|b|^4}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)}\right].$$

Repeating the protocol exactly as above, the probability of success for getting the maximally entangled cluster state after n repetition of the ECP is

$$P = \sum_{i=1}^{n} p_{s_i} = 2\left[\frac{|a|^2|b|^2}{(|a|^2+|b|^2)^2} + \frac{|a|^4|b|^4}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)} + \frac{|a|^8|b|^8}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)(|a|^8+|b|^8)} + \cdots + \frac{|a|^{2^n}|b|^{2^n}}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)\cdots(|a|^{2^n}+|b|^{2^n})}\right].$$

3. ECP for partially entangled 2N-qubit cluster states

Let us assume that a partially entangled 2N-qubit cluster state is given by

$$\begin{aligned}|\phi_{2\mathrm{N}}\rangle_{12345\cdots 2\mathrm{N}} = a(&|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}+\cdots\\ &+|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}})\\ +b(&|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}+\cdots\\ &-|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}),\end{aligned}$$

where $2^{\mathrm{N}-1}(|a|^2+|b|^2)=1$.

Here Dick holds the 2Nth qubit and prepares a single photon E whose initial state is given by $|H\rangle_{\mathrm{E}}$ and then performs a CNOT gate operation, where E is the target qubit after which the composite system is in the following state:

$$\begin{aligned}|\Psi'\rangle_{12345\cdots 2\mathrm{NE}} = {} & a(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}+\cdots \\ & +|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}})|H\rangle_{\mathrm{E}} \\ & +b(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}+\cdots \\ & -|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}})|V\rangle_{\mathrm{E}}.\end{aligned}$$

Define a transformation, which is a rotation R_{θ_1}, given by

$$|H\rangle \to \frac{1}{\sqrt{|a|^2+|b|^2}}(b|V\rangle - a|H\rangle),$$

$$|V\rangle \to \frac{1}{\sqrt{|a|^2+|b|^2}}(a|V\rangle + b|H\rangle),$$

where a and b are complex numbers.

Dick applies the transformation R_{θ_1} on this single qubit E to the composite system and obtains the following state:

$$\begin{aligned}|\Psi''\rangle_{1234\cdots 2\mathrm{NE}} = {} & \frac{1}{\sqrt{(|a|^2+|b|^2)}}\{ab[|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}+\cdots \\ & +|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}+\cdots \\ & +|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}+\cdots \\ & -|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}]|V\rangle_{\mathrm{E}} \\ & +[-a^2(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}+\cdots \\ & +|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|H\rangle_{2\mathrm{N}-1}|H\rangle_{2\mathrm{N}}) \\ & +b^2(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2\mathrm{N}-3}|H\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}} \\ & +|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}}+\cdots \\ & -|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2\mathrm{N}-3}|V\rangle_{2\mathrm{N}-2}|V\rangle_{2\mathrm{N}-1}|V\rangle_{2\mathrm{N}})]|H\rangle_{\mathrm{E}}\}.\end{aligned}$$

When Dick obtains the state $|V\rangle_{\rm E}$, the system 1234, ..., 2N qubits is in the standard 2N-qubit cluster state

$$\begin{aligned}|\Psi_{\rm CL}\rangle_{1234\cdots 2N} = \frac{1}{2^{N/2}}(&|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2N-1}|H\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2N-1}|H\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|H\rangle_{2N-1}|H\rangle_{2N}+\cdots\\ &+|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|H\rangle_{2N-1}|H\rangle_{2N}+\cdots\\ &+|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2N-3}|H\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2N-3}|H\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}+\cdots\\ &-|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}).\end{aligned}$$

Probability of success of getting cluster state is equal to

$$p_{s_1} = \frac{2|a|^2|b|^2}{(|a|^2+|b|^2)^2}.$$

When Dick obtains the state $|H\rangle_{\rm E}$, the system 1234, ..., 2N qubits is in the 2N-qubit cluster class state with partial entanglement, given by

$$\begin{aligned}|\Psi_L\rangle_{\rm ABCD} = \frac{1}{\sqrt{2^{N-1}(a^4+b^4)}}\{&-a^2(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2N-1}|H\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2N-1}|H\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|H\rangle_{2N-1}|H\rangle_{2N}+\cdots\\ &+|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|H\rangle_{2N-1}|H\rangle_{2N})\\ &+b^2(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4\cdots|H\rangle_{2N-3}|H\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|H\rangle_{2N-3}|H\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}\\ &+|H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N}+\cdots\\ &-|V\rangle_1|V\rangle_2|V\rangle_3|V\rangle_4\cdots|V\rangle_{2N-3}|V\rangle_{2N-2}|V\rangle_{2N-1}|V\rangle_{2N})\}.\end{aligned}$$

The probability of obtaining the above state is equal to

$$p_{f_1} = \frac{|a|^4+|b|^4}{(|a|^2+|b|^2)^2}.$$

Then the probability of success of ECP after the execution of two steps is given by

$$P = \sum_{i=1}^{2} p_{s_i} = 2\left[\frac{|a|^2|b|^2}{(|a|^2+|b|^2)^2} + \frac{|a|^4|b|^4}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)}\right].$$

Now the total probability of success of getting maximally entangled cluster state after n repetition of concentration protocol over 2N-qubits is

$$\begin{aligned}P = \sum_{i=1}^{n} p_{s_i} = 2\Bigg[&\frac{|a|^2|b|^2}{(|a|^2+|b|^2)^2} + \frac{|a|^4|b|^4}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)}\\ &+\frac{|a|^8|b|^8}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)(|a|^8+|b|^8)}+\cdots\\ &+\frac{|a|^{2^n}|b|^{2^n}}{(|a|^2+|b|^2)^2(|a|^4+|b|^4)\cdots(|a|^{2^n}+|b|^{2^n})}\Bigg].\end{aligned}$$

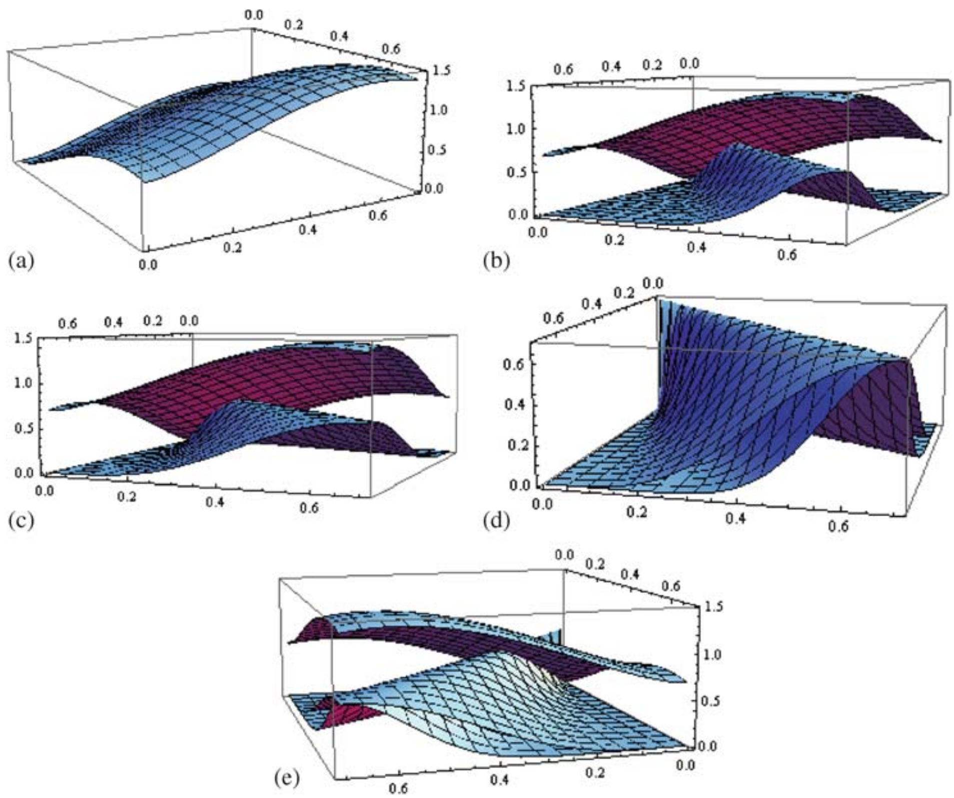

Figure 2. (**a**) The amount of entanglement of eq. (1), given by eq. (5) in the Appendix; (**b**) the amount of entanglement of eqs (1) and (2), given by eqs (5) and (4); (**c**) the amount of entanglement of eqs (1) and (3), given by eqs (5) and (6); (**d**) the amount of entanglement of eqs (2) and (3), given by eqs (4) and (6); (**e**) the amount of entanglement of eqs (1), (2) and (3), given by eqs (5), (4) and (6). Here the cases with real coefficients are considered.

4. Discussion and conclusion

We show that the change of the amounts of entanglement of the states which are taken as the initial states for each iteration gradually decrease in each step of the protocol compared to the previous step. When referred to the four-qubit case, the amounts of entanglement in the states (1), (2) and (3), which are the initial states of the first three steps respectively of the protocol, gradually decrease. We observe it by plotting the graph given in figure 2. We use entropy measure of entanglement, the details of which are given in the Appendix.

We observe from figure 2 that the amount of entanglement is gradually decreased as the iteration is progressed, i.e., the amount of entanglement of the initial state is higher than the next iterative states. Here these states are represented by eqs (1)–(3). So, as the iteration progresses, the amount of entanglement of the partially entangled cluster states, which are considered as the initial state in each step of the iteration, decreases. In our previous work [24], we have introduced an ECP of the cluster state where we used single qubit case and PBS, but in this paper we use one ancillary qubit and, instead of PBS, we use the CNOT gate here.

Remark 1. Here is a comparison between the present work and our previous work [24]. In both the works, we consider the same problem of constructing cluster states. We use two different ECPs for this purpose in the two respective works. In [24], we use the ancilla $|\psi\rangle_{\mathrm{E}} = \frac{1}{\sqrt{2}}(|H\rangle_{\mathrm{E}} + |V\rangle_{\mathrm{E}})$, while in the present context it is $|H\rangle_{\mathrm{E}}$. In the present paper, we use a CNOT gate operation which is not done in [24]. The present ECP is an iterative scheme which is not the case with that in [24]. Either of the two ECPs is not reducible to the other, that is, they are independent.

Remark 2. There are several measures of multipartite entanglement. One is the entropic measure which is used in this work. With respect to this measure of entanglement, the cluster states are maximally entangled. For this reason the initial state is called a partially entangled state.

Acknowledgements

This work is supported by the University Grants Commission of India. The support is gratefully acknowledged. The authors also gratefully acknowledge the valuable suggestions made by the referee.

Appendix

To measure the amount of entanglement of eq. (2), we use the entropic measure here. We restrict our attention to states with real coefficient. This is with a view to the numerical study which we display through figure 2.

Let

$$\rho_{\mathrm{ABCD}} = |\phi'\rangle_{\mathrm{ABCD}}\langle\phi'|.$$

Then

$$\begin{aligned}\rho_{\mathrm{AB}} &= \mathrm{Tr}_{\mathrm{CD}}(\rho_{\mathrm{ABCD}})\\ &= \frac{1}{2}|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{BA}}\langle H|_{\mathrm{B}}\langle H| + \frac{a^4-b^4}{2(a^4+b^4)}\\ &\quad\times\{|H\rangle_{\mathrm{A}}|H\rangle_{\mathrm{BA}}\langle V|_{\mathrm{B}}\langle V| + |V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{B}\ \mathrm{A}}\langle H|_{\mathrm{B}}\langle H|\}\\ &\quad+\frac{1}{2}|V\rangle_{\mathrm{A}}|V\rangle_{\mathrm{BA}}\langle V|_{\mathrm{B}}\langle V|.\end{aligned}$$

The eigenvalues of ρ_{AB} are $a^4/(a^4+b^4)$, $b^4/(a^4+b^4)$, 0, 0. Then the amount of entanglement of ρ_{AB} is

$$\begin{aligned}S(\rho_{\mathrm{AB}}) &= -\frac{a^4}{a^4+b^4}\ln\left(\frac{a^4}{a^4+b^4}\right) - \frac{b^4}{a^4+b^4}\ln\left(\frac{b^4}{a^4+b^4}\right)\\ &= \ln(a^4+b^4) - \frac{1}{a^4+b^4}\{a^4\ln(a^4) + b^4\ln(b^4)\}.\end{aligned}$$

Now,

$$\rho_{\rm CD}{=}{\rm Tr}_{\rm AB}(\rho_{\rm ABCD}) = \frac{a^4}{a^4+b^4}|H\rangle_{\rm C}|H\rangle_{\rm D}\,{}_{\rm C}\langle H|_{\rm D}\langle H| + \frac{b^4}{a^4+b^4}|V\rangle_{\rm C}|V\rangle_{\rm D}\,{}_{\rm C}\langle V|_{\rm D}\langle V|.$$

The eigenvalues of $\rho_{\rm CD}$ are $a^4/(a^4+b^4), b^4/(a^4+b^4), 0, 0$. Then the amount of entanglement of $\rho_{\rm CD}$ is

$$\begin{aligned} S(\rho_{\rm CD}) &= -\frac{a^4}{a^4+b^4}\ln\left(\frac{a^4}{a^4+b^4}\right) - \frac{b^4}{a^4+b^4}\ln\left(\frac{b^4}{a^4+b^4}\right) \\ &= \ln(a^4+b^4) - \frac{1}{a^4+b^4}\{a^4\ln(a^4) + b^4\ln(b^4)\}. \end{aligned}$$

So the amount of entanglement of $\rho_{\rm ABCD}$ is equal to

$$\begin{aligned} S(\rho_{\rm ABCD}) &= S(\rho_{\rm AB}) = S(\rho_{\rm CD}) = -\frac{a^4}{a^4+b^4}\ln\left(\frac{a^4}{a^4+b^4}\right) \\ &\quad -\frac{b^4}{a^4+b^4}\ln\left(\frac{b^4}{a^4+b^4}\right) \\ &= \ln(a^4+b^4) - \frac{1}{a^4+b^4}\{a^4\ln(a^4) + b^4\ln(b^4)\} \end{aligned} \tag{4}$$

and this represents the amount of entanglement of eq. (2).

In the case of eq. (1), if we denote its mixed state i.e., the density operator by $\rho'_{\rm ABCD}$, then the amount of entanglement of the state represented by eq. (1) is given by

$$\begin{aligned} S(\rho'_{\rm ABCD}) &= -2a^2\ln(2a^2) - 2b^2\ln(2b^2) \\ &= -4a^2\ln(a) - 4b^2\ln(b) - \ln(2). \end{aligned} \tag{5}$$

In the case of eq. (3), if we denote its mixed state, i.e., the density operator by $\rho''_{\rm ABCD}$, then the amount of entanglement of the state represented by eq. (3) is given by

$$\begin{aligned} S(\rho''_{\rm ABCD}) &= -\frac{a^8}{a^8+b^8}\ln\left(\frac{a^8}{a^8+b^8}\right) - \frac{b^8}{a^8+b^8}\ln\left(\frac{b^8}{a^8+b^8}\right) \\ &= \ln(a^8+b^8) - \frac{1}{a^8+b^8}\{a^8\ln(a^8) + b^8\ln(b^8)\}. \end{aligned} \tag{6}$$

References

[1] M A Nielsen and I L Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000)
[2] D McMahon, *Quantum computing explained* (John Wiley and Sons, 2007)
[3] A Einstein, B Podolsky and N Rosen, *Phys. Rev.* **47**, 777 (1935)
[4] B S Choudhury and A Dhara, *Int. J. Theor. Phys.*, DOI:10.1007/s10773-013-1709-x (2013)
[5] B Gu, D H Quan and S R Xiao, *Int. J. Theor. Phys.* **51**, 2966 (2012)

[6] C H Bennett, D P DiVincenzo, P W Shor, J A Smolin, B M Terhal and W K Wootters, *Phys. Rev. Lett.* **87**, 077902 (2001)
[7] H J Briegel and R Raussendorf, *Phys. Rev. Lett.* **86**, 910 (2001)
[8] C H Bennett, G Brassard, S Popescu and B Schumacher, *Phys. Rev. A* **53**, 2046 (1996)
[9] D Deutsch, A Ekert, R Jozsa, C Macchiavello, S Popescu and A Sanpera, *Phys. Rev. Lett.* **77**, 2818 (1996)
[10] J W Pan, S Gasparonl, R Ursin, G Weihs and A Zellinger, *Nature (London)* **423**, 417 (2003)
[11] Y B Sheng, F G Deng and H Y Zhou, *Eur. Phys. J. D* **55**, 235 (2009)
[12] Y B Sheng, F G Deng and G L Long, *Phys. Lett. A* **375**, 396 (2011)
[13] M Horodecki, P Horodecki and R Horodecki, *Phys. Rev. Lett.* **80**, 5239 (1998)
[14] P G Kwiat, S B Lopez, A Stefanov and N Gisin, *Nature* **409**, 1014 (2001)
[15] S Bose, V Vedral and P L Knight, *Phys. Rev. A* **60**, 194 (1999)
[16] B S Shi, Y K Jiang and G C Guo, *Phys. Rev. A* **62**, 054301 (2000)
[17] T Yamamoto, M Koashi and N Imoto, *Phys. Rev. A* **64**, 012304 (2001)
[18] Z Zhao, T Yang, Y A Chen, A N Zhang and J W Pan, *Phys. Rev. Lett.* **90**, 207901 (2003)
[19] Y B Sheng, F G Deng and H Y Zhou, *Phys. Rev. A* **77**, 062325 (2008)
[20] Y B Sheng, F G Deng and H Y Zhou, *Phys. Lett. A* **373**, 1823 (2009)
[21] Y B Sheng, F G Deng and H Y Zhou, *Quantum Inf. Comput.* **10**, 0272 (2010)
[22] Y B Sheng, F G Deng and H Y Zhou, *Phys. Rev. A* **85**, 012307 (2012)
[23] F G Deng, *Phys. Rev. A* **85**, 022311 (2012)
[24] B S Choudhury and A Dhara, *Quantum Inf. Process* **12**, 2577 (2013)
[25] S Y Zhao, J Liu, L Zhou and Y B Sheng, *Quantum Inf. Process*, DOI:10.1007/s11128-013-0623-8 (2013)
[26] Q Zhang, C Li, Y Li and Y Nie, *Int. J. Theor. Phys.*, DOI:10.1007/s10773-012-1294-4 (2012)
[27] P Dong, Z Y Xue, M Yang and Z L Cao, *Phys. Rev. A* **73**, 033818 (2006)