PRAMANA
— journal of
physics

Vol. 86, No. 5
May 2016
pp. 965–972

Analytical results for non-Hermitian parity–time-symmetric and Hermitian asymmetric volcano potentials

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MS received 31 March 2015; accepted 1 June 2015
DOI: 10.1007/s12043-015-1139-9; ***e*Publication:** 16 February 2016

Abstract. We investigate both the non-Hermitian parity–time-(PT-)symmetric and Hermitian asymmetric volcano potentials, and present the analytical solution in terms of the confluent Heun function. Under certain special conditions, the confluent Heun function can be terminated as a polynomial, thereby leading to certain exact analytical results. It is found that the non-Hermitian PT-symmetric volcano potentials support the normalizable and non-normalizable reflectionless states with real energies. The Hermitian asymmetric volcano potentials allow normalizable reflectionless states with complex energies.

Keywords. Volcano potentials; Heun confluent equation; Heun confluent function; reflectionless scattering state.

PACS Nos 03.65.Ge; 03.65.Fd; 11.30.Er

1. Introduction

In recent years, a class of quasiexactly solvable (QES) non-Hermitian parity–time-(PT-) symmetric Hamiltonians have attracted extensive research interest [1–10]. These systems include two important concepts: quasiexact solvability and non-Hermitian PT-symmetric Hamiltonian. Quasiexact solvability means that a finite portion of the energy spectrum and the associated eigenfunctions can be solved algebraically [11–15]. The non-Hermitian PT-symmetric Hamiltonians are viewed as the extension of quantum mechanics based on Hermitian Hamiltonians [16]. It has been shown that a wide class of non-Hermitian PT-symmetric Hamiltonians can still possess entirely real eigenvalue spectra below a phase transition symmetry-breaking point. In the last few years, many interesting phenomena related to non-Hermitian PT-symmetric Hamiltonians have been observed [17–24].

Volcano potentials are one of the important QES potentials which are unbounded from below [25–30]. Compared to many regular quantum potentials, volcano potentials exhibit several new features. For example, certain symmetric volcano potentials can support parity-paired degenerate bound states with real energies [25,26] and the normalizable scattering states with perfect transmission [27–29]. The reflectionless scattering states and bound states have the same energies. In particular, under special conditions, the energy of the ground bound state is even above the maximum of the volcano potentials [30]. Therefore, these bound states may represent von Neumann–Wigner states in the continuum of scattering states [31,32]. Now the von Neumann–Wigner states have been observed in different systems [33–35].

In this work, we consider two kinds of volcano potentials. The first one is the non-Hermitian PT-symmetric volcano potential and the second one is the Hermitian asymmetric volcano potential. It is found that the two kinds of volcano potentials are still QES. We present an analytical exact solution in terms of the Heun function confluent. When the system parameters satisfy a certain relation, the Heun function confluent can be terminated as a polynomial. Such truncation leads to exact analytical results in an explicit form. We show that due to the presence of non-Hermitian terms, the non-Hermitian PT-symmetric volcano potentials can also allow non-normalizable reflectionless states with real energies. The parameter conditions for both normalizable and non-normalizable reflectionless states are obtained analytically. In addition, it is found that the Hermitian asymmetric volcano potentials support normalizable reflectionless states with complex energies.

2. Exact solutions of the non-Hermitian PT-symmetric volcano potentials

We first consider the non-Hermitian complex volcano potential $V(x)$

$$V(x) = V_1 \sinh^2 x + V_2\, \mathrm{sech}^2 x + i(V_3 \sinh x + V_4\, \mathrm{sech}\, x \tanh x), \tag{1}$$

with $V_1 < 0$. When $V_3 = V_4 = 0$, the resulting Hermitian volcano potential is unbounded from below [25–30]. The Schrödinger equation for this complex volcano potential $V(x)$ is given as ($\hbar = 2m = 1$)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{2}$$

This non-Hermitian complex system is PT-symmetric, as the Hamiltonian $p^2 + V(x)$ is invariant under PT reflection with the parity reflection operator (P: $x \to -x$, $p \to -p$) and the time reversal operator (T: $x \to x$, $p \to -p$, $i \to -i$) [16].

In the Hermitian case of $V_{3,4} = 0$ and $V_2 = 1/4 - (n+1)^2$ with $n \geq 0$ being the integer, exact analytical results for the bound states and the corresponding normalizable reflectionless states have been found [25–29]. We note that the corresponding Hermitian potential $V_{\mathrm{H}}(x) = V_1 \sinh^2 x + V_2\, \mathrm{sech}^2 x + V_3 \sinh x + V_4\, \mathrm{sech}\, x \tanh x$ has been studied in ref. [36]. It has been found that the eigenfunctions of $V_{\mathrm{H}}(x)$ may be normalizable in the following two cases: (1) $V_1 = V_3 = 0$ [36] and (2) $V_3 = V_4 = 0$ [27,28]. In the present work, we show that the eigenfunctions for the complex potential $V(x)$ may be normalizable under certain special parameter conditions. In addition, the complex potential $V(x)$ supports the reflectionless states which can be either normalizable or non-normalizable.

We first make the transformations

$$z = \frac{1 - i \sinh x}{2}, \quad \psi(x) = e^{i\sqrt{-V_1}\sinh x}(1 - i \sinh x)^{\lambda}(1 + i \sinh x)^{\rho}\phi(z), \quad (3)$$

where $\lambda = (1 - \sqrt{1 - 4V_2 - 4V_4})/4$ and $\rho = (1 - \sqrt{1 - 4V_2 + 4V_4})/4$, and then find that $\phi(z)$ satisfies the Heun confluent equation [37,38]

$$\frac{d^2\phi}{dz^2} + \left(\alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1}\right)\frac{d\phi}{dz} + \frac{pz + q}{z(z - 1)}\phi = 0, \quad (4)$$

where $p = \delta + \alpha(\beta + \gamma + 2)/2$ and $q = \eta + \beta/2 + (\gamma - \alpha)(\beta + 1)/2$. The parameters α, β, γ, δ and η read as $\alpha = -4\sqrt{-V_1}$, $\beta = -\sqrt{1 - 4V_2 - 4V_4}/2$, $\gamma = -\sqrt{1 - 4V_2 + 4V_4}/2$, $\delta = 2V_3$, $\eta = 3/8 + V_1 - V_2/2 - V_3 + E$. Equation (4) has two regular singularities at $z = 0$ and 1, and an irregular singularity at $z = \infty$. The Heun confluent equation has a local Frobenius solution around $z = 0$ of the form [37,38]

$$\phi(z) = \text{HC}(\alpha, \beta, \gamma, \delta, \eta, z) = \sum_{n=0}^{\infty} h_n z^n, \quad (5)$$

where $\text{HC}(\alpha, \beta, \gamma, \delta, \eta, z)$ is the Heun confluent function. The coefficients h_n in the Heun confluent function are defined by the three-term recurrence relation, $A_n h_n = B_n h_{n-1} + C_n h_{n-2}$ $(n \geq 1)$ with the initial conditions $h_0 = 1$ and $h_{-1} = 0$. Here, $A_n = 1 + \beta/n$, $B_n = 1 + (\beta + \gamma - \alpha - 1)/n + (\eta - \beta/2 + (\gamma - \alpha)(\beta + 1)/2)/n^2$ and $C_n = (\delta + \alpha(\beta + \gamma)/2 + \alpha(n - 1))/n^2$. The solution to eq. (2) is thus given as

$$\psi(x) = e^{i\sqrt{-V_1}\sinh x}(1 - i \sinh x)^{\lambda}(1 + i \sinh x)^{\rho}\text{HC}(\alpha, \beta, \gamma, \delta, \eta, z). \quad (6)$$

Mathematically, the radius of convergence of the confluent Heun function is $|z| < 1$. It follows from $|z| = \cosh(x)/2$ that the solution $\psi(x)$ may not be valid in a certain spatial region with $|z| > 1$. However, there exist necessary conditions [37,38]

$$\delta = -\left(N + \frac{\gamma + \beta + 2}{2}\right)\alpha, \quad N = 0, 1, 2, \ldots, \quad (7)$$

$$h_{N+1} = 0, \quad (8)$$

under which the Heun confluent function $\text{HC}(\alpha, \beta, \gamma, \delta, \eta, z)$ can be terminated as a polynomial of degree N for z. By applying conditions (7) and (8) to $\psi(x)$, we can find certain exact analytical results in closed form. The application of condition (7) to $\text{HC}(\alpha, \beta, \gamma, \delta, \eta, z)$ in $\psi(x)$ leads to the relation between the parameters $V_{1,2,3,4}$

$$V_3 = \sqrt{-V_1}\left(2N + 2 - \frac{\sqrt{1 - 4V_2 + 4V_4} + \sqrt{1 - 4V_2 - 4V_4}}{2}\right). \quad (9)$$

When $V_3 = 0$ and $4|V_4| \leq (1 - 4V_2)$, the parameter relation (9) is simplified as $V_2 = 1/4 - (N + 1)^2 - V_4^2/4(N + 1)^2$. In [25–30], researchers have mainly discussed the case of $V_{3,4} = 0$. Our results contain the previous results as special cases. Because the coefficient h_{N+1} is a polynomial of degree $N + 1$ in E, the eigenvalues E

can be obtained from the condition (8) $h_{N+1}(E) = 0$. By making use of truncation of the Heun confluent function, we have the exact solutions

$$\psi_N^m(x) = \mathrm{e}^{i\sqrt{-V_1}\sinh x}(1 - i\sinh x)^{(1-\sqrt{1-4V_2-4V_4})/4}(1 + i\sinh x)^{(1-\sqrt{1-4V_2+4V_4})/4} \times \sum_{n=0}^{N} h_n(E_N^m)\frac{(1 - i\sinh x)^n}{2^n}. \tag{10}$$

Here E_N^m corresponds to the mth zeroes of $h_{N+1}(E) = 0$. To further give physically acceptable solutions, we must have

$$F_N \equiv N + (2 - \sqrt{1 - 4V_2 - 4V_4} - \sqrt{1 - 4V_2 + 4V_4})/4 \leq 0 \tag{11}$$

for $\psi_N^m(x)$. The resulting exact solutions represent right-moving reflectionless states [27,28]. $F_N<0$ corresponds to the normalizable reflectionless states and $F_N = 0$ corresponds to the non-normalizable reflectionless states. In addition, by using relations $1 \pm i\sinh x = (\cosh(x/2) \pm i\sinh(x/2))^2 = \cosh(x)\exp[\pm i2\theta]$ with $\theta = \arctan \times(\tanh(x/2))$, eq. (10) can be rewritten as

$$\psi_N^m(x) = \mathrm{e}^{i\sqrt{-V_1}\sinh x + i\kappa\theta}(\cosh x)^{-N-1/2+\frac{V_3}{2\sqrt{-V_1}}}\sum_{n=0}^{N} h_n(E_N^m)\frac{\cosh^n(x)\mathrm{e}^{-i2n\theta}}{2^n} \tag{12}$$

with

$$\kappa = \frac{\sqrt{1 - 4V_2 - 4V_4} - \sqrt{1 - 4V_2 + 4V_4}}{2}. \tag{13}$$

When $V_3 = 0$ in the exact solution (12), the highest-power term of $\cosh x$ is $(\cosh x)^{-1/2}$. The resulting exact solution is localized and normalizable, and represents right-moving reflectionless states [27,28]. The presence of V_3 allows the existence of non-normalizable reflectionless states. In the following, we present several simple solutions under certain special conditions.

Case I.

We first consider the case of $N = 0$ and $V_4 = 0$ for $\psi_N^m(x)$. From eq. (9), we have

$$V_3 = \sqrt{-V_1}(2 - \sqrt{1 - 4V_2}). \tag{14}$$

From $h_1(E) = (2V_2 - 1 - 2V_1 - 2E - \sqrt{1 - 4V_2})/(\sqrt{1 - 4V_2} - 2) = 0$, we obtain the eigenvalue

$$E_0^1 = V_2 - V_1 - \frac{1}{2}\sqrt{1 - 4V_2} - \frac{1}{2}. \tag{15}$$

The solution $\psi_N^m(x)$ is thus given as

$$\psi_0^1(x) = \frac{\mathrm{e}^{i\sqrt{-V_1}\sinh x}}{(\cosh x)^{(\sqrt{1-4V_2}-1)/2}}. \tag{16}$$

It represents a physically acceptable state for $V_2 \leq 0$. $\psi_0^1(x)$ denotes the normalizable reflectionless state for $V_2 < 0$ and non-normalizable reflectionless state for $V_2 = 0$. If we take $V_3 = 0$, $V_2 = -3/4$ and $V_1 = -1/4$, the lowest reflectionless states can be reproduced [27,28].

In the same case of $N = 0$, if we now take $V_4 = \sqrt{-V_2}$, we have $\sqrt{1-4V_2-4V_4} = 1-2\sqrt{-V_2}$ under the condition $-1/4 \le V_2 < 0$. From eq. (9), we have

$$V_3 = \sqrt{-V_1}. \tag{17}$$

From $h_1(E) = 0$, we obtain the eigenvalue

$$E_0^1 = -2\sqrt{V_1 V_2} - V_1. \tag{18}$$

In this case, we have the non-normalizable reflectionless state

$$\psi_0^1(x) = \mathrm{e}^{i\sqrt{-V_1}\sinh x - i2\sqrt{-V_2}\theta}. \tag{19}$$

Case II.
The next case is $N = 1$ and $V_4 = 0$ for $\psi_N^m(x)$. From eq. (9), we have

$$V_3 = \sqrt{-V_1}(4 - \sqrt{1-4V_2}). \tag{20}$$

From the condition $h_2(E) = 0$, we have two roots for E

$$E_1^1 = V_2 - V_1 + \sqrt{1-4V_2} - \frac{3}{2} - \frac{\sqrt{C_1}}{4}, \tag{21}$$

$$E_1^2 = V_2 - V_1 + \sqrt{1-4V_2} - \frac{3}{2} + \frac{\sqrt{C_1}}{4}, \tag{22}$$

where $C_1 = (1-4V_2)^{3/2} + (4V_2-17)\sqrt{1-4V_2} + 20 - 16V_2 - 64V_1$. The corresponding solutions are given as

$$\psi_1^1(x) = \frac{\mathrm{e}^{i\sqrt{-V_1}\sinh x}}{(\cosh x)^{(\sqrt{1-4V_2}-1)/2}} \times \left(1 + \frac{4 - 2\sqrt{1-4V_2} + 8\sqrt{-V_1} + C_1}{4\sqrt{1-4V_2} - 8}(1 - i\sinh x)\right), \tag{23}$$

$$\psi_1^2(x) = \frac{\mathrm{e}^{i\sqrt{-V_1}\sinh x}}{(\cosh x)^{(\sqrt{1-4V_2}-1)/2}} \times \left(1 + \frac{4 - 2\sqrt{1-4V_2} + 8\sqrt{-V_1} - C_1}{4\sqrt{1-4V_2} - 8}(1 - i\sinh x)\right). \tag{24}$$

In this situation, the chosen values of $V_3 = 0$, $V_2 = -15/4$ and $V_1 = -1/4$ lead to the results in [27,28]. It is clearly seen that under the condition $(\sqrt{1-4V_2}-1)/2 > 1$, $\psi_1^1(x)$ and $\psi_1^2(x)$ are normalizable. In addition, in $\psi_1^1(x)$, for $V_2 = -2$, we have $(\sqrt{1-4V_2}-1)/2 = 1$ and

$$\psi_1^1(x) = \mathrm{e}^{i\sqrt{-V_1}\sinh x}(A\,\mathrm{sech}\,x + iB\tanh x) \tag{25}$$

with

$$A = 1 + (4\sqrt{-V_1} + \sqrt{1 - 16V_1} - 1)/2,$$
$$B = -(4\sqrt{-V_1} + \sqrt{1 - 16V_1} - 1)/2. \tag{26}$$

It is evident that this solution represents a non-normalizable reflectionless state.

3. Exact solutions of the Hermitian asymmetric volcano potentials

In this section, we consider the Hermitian volcano potentials

$$V(x) = V_1 \sinh^2 x + V_2 \operatorname{sech}^2 x + V_4 \operatorname{sech} x \tanh x. \tag{27}$$

As $V(-x) \neq V(x)$, the volcano potentials are asymmetric. By replacing V_4 by $-iV_4$ and setting $V_3 = 0$ in eq. (12), we have the solution

$$\psi_N^m(x) = \mathrm{e}^{i\sqrt{-V_1}\sinh x - V_4\theta/(N+1)} (\cosh x)^{-N-1/2} \sum_{n=0}^{N} h_n(E_N^m) \frac{\cosh^n(x)\mathrm{e}^{-i2n\theta}}{2^n}. \tag{28}$$

As the highest term for the power of $\cosh x$ is given as $(\cosh x)^{-1/2}$, the resulting exact solution is localized and normalizable, and represents right-moving reflectionless states [27,28]. From eq. (9), we obtain the parameter condition for the solution to be valid

$$2N + 2 - \frac{\sqrt{1 - 4V_2 - 4iV_4} + \sqrt{1 - 4V_2 + 4iV_4}}{2} = 0. \tag{29}$$

From the above equation, we further have

$$V_2 = \frac{1}{4} - (N+1)^2 + \frac{V_4^2}{4(N+1)^2}. \tag{30}$$

Now, we consider only the two simplest cases: $N = 0$ and $N = 1$.

When $N = 0$, from eq. (30), we have the parameter relation

$$V_2 = -\frac{3}{4} + \frac{V_4^2}{4}. \tag{31}$$

From $h_1(E)$=0, we obtain the eigenvalue

$$E = -\frac{1}{4} - V_1 + i\sqrt{-V_1}V_4. \tag{32}$$

It is evident that as $V_1 < 0$, the eigenvalue E is complex. The corresponding solution is given as

$$\psi_0^1(x) = \frac{\mathrm{e}^{i\sqrt{-V_1}\sinh(x) - V_4\theta}}{\sqrt{\cosh(x)}}. \tag{33}$$

When $N = 1$, from eq. (30), we have the parameter relation

$$V_2 = -\frac{15}{4} + \frac{V_4^2}{16}. \tag{34}$$

From the condition $h_2(E) = 0$, we have two roots for E

$$E_1^1 = -5/4 - V_1 - \sqrt{1 - 4V_1 + i\sqrt{-V_1}V_4} + i\frac{\sqrt{-V_1}V_4}{2}, \tag{35}$$

$$E_1^2 = -5/4 - V_1 + \sqrt{1 - 4V_1 + i\sqrt{-V_1}V_4} + i\frac{\sqrt{-V_1}V_4}{2}. \tag{36}$$

It is evident that as $V_1 < 0$, the two eigenvalues $E_1^{1,2}$ are complex. The corresponding solutions are given as

$$\psi_{1,1}^1(x) = \frac{e^{i\sqrt{-V_1}\sinh(x) - V_4\theta/2}}{(\cosh x)^{3/2}} \times \left(1 + \frac{8\sqrt{-V_1} - 4 + 4\sqrt{1 - 4V_1 + i\sqrt{-V_1}V_4}}{8 + i8V_4}(1 - i\sinh x)\right), \tag{37}$$

$$\psi_{1,1}^2(x) = \frac{e^{i\sqrt{-V_1}\sinh(x) - V_4\theta/2}}{(\cosh x)^{3/2}} \times \left(1 + \frac{8\sqrt{-V_1} - 4 - 4\sqrt{1 - 4V_1 + i\sqrt{-V_1}V_4}}{8 + i8V_4}(1 - i\sinh x)\right). \tag{38}$$

In principle, for a given value of N, we have $N+1$ complex energies. It is clearly seen that these complex energy solutions represent the normalizable reflectionless states. They are also related to the quasinormal modes [39,40]. In refs [39,40], several forms of volcano potentials have been shown to support normalizable reflectionless states with complex energies.

4. Conclusion

In conclusion, we have presented certain exact analytical results for two different types of volcano potentials. It is shown that the analytical solutions are expressed in terms of the Heun confluent function. Under certain conditions, the Heun confluent functions can be terminated as a polynomial. With such truncation of the Heun confluent function, we have obtained exact analytical results in an explicit form. In the non-Hermitian PT-symmetric volcano potentials, the normalizable and non-normalizable reflectionless states with real energies are obtained. In the Hermitian asymmetric volcano potentials, the normalizable reflectionless states with complex energies are found.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under Grant No. 11375059 and the Scientific Research Fund of Hunan Provincial Education Department under Grant No. 13A058.

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