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Reduction of the Bethe–Salpeter wave function: Fermion–scalar case and scalar–scalar case

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Abstract. In this paper, the general forms of the nonrelativistic Bethe–Salpeter wave functions for fermion–scalar bound state and scalar–scalar bound state are presented. Using the obtained normalization conditions and the corresponding Schrödinger equations for these bound states, the nonrelativistic Bethe–Salpeter wave functions can be calculated and can be used to compute the amplitude for the process involving these bound states.

Keywords. Bethe–Salpeter wave function; nonrelativistic reduction; fermion–scalar; scalar–scalar.

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1. Introduction

Supersymmetry theories predict the existence of massive scalar quarks and scalar leptons, and there are possibilities for these scalar particles to form bound states with ordinary particles or with themselves, such as $c\bar{\tilde{b}}$, $\tilde{b}\bar{\tilde{b}}$, $\tilde{t}\bar{\tilde{t}}$ [1,2]. For example, the light stop, which is generally expected in the minimal supersymmetric Standard Model, will be stable enough to form bound states provided that it has no flavour-preserving two-body decays. A light stop is very heavy and its mass can be between 200 and 400 GeV [3]. Therefore, the stop as a constituent of a bound state will be nonrelativistic. When decays or productions of a bound state of this kind are considered, the wave function, often the Bethe–Salpeter wave function [2,4–7], of the bound state is needed, with which the amplitude of the process involving bound state can be calculated.

In ref. [8], we have presented the general form of the nonrelativistic Bethe–Salpeter wave function for the bound state comprised of two fermions, with which the amplitude for the process involving bound state can be calculated. But the cases of bound states composed of one fermion and one scalar particle or of two scalar constituents are neglected. In this paper, we use the procedure applied in ref. [8] to obtain the general form of reduction of the Bethe–Salpeter wave functions for fermion–scalar bound state and scalar–scalar bound state.

The paper is organized as follows. The general form of reduction of the Bethe–Salpeter wave function for the case of fermion–scalar bound state is presented in §2, and the case of scalar–scalar bound state is discussed in §3. The conclusion is in §4.

2. Fermion–scalar case

2.1 *Reduction of the Bethe–Salpeter equation to the Schrödinger equation*

The Bethe–Salpeter wave function [5,9] for the bound state composed of one fermion and one scalar constituent is defined as

$$\chi_P(x_1, x_2) = \langle 0|T\, Q(x_1)\bar{S}(x_2)|P\rangle, \tag{1}$$

where $P^2 = M^2$, M is the mass of the bound state. T denotes the time ordering, Q represents a fermion field and $\bar{S}$ represents an antiscalar field, while other indices are suppressed.

Due to translational invariance the Fourier transformation can be written as

$$\chi_P(x_1, x_2) = \mathrm{e}^{-iPX}\chi_P(x), \qquad \chi_P(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{-ipx}\,\chi_P(p), \tag{2}$$

where

$$\begin{aligned} x &= x_1 - x_2, \quad X = \eta_1 x_1 + \eta_2 x_2, \quad \eta_1 + \eta_2 = 1, \\ p &= \eta_2 p_1 - \eta_1 p_2, \quad P = p_1 + p_2, \quad \eta_i = \frac{m_i}{m_1 + m_2}. \end{aligned} \tag{3}$$

The Bethe–Salpeter equation [5,9] for the bound state consisting of one fermion and one scalar constituent reads as

$$\chi_P(p) = S^F(p_1)\int \frac{\mathrm{d}^4 p'}{(2\pi)^4}\,K(P, p, p')\,\chi_P(p')\Delta^F(-p_2), \tag{4}$$

where $S_1^F(p_1)$ is the full fermion propagator and $\Delta^F(-p_2)$ is the full scalar particle propagator. We shall approximate the full propagators $S^F(p_1)$ and $\Delta^F(-p_2)$ by the free propagators

$$S(p_1) = \frac{i}{\not{p}_1 - m_1 + i\epsilon}, \qquad \Delta(-p_2) = \frac{i}{p_2^2 - m_2^2 + i\epsilon}, \tag{5}$$

where m_1 and m_2 are interpreted as effective masses for the fermion and the scalar particle. In hadron physics, this approximation has been criticized because free propagators might be incompatible with a confining kernel. On the other hand, one can argue that this choice naturally leads to nonrelativistic potential models that have been applied successfully to heavy quarkonia. The normalization condition for the Bethe–Salpeter amplitude reads as

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{d}^4 p'}{(2\pi)^4}\mathrm{Tr}\left[\bar{\chi}(p)P^{\mu}\frac{\mathrm{d}}{\mathrm{d}P^{\mu}}(I(P, p, p') - K(P, p, p'))\chi(p')\right] = 2iM^2, \tag{6}$$

where

$$\bar{\chi}_P(p) = \chi_P^\dagger(p)\gamma^0$$

and

$$I(P, p, p') = (2\pi)^4\delta^4(p - p')S^{F^{-1}}(p_1)\Delta^{F^{-1}}(-p_2). \tag{7}$$

We introduce components of the relative momentum $p = p_\parallel + p_\perp$, parallel and perpendicular to the meson momentum P by [6]

$$\hat{P} = \frac{P}{M}, \quad M = \sqrt{P^2}, \quad p_l = p \cdot \hat{P},$$
$$p = p_\parallel + p_\perp, \quad p_\parallel = p_l\hat{P}, \quad p_\perp = p - p_l\hat{P}, \quad \mathrm{d}^4p = \mathrm{d}p_l\mathrm{d}^3p_\perp, \tag{8}$$

where $p_\parallel$ is the longitudinal part and $p_\perp$ is its transverse part. In the rest frame of the bound state with momentum $P = (M, \mathbf{0})$, $p_l = p^0$, $p_\parallel = (p^0, \mathbf{0})$ and $p_\perp = (0, \mathbf{p})$. The projection operators can be written in covariant form,

$$\Lambda^\pm(p_\perp) = \frac{\omega_1 \pm H(p_\perp)}{2\omega_1}, \quad \omega_i = \sqrt{m_i^2 - p_\perp^2},$$
$$H(p_\perp) = \hat{\not{P}}(m_1 - \not{p}_\perp). \tag{9}$$

In this paper, the covariant instantaneous approximation is employed [7], in which the approximated kernel is independent of the change of the longitudinal component of the relative momentum,

$$K(P, p, p') \to K(p_\perp, p'_\perp) = -iV(p_\perp, p'_\perp). \tag{10}$$

It is a good approximation for a system composed of heavy and light constituents or two heavy constituents which can move relativistically as a whole. It will reduce to instantaneous approximation in the rest frame of the bound state.

Introduce the notation for later convenience

$$\psi_P(p_\perp) = \int \frac{\mathrm{d}p_l}{2\pi}\chi_P(p), \qquad \Gamma(p_\perp) = \int \frac{\mathrm{d}^3p'_\perp}{(2\pi)^3}V(p_\perp, p'_\perp)\psi_P(p'_\perp), \tag{11}$$

where $\psi_P(p_\perp)$ is the Salpeter wave function. Using eqs (5), (9) and (10), the Bethe–Salpeter equation (4) becomes

$$\chi(p) = \left[\frac{\Lambda^+(p_\perp)}{\eta_1 M + p_l - \omega_1 + i\epsilon} + \frac{\Lambda^-(p_\perp)}{\eta_1 M + p_l + \omega_1 - i\epsilon}\right]\hat{\not{P}}(-i)\Gamma(p_\perp)$$
$$\times\frac{1}{2\omega_2}\left[\frac{1}{\eta_2 M - p_l + \omega_2 - i\epsilon} - \frac{1}{\eta_2 M - p_l - \omega_2 + i\epsilon}\right]. \tag{12}$$

Performing the p_l integrals in the Bethe–Salpeter equation (12) yields the Salpeter equation

$$\psi_P(p_\perp) = \frac{\Lambda^+(p_\perp)\ \hat{\not{P}}\Gamma(p_\perp)}{2\omega_2(M - \omega_1 - \omega_2)} + \frac{\Lambda^-(p_\perp)\ \hat{\not{P}}\Gamma(p_\perp)}{2\omega_2(M + \omega_1 + \omega_2)}. \tag{13}$$

Applying the energy projectors $\Lambda^\pm(p_\perp)$ from the left-hand side to the Salpeter equation (13) leads to

$$(M - \omega_1 - \omega_2)\psi_P^+(p_\perp) = \frac{1}{2\omega_2}\Lambda^+(p_\perp)\ \hat{\not{P}}\Gamma(p_\perp),$$
$$(M + \omega_1 + \omega_2)\psi_P^-(p_\perp) = \frac{1}{2\omega_2}\Lambda^-(p_\perp)\ \hat{\not{P}}\Gamma(p_\perp), \tag{14}$$

where $\psi_P^{\pm}(p_\perp) = \Lambda^{\pm}(p_\perp)\psi_P(p_\perp)$. The normalization condition for the Salpeter wave function (6) reduces to

$$\int \frac{\mathrm{d}^3 p_\perp}{(2\pi)^3}\omega_2 \mathrm{Tr}\left\{\bar{\psi}(p_\perp)\ \hat{P}\Lambda^{+}(p_\perp)\psi(p_\perp) + \bar{\psi}(p_\perp)\hat{P}\Lambda^{-}(p_\perp)\psi(p_\perp)\right\} = M. \tag{15}$$

For weakly bound states with $M \approx m_1 + m_2$, one has [10]

$$M - \omega_1 - \omega_2 \ll M + \omega_1 + \omega_2. \tag{16}$$

Therefore, the second term on the right side of the Salpeter equation (13) can be dropped. This leads to the reduced Salpeter equation,

$$(M - \omega_1 - \omega_2)\psi_P^{+}(p_\perp) = \frac{1}{2\omega_2}\Lambda^{+}(p_\perp)\ \hat{P}\Gamma(p_\perp). \tag{17}$$

The reduction of the Salpeter equation (13) to the reduced Salpeter equation (17) may be affected by imposing a further constraint

$$\Lambda^{-}(p_\perp)\psi_P(p_\perp) = 0. \tag{18}$$

The normalization condition (15) then reduces to

$$\int \frac{\mathrm{d}^3 p_\perp}{(2\pi)^3}\omega_2 \mathrm{Tr}\left\{\bar{\psi}(p_\perp)\ \hat{P}\Lambda^{+}(p_\perp)\psi(p_\perp)\right\} = M. \tag{19}$$

By neglecting all the negative-energy contributions as well as the spin degrees of freedom of the constituents, which may be justified for semirelativistic and weakly bound heavy constituents, the spinless Salpeter equation can be obtained from eq. (17)

$$(M - \omega_1 - \omega_2)\phi(p_\perp) = \frac{1}{2\omega_2}\int \frac{\mathrm{d}^3 p'_\perp}{(2\pi)^3} V(p_\perp, p'_\perp)\phi(p'_\perp), \tag{20}$$

which is obtained by simplifying the Bethe–Salpeter equation greatly, and can also be regarded as a relativistic generalization of the Schrödinger equation. By expanding ω_i as

$$\omega_i = m_i - \frac{p_\perp^2}{2m_i} + \cdots, \tag{21}$$

eq. (20) reduces to the Schrödinger equation [6,9,11,12]

$$\left(\epsilon + \frac{p_\perp^2}{2\mu}\right)\phi(p_\perp) = \frac{1}{2m_2}\int \frac{\mathrm{d}^3 p'_\perp}{(2\pi)^3} V(p_\perp, p'_\perp)\phi(p'_\perp), \tag{22}$$

where $\epsilon = M - m_1 - m_2$, $\mu = m_1 m_2/(m_1 + m_2)$.

2.2 *Reduction of the Bethe–Salpeter wave function*

As discussed in §2.1, by discarding the negative part, we can obtain from eq. (12)

$$\chi_P(p) = \frac{\Lambda^{+}(p_\perp)\ \not{P} i\Gamma(p_\perp)}{2\omega_2(\eta_1 M + p_l - \omega_1 + i\epsilon)(\eta_2 M - p_l - \omega_2 + i\epsilon)}. \tag{23}$$

Then using eq. (17), the above equation becomes

$$\chi_P(p) = \frac{\Lambda^+(p_\perp)(M - \omega_1 - \omega_2) i\psi(p_\perp)}{(\eta_1 M + p_l - \omega_1 + i\epsilon)(\eta_2 M - p_l - \omega_2 + i\epsilon)}. \tag{24}$$

The corresponding normalization condition is eq. (19).

2.3 *General form of the nonrelativistic Bethe–Salpeter wave function*

The general form of the Salpeter wave function reads for state with parity $\eta_P = (-1)^{j-1/2}$ [9] as

$$\psi^{i=j-1/2}(p_\perp) = (g_1 + \not{p}_\perp g_2)\hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P), \tag{25}$$

and, for state with parity $\eta_P = (-1)^{j+1/2}$,

$$\psi^{i=j+1/2}(p_\perp) = (g_1 + \not{p}_\perp g_2)\gamma^5 \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P), \tag{26}$$

where $\hat{p}_\perp^\mu = p_\perp^\mu/\omega_0$, $\omega_0 = \sqrt{-p_\perp^2}$, $g_i \equiv g_i(\omega_0)$. $U^{\mu_1\mu_2\cdots\mu_i}(P)$ is the tensor-spinor (see Appendix for details). From eqs (25) and (26), it is found that the states for the scalar–fermion bound state should be ${}^{2s+1}(L_j)^{\eta_P}$ with spin j and parity $\eta_P = (-1)^{j-1/2}$ or $(-1)^{j+1/2}$, i.e., ${}^2(S_{1/2})^+$, ${}^2(P_{1/2})^-$, ${}^2(P_{3/2})^-$, ${}^2(D_{3/2})^+, \ldots$.

Using eqs (9) and (A.2), the Salpeter wave functions (25) and (26) can be rewritten in the following form:

$$\psi^{i=j-1/2}(p_\perp) = \left[\Lambda^+(p_\perp) f_1 + \Lambda^-(p_\perp) f_2\right] \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P) \tag{27}$$

and

$$\psi^{i=j+1/2}(p_\perp) = \left[\Lambda^+(p_\perp) f_1 + \Lambda^-(p_\perp) f_2\right] \gamma^5 \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P), \tag{28}$$

where

$$g_1 = \frac{\omega_1 + m_1}{2\omega_1} f_1 + \frac{\omega_1 - m_1}{2\omega_1} f_2, \qquad g_2 = \frac{1}{2\omega_1}(f_2 - f_1). \tag{29}$$

The constraints (18) will reduce eqs (27) and (28) respectively to

$$\psi^{i=j-1/2}(p_\perp) = \Lambda^+(p_\perp)\hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P) f,$$
$$\psi^{i=j+1/2}(p_\perp) = \Lambda^+(p_\perp)\gamma^5 \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_i} U_{\mu_1\mu_2\ldots\mu_i}(P) f. \tag{30}$$

In the rest frame of the bound state, the normalization condition for (30) reads as

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i+1}{2i+3} S_i f^2 = M, \tag{31}$$

where $S_i = \mathbb{P}^{\mu\mu_1\cdots\mu_i\nu\nu_1\cdots\nu_i} g^{\mu\nu} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\nu_1} \cdots$.

The potential corresponds to the Bethe–Salpeter interaction kernel, see eq. (10), which is defined as the sum of all two-particle irreducible graphs. In the fermion–fermion case, the interaction between constituents usually is vector, for example, the one-photon-exchange interaction for atom and the perturbative one-gluon-exchange interaction plus phenomenological nonperturbative part for hadrons. In the scalar–fermion case, the exchange could be bosonic or fermionic, and the latter will be small. For simplicity, the potential takes the form $V = 1 \otimes 1V_s + \gamma_\mu \otimes q^\mu V_v$ as an example, and other forms can be treated similarly. The nonrelativistic equation reads as

$$\left(\epsilon - \frac{\mathbf{p}^2}{2\mu}\right) S_i f(|\mathbf{p}|) = \int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} \left[V_s + V_v q_0\right] T_i f(|\mathbf{p}'|), \tag{32}$$

where $T_i = \mathbb{P}^{\mu\mu_1\cdots\mu_i\nu\nu_1\cdots\nu_i} g^{\mu\nu} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\prime\nu_1} \cdots$. From eqs (30)–(32), it is obvious that in the nonrelativistic limit there are degenerate doublets with the same parity $(-1)^i$ and angular momentum L and different spin $j = i - 1/2$, $j = i + 1/2$.

The Schrödinger equation (32) together with the normalization condition (31) determine the wave function. Substituting eqs (27), (28) and (30) into (24) yields the nonrelativistic Bethe–Salpeter wave function.

3. Scalar–scalar case

3.1 *Reduction of the Bethe–Salpeter equation to the Schrödinger equation*

The Bethe–Salpeter wave function for the bound state composed of two scalar constituents is defined by

$$\chi_P(x_1, x_2) = \langle 0|T S_1(x_1)\bar{S}_2(x_2)|P\rangle, \tag{33}$$

where S_1 is a scalar field, $\bar{S}_2$ is an antiscalar field and other indices are suppressed. The Bethe–Salpeter equation [5] for the bound state comprising two scalar constituents reads as

$$\chi_P(p) = \Delta_1^F(p_1) \int \frac{\mathrm{d}^4 p'}{(2\pi)^4} K(P, p, p')\, \chi_P(p')\Delta_2^F(-p_2), \tag{34}$$

where $\Delta_i^F(p_i)$ are the full scalar particle propagators. We shall approximate the full propagators by free propagators

$$\Delta_i(p_i) = \frac{i}{p_i^2 - m_i^2 + i\epsilon}. \tag{35}$$

The normalization condition for the Bethe–Salpeter wave function (33) reads as

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 p'}{(2\pi)^4} \mathrm{Tr}\left[\bar{\chi}(p) P^\mu \frac{\mathrm{d}}{\mathrm{d}P^\mu}(I(P, p, p') - K(P, p, p'))\chi(p')\right] = 2iM^2, \tag{36}$$

where $\bar{\chi}_P(p) = \chi_P^\dagger(p)$ and

$$I(P, p, p') = (2\pi)^4 \delta^4(p - p')\Delta_1^{F^{-1}}(p_1)\Delta_2^{F^{-1}}(-p_2). \tag{37}$$

Rewrite the Bethe–Salpeter equation (34) as

$$\chi(p) = \frac{1}{2\omega_1}\left[\frac{1}{\eta_1 M + p_l - \omega_1 + i\epsilon} - \frac{1}{\eta_1 M + p_l + \omega_1 - i\epsilon}\right](-i)\Gamma(p_\perp) \times \frac{1}{2\omega_2}\left[\frac{1}{\eta_2 M - p_l + \omega_2 - i\epsilon} - \frac{1}{\eta_2 M - p_l - \omega_2 + i\epsilon}\right]. \tag{38}$$

Performing the p_l integrals in the Bethe–Salpeter equation (12) yields the Salpeter equation

$$[M^2 - (\omega_1 + \omega_2)^2]\psi_P(p_\perp) = \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\Gamma(p_\perp), \tag{39}$$

where

$$\Gamma(p_\perp) = \int \frac{d^3 p'_\perp}{(2\pi)^3} V(p_\perp, p'_\perp)\psi_P(p'_\perp), \quad K(p_\perp, p'_\perp) = -iV(p_\perp, p'_\perp) \tag{40}$$

and the normalization condition (36) reduces to

$$\int \frac{d^3 p_\perp}{(2\pi)^3} \frac{2\omega_1\omega_2}{\omega_1 + \omega_2} \bar{\psi}(p_\perp)\psi(p_\perp) = 1. \tag{41}$$

Neglecting all the negative-energy contributions, from eq. (39) we obtain

$$(M - \omega_1 - \omega_2)\psi_P(p_\perp) = \frac{1}{4\omega_1\omega_2}\Gamma(p_\perp), \tag{42}$$

with the normalization condition

$$\int \frac{d^3 p_\perp}{(2\pi)^3} 2\omega_1\omega_2 \bar{\psi}(p_\perp)\psi(p_\perp) = M. \tag{43}$$

3.2 *Reduction of the Bethe–Salpeter wave function*

By discarding the negative part, from eq. (38), we obtain

$$\chi(p) = \frac{1}{4\omega_1\omega_2} \frac{1}{(\eta_1 M + p_l - \omega_1 + i\epsilon)(\eta_2 M - p_l - \omega_2 + i\epsilon)} i\Gamma(p_\perp). \tag{44}$$

Then using eq. (42), the above equation becomes

$$\chi_P(p) = \frac{(M - \omega_1 - \omega_2) i\psi(p_\perp)}{(\eta_1 M + p_l - \omega_1 + i\epsilon)(\eta_2 M - p_l - \omega_2 + i\epsilon)} \tag{45}$$

and the corresponding normalization condition is eq. (43).

3.3 *General form of the nonrelativistic Bethe–Salpeter wave function*

The general form of the Salpeter wave function for state with parity $\eta_P = (-1)^j$ [9] is

$$\psi^j(p_\perp) = \epsilon_{\mu_1 \cdots \mu_j} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_j} f(p_\perp^2), \tag{46}$$

where $\epsilon_{\mu_1 \cdots \mu_j}$ is the polarization tensor which is totally symmetric, traceless and transverse,

$$\epsilon_{\mu_1\mu_2\cdots\mu_j} = \epsilon_{\mu_2\mu_1\cdots\mu_j}, \qquad \epsilon^{\nu}_{\nu\mu_2\cdots} = 0, \qquad P^{\mu_1}\epsilon_{\mu_1\cdots\mu_j} = 0. \tag{47}$$

For the bound states composed of one scalar constituent and one antiscalar constituent, there exists only $L_j^{\eta_P}$ state with spin $j = L$ arising from the relative angular momentum of components in the bound state, and parity $\eta_P = (-1)^L$, i.e., S_0^+, P_1^-, ….

In the rest frame of the bound state, if the potential takes the form $V = 1 \otimes 1 V_s + q_\mu \otimes q^\mu V_v$, the nonrelativistic equation reads as

$$\left(\epsilon - \frac{\mathbf{p}^2}{2\mu}\right) S'_j f(|\mathbf{p}|) = \int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} \left[V_s + V_v q^2\right] T'_j f(|\mathbf{p}'|), \tag{48}$$

where

$$\begin{aligned} S'_j &= \mathbb{P}^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\nu_1} \cdots , \\ T_j &= \mathbb{P}^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\prime\nu_1} \cdots . \end{aligned} \tag{49}$$

The normalization condition for (46) reads as

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} 2\omega_1\omega_2 S'_j f^2 = M. \tag{50}$$

The Schrödinger equation (48) together with the normalization condition (50) determine the wave function. Substituting eq. (46) into (45) gives the nonrelativistic Bethe–Salpeter wave function.

4. Conclusion

In this paper, we have obtained the general form of the nonrelativistic Bethe–Salpeter wave functions for bound states of arbitrary spin and definite parity which are composed of one fermion and one scalar constituent or of two scalar constituents. Using the obtained normalization conditions and the corresponding Schrödinger equations, the nonrelativistic Bethe–Salpeter wave functions can be calculated.

The general formalism obtained can be applied to investigate various topics involving heavy bound states, for which the nonrelativistic bound-state picture can be adopted. For example, when decays or productions of a bound state comprising one fermion and one scalar constituent or of two scalar constituents are considered, the wave function is needed to compute the corresponding amplitudes of the concerning processes.

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Appendix A. Formulas

For completeness and convenience we list some useful formulas for tensor-spinor. The tensor-spinor [13] $U^{\mu_1\mu_2\cdots\mu_i}(P)$ is totally symmetric, traceless, and transverse,

$$\begin{aligned} &U^{\mu_1\mu_2\cdots\mu_i}(P) = U^{\mu_2\mu_1\cdots\mu_i}(P), \qquad U^{\mu_1\mu_2\cdots\mu_i}_{\mu_1}(P) = 0, \\ &P_{\mu_1} U^{\mu_1\mu_2\cdots\mu_i}(P) = 0. \end{aligned} \tag{A.1}$$

The tensor–spinor $U^{\mu_1\mu_2\cdots\mu_i}(P)$ has other properties also: obeying the Dirac equation and satisfying the spinor trace condition,

$$(\not{P} - M) U^{\mu_1\mu_2\cdots\mu_i}(P) = 0, \qquad \gamma_{\mu_1} U^{\mu_1\mu_2\cdots\mu_i}(P) = 0. \tag{A.2}$$

For half-odd spin j, there is the relation [14]

$$\sum_{j_m} U^{\mu_1\mu_2\ldots\mu_{j-\frac{1}{2}}} \bar{U}^{\nu_1\nu_2\ldots\nu_{j-\frac{1}{2}}} = \Lambda_P^{\pm} \mathbb{P}^{\mu_1\cdots\mu_{j-\frac{1}{2}}\nu_1\cdots\nu_{j-\frac{1}{2}}}(j), \tag{A.3}$$

where $\Lambda_P^{\pm}$ are the energy projectors for the bound state, and

$$\mathbb{P}^{\mu_1\cdots\mu_{j-\frac{1}{2}}\nu_1\cdots\nu_{j-\frac{1}{2}}}(j) = \frac{j+\frac{1}{2}}{2j+2}\gamma_\mu\gamma_\nu \mathbb{P}^{\mu\mu_1\cdots\mu_{j-\frac{1}{2}}\nu\nu_1\cdots\nu_{j-\frac{1}{2}}}\left(j+\frac{1}{2}\right). \tag{A.4}$$

For integer spin j, there is the expression [14]

$$\begin{aligned}
&\mathbb{P}^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j}(j,P) = \sum_{j_z} \epsilon^{*\mu_1\cdots\mu_j}\epsilon^{\nu_1\cdots\nu_j} \\
&= \left(\frac{1}{j!}\right)^2 \sum_{\substack{P(\mu)\\P(\nu)}} \left[\prod_{i=1}^{j}\mathbb{P}^{\mu_i\nu_i} + a_1^j \mathbb{P}^{\mu_1\mu_2}\mathbb{P}^{\nu_1\nu_2}\prod_{i=3}^{j}\mathbb{P}^{\mu_i\nu_i} + \cdots\right. \\
&\quad + a_r^j \mathbb{P}^{\mu_1\mu_2}\mathbb{P}^{\nu_1\nu_2}\ldots\mathbb{P}^{\mu_{2r-1}\mu_{2r}}\mathbb{P}^{\nu_{2r-1}\nu_{2r}}\prod_{i=2r+1}^{j}\mathbb{P}^{\mu_i\nu_i} + \cdots \\
&\quad \left. + \begin{cases} a_{j/2}^j \mathbb{P}^{\mu_1\mu_2}\mathbb{P}^{\nu_1\nu_2}\ldots\mathbb{P}^{\mu_{j-1}\mu_j}\mathbb{P}^{\nu_{j-1}\nu_j}, & \text{for even j} \\ a_{(j-1)/2}^j \mathbb{P}^{\mu_1\mu_2}\mathbb{P}^{\nu_1\nu_2}\ldots\mathbb{P}^{\mu_{j-2}\mu_{j-1}}\mathbb{P}^{\nu_{j-2}\nu_{j-1}}\mathbb{P}^{\mu_j\nu_j}, & \text{for odd j} \end{cases}\right],
\end{aligned} \tag{A.5}$$

where the sum is over all permutations of μ and ν,

$$\mathbb{P}^{\mu\nu} \equiv -g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2} \tag{A.6}$$

and

$$a_r^j = \left(-\frac{1}{2}\right)^r \frac{j!(2j-2r-1)!!}{r!(j-2r)!(2j-1)!!}. \tag{A.7}$$

In the above equation, $n!$ gives the factorial of n, $n! = n(n-1)\cdots$, and $n!!$ gives the double factorial of n, $n!! = n(n-2)\ldots$.

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