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A novel approach for solving fractional Fisher equation using differential transform method

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Abstract. In the present paper, an analytic solution of nonlinear fractional Fisher equation is deduced with the help of the powerful differential transform method (DTM). To illustrate the method, two examples have been prepared. The method for this equation has led to an exact solution. The reliability, simplicity and cost-effectiveness of the method are confirmed by applying this method on different forms of functional equations.

Keywords. Differential transform method; fractional Fisher equation.

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1. Introduction

In recent years, fractional differential equations have attracted lots of interest due to their numerous applications in areas of physics and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order [2–5].

The Fisher equation, as a nonlinear model for a physical system involving linear diffusion and nonlinear growth, takes the non-dimensional form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{1}$$

with initial condition $u(x,0) = g(x)$. When $\alpha = 1$, eq. (1) reduces to the classical Fisher equation [6,7]. The purpose of this paper is to obtain analytic solution of this equation by DTM.

Fisher proposed eq. (1) as a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions and branching Brownian motion processes.

The differential transform method was first introduced by Zhou [8] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of polynomial expressions such as Taylor series expansion. But the procedure is easier than the traditional higher-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive for higher orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

2. Basic definitions

In this section, we give some basic definitions and properties of the fractional calculus theory which can be found in [4,5].

DEFINITION 2.1

A real function $f(x), x > 0$, is in space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty]$ and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu, m \in N$.

DEFINITION 2.2

The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0,\ x > 0,\ J^0 f(x) = f(x).$$

The properties of the operator J^α can be found in [9], and we only mention the following (in this case, $f \in C_\mu, \mu \geq -1,\ \alpha, \beta \geq 0$ and $\gamma > -1$):

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
(3) $J^\alpha x^\gamma = [\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)] x^{\alpha+\gamma}$.

Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo, in his work [5] on the theory of viscoelasticity.

DEFINITION 2.3

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t,$$

for $m - 1 < \alpha \leq m,\ m \in N,\ x > 0,\ f \in C_{-1}^m$.

The following two properties of this operator will be used in what follows.

Lemma 2.4. *If* $m-1<\alpha\leq m$ *and* $f\in C_{\mu}^{m}$, $\mu\geq -1$, *then*

$$D_{*}^{\alpha}J^{\alpha}f(x)=f(x)$$

and

$$J^{\alpha}D_{*}^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f(0^{+})\frac{x^{k}}{k!},\quad x>0.$$

The Caputo fractional derivative is considered here, because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we have considered nonlinear fractional gas dynamics equation, where the unknown function $u=u(x,t)$, assumed to be a causal function of fractional derivatives, is taken in Caputo sense as follows:

DEFINITION 2.5

For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha>0$, is defined as

$$D_{*t}^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{0}^{t}(t-\tau)^{m-\alpha-1}\frac{\partial^{m}u(x,\tau)}{\partial\tau^{m}}\mathrm{d}\tau, & m-1<\alpha<m,\\ \dfrac{\partial^{m}u(x,t)}{\partial t^{m}}, & \alpha=m\in N.\end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult refs [1–7,10].

3. Generalized two-dimensional differential transform method

DTM is an analytic method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high-order Taylor series method requires symbolic computation. However, DTM obtains a polynomial series solution by means of an iterative procedure. This method is well addressed in [8]. The proposed method is based on a combination of classical two-dimensional DTM and generalized Taylor's formula.

Consider a function of two variables $u(x,y)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,y)=f(x)g(y)$. Based on the properties of generalized two-dimensional differential transform [11–14], the function $u(x,y)$ can be represented as

$$\begin{aligned}u(x,y)&=\sum_{k=0}^{\infty}F_{\alpha}(k)(x-x_{0})^{k\alpha}\sum_{h=0}^{\infty}G_{\beta}(h)(y-y_{0})^{h\beta}\\&=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,\beta}(k,h)(x-x_{0})^{k\alpha}(y-y_{0})^{h\beta},\end{aligned}\quad(2)$$

Table 1. Basic properties of the two-dimensional differential transform.

Original function	Transformed function
$u(x,y)=g(x,y)\pm h(x,y)$	$U_{\alpha,\beta}(k,h)=G_{\alpha,\beta}(k,h)\pm H_{\alpha,\beta}(k,h)$
$u(x,y)=\lambda g(x,y)$	$U_{\alpha,\beta}(k,h)=\lambda G_{\alpha,\beta}(k,h)$
$u(x,y)=g(x,y)h(x,y)$	$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}G_{\alpha,\beta}(r,h-s)H_{\alpha,\beta}(k-r,s)$
$u(x,y)=(x-x_0)^{m\alpha}(y-y_0)^{n\beta}$	$U_{\alpha,\beta}(k,h)=\delta(k-m,h-n)=\begin{cases}1, & k=m, h=n\\ 0, & \text{otherwise}\end{cases}$
$u(x,y)=g(x,y)h(x,y)v(x,y)$	$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}G_{\alpha,\beta}(r,h-s-p)H_{\alpha,\beta}(t,s) \times V_{\alpha,\beta}(k-r-t,p)$
$u(x,y)=D^{\alpha}_{*x_0}g(x,y),\ 0<\alpha\le 1$	$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}G_{\alpha,\beta}(k+1,h)$

where $0<\alpha,\beta\le 1$, $U_{\alpha,\beta}(k,h)=F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[(D^{\alpha}_{*x_0})^{k}(D^{\beta}_{*y_0})^{h}u\,(x,y)\right]_{(x_0,y_0)}, \tag{3}$$

where

$$(D^{\alpha}_{*x_0})^{k}=\underbrace{D^{\alpha}_{*x_0}\ldots D^{\alpha}_{*x_0}}_{k}.$$

When $\alpha=1$ and $\beta=1$, the generalized two-dimensional differential transform (2) reduces to the classical two-dimensional differential transform.

Let $U_{\alpha,\beta}(k,h)$, $G_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$ and $H_{\alpha,\beta}(k,h)$ be the differential transformations of the functions $u(x,y)$, $g(x,y)$, $v(x,y)$ and $h(x,y)$. From eqs (2) and (3), some basic properties of the two-dimensional differential transform are introduced in table 1.

Then the generalized differential transform (3) becomes

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D^{\alpha k}_{*x_0}(D^{\beta}_{*y_0})^{h}u(x,y)\right]_{(x_0,y_0)}.$$

If

$$u(x,y)=D^{\gamma}_{x_0}v(x,y),\quad m-1<\gamma\le m \quad\text{and}\quad v(x,y)=f(x)f(y),$$

then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h).$$

The proofs of these properties can be found in [9,11,15].

4. Numerical example

In this section, differential transform method (DTM) will be applied for solving Fisher equation. The results reveal that the method is very effective and simple.

Example 1. Consider the following Fisher equation with the initial condition

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + u(1-u), \tag{4}$$

$$u(x,0) = \lambda. \tag{5}$$

Taking the differential transform of (4), leads to

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1) - (k+1)(k+2)U_{\alpha,1}(k+2,h) - U_{\alpha,1}(k,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r,s) = 0.$$

From the initial condition given by eq. (5), we obtain

$$U_{\alpha,1}(k,0) = \begin{cases} \lambda, & k=0, \\ 0, & k=1,2,\dots. \end{cases}$$

Substituting all $U(k,h)$ into eq. (2), the series solution form will be obtained

$$u(x,t) = \lambda\left(1 + (1-\lambda)\frac{t^\alpha}{\Gamma(\alpha+1)} + (1-\lambda)(1-2\lambda)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right).$$

As $\alpha = 1$, this series has the closed form $\lambda e^t/(1-\lambda+\lambda e^t)$, which is an exact solution of the classical Fisher equation (figures 1 and 2).

Example 2. In this example, consider the inhomogeneous fractional Fisher equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + 6u(1-u), \tag{6}$$

with initial condition,

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{7}$$

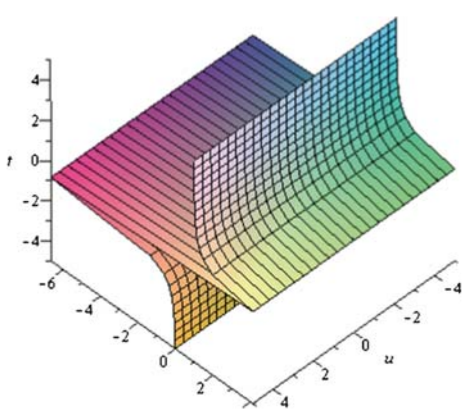

Figure 1. Plots of solution of Example 1, for $\lambda = 2$, $\alpha = 1$.

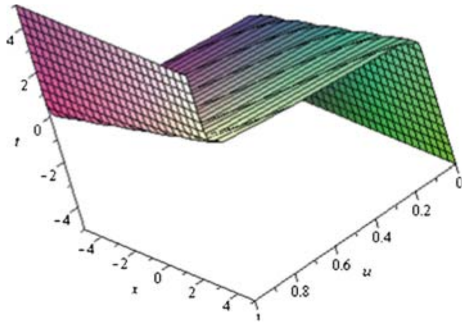

Figure 2. Plots of solution of Example 2, for $\alpha = 1$.

One can readily find the differential transform of (6), as follows:

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1) - (k+1)(k+2)U_{\alpha,1}(k+2,h)$$
$$-6U_{\alpha,1}(k,h) + 6\sum_{r=0}^{k}\sum_{s=0}^{h} U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r,s) = 0.$$

For the special case $\alpha = 1$, the solution will be as follows:

$$u(x,t) = \frac{1}{(1+\mathrm{e}^{x-5t})^2},$$

which is an exact solution.

5. Conclusion

In this paper, application of DTM to fractional Fisher equation has been presented successfully. The results show that differential transform method is a powerful and efficient technique for finding analytical solutions for nonlinear partial differential equations of fractional order. The lesser computation time required for this method compared to the time required in Adomian decomposition method (ADM) and the rapid convergence shows that the method is reliable. DTM also gives a significant improvement in solving partial differential equations over existing methods. In comparison with ADM, the main advantage of DTM is that this method provides solution of the problem without calculating Adomian's polynomials. In this paper, we used the Maple Package, to calculate the series obtained by differential transform method.

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