PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 86, No. 4
April 2016
pp. 747–761

Deriving relativistic Bohmian quantum potential using variational method and conformal transformations

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MS received 26 October 2014; revised 20 January 2015; accepted 23 February 2015
DOI: 10.1007/s12043-015-1076-7; ***e*Publication:** 18 November 2015

Abstract. In this paper we shall argue that conformal transformations give some new aspects to a metric and changes the physics that arises from the classical metric. It is equivalent to adding a new potential to relativistic Hamilton–Jacobi equation. We start by using conformal transformations on a metric and obtain modified geodesics. Then, we try to show that extra terms in the modified geodesics are indications of a background force. We obtain this potential by using variational method. Then, we see that this background potential is the same as the Bohmian non-local quantum potential. This approach gives a method stronger than Bohm's original method in deriving Bohmian quantum potential. We do not use any quantum mechanical postulates in this approach.

Keywords. Bohmian quantum mechanics; quantum potential; non-locality; conformal transformations.

PACS Nos 03.65.Ta; 03.65.Ca; 45.20.Jj; 03.65.Ge; 03.65.Pm

1. Introduction

According to the general theory of relativity, the geometry of space-time determines the dynamics of a particle. Although other forces are not unified with gravitation, this is not a problem to dissuade us from using general relativity and its strong framework. Quantum phenomena are somewhat strange for us because we do not know how they take place. This statement may be meaningless in the framework of the standard quantum mechanics, because, in this formalism, there is an intrinsic uncertainty in the position and momentum of a particle, which causes particle to behave quantum mechanically in a stochastic manner. We can only talk about the probability of detecting a particle in a special region at a specific time or with a special eigenvalue of energy or other quantities of a particle. In Bohmian quantum mechanics, we talk about a quantum potential which gives quantum behaviour to a particle in a causal manner, but the origin of this potential is undetected yet. Now, we want to look for the quantum mechanical behaviour of a particle in the geometry of space-time. Our classical (relativistic) world has a special metric and influences particles in a deterministic manner, and we say that particles behave classically. As the classical metric cannot cause quantum behaviours, we should change the metric. Thus, we need special transformations unlike coordinate transformations which change the metric to a new metric with new aspects. Using this argument, we consider conformal transformations which are capable of producing new effects.

In the following, we explain Bohmian quantum mechanics briefly and then argue about the criticisms against it and demonstrate that these criticisms can be removed using a variationl approach. In §2, we review non-relativistic and relativistic Bohmian mechanics briefly. In §3, we explain the relation between background potential and the extra terms in the modified geodesics equation, which result from conformal transformations. Then, in §4 we exert a conformal transformation on relativistic classical Hamilton–Jacobi to attribute a background potential to these extra terms and to obtain a formula for this potential via variational method, without using quantum mechanical postulates. In §5, we consider the energy of a relativistic particle moving on the modified geodesics, and then investigate its non-relativistic limit, without using the Schrödinger equation. In this paper, by classical mechanics, we mean the relativistic classical mechanics, not the Newtonian mechanics.

2. A brief review of Bohmian mechanics

2.1 *Non-relativistic Bohmian mechanics*

Bohm considered a guiding field $\psi(\mathbf{x}, t)$ which guides a particle in a complex manner and postulated that the dynamical properties of the particle are well-defined and real, like in classical mechanics, and they are obtained from a quantum wave function. In this theory, evolution of a system is obtained from Schrödinger's equation, and its position and momentum are determined from the wave function. By writing the wave function in a polar form

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) \exp\left(i \frac{S(\mathbf{x}, t)}{\hbar}\right)$$

and substituting it into Schrödinger's equation, we obtain the generalized Hamilton–Jacobi and continuity equations:

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}) + Q(\mathbf{x}) = 0 \tag{1}$$

$$\frac{\partial R^2}{\partial t} + \frac{1}{m}\nabla \cdot (R^2 \nabla S) = 0. \tag{2}$$

The position of the particle is obtained from the following equation:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \left(\frac{\nabla S(\mathbf{x}, t)}{m}\right)_{\mathbf{X}=\mathbf{x}(t)}, \tag{3}$$

where $\nabla S(\mathbf{x}, t)$ is the momentum of the particle. By knowing the initial position $\mathbf{x}$ and wave function $\psi(\mathbf{x}, t)$, the future of the system is obtained. The expression $\mathbf{X} = \mathbf{x}(t)$ means that among all possible trajectories, in an ensemble of particles, we choose one of them. The probabilistic character of Bohmian mechanics is a secondary aspect of the guiding wave function and is not an intrinsic property, as is in Born's statistical interpretation of the wave function. This probabilistic aspect is due to the effects of the environment on the system, and it does not allow us to know the initial values with infinite precision [1,2].

The quantity Q in (1) is called quantum potential and is given by

$$Q = -\frac{\hbar^2 \nabla^2 R(\mathbf{x}, t)}{2mR(\mathbf{x}, t)}. \tag{4}$$

If we multiply R by a real constant α, the quantum potential does not alter. This is an essential property which implies that this potential does not decrease with distance necessarily, and it depends on the form of the wave function, rather than its amplitude; and its form depends on the boundary conditions. This property gives quantum mechanical behaviour to a particle. Physicist usually use the word 'non-locality' in many particle processes. In Bohm's deterministic quantum mechanics, there is a sort of non-locality in the one-particle case which was first pointed out by Bohr, but he used the world wholeness instead of non-locality [2–4]. This means that all boundary conditions affect the motion of a particle through a potential which resembles Bohmian quantum potential. There are some articles which show that in the one-particle case too, non-locality is significant [5]. In some of his famous statements, Bohm likens this property with the information carried by a radar [2]. So, in this paper, we use the word 'non-locality' in a more general sense than is used for the case of many-particle systems.

Some researchers prefer to develop deterministic quantum mechanics without using the concept of quantum potential and they simply work with the guidance equation (3) (see refs [6–8]). This is because they believe that with the initial wave equation and position, and by using guidance equation, the evolution of the system is predictable, and quantum potential is an extra non-essential concept. Furthermore, as this term, in the Hamilton–Jacobi equation, is obtained by writing the wave function in a polar form, it has no originality. We shall demonstrate that this criticism is removed by variational method.

Non-locality in many-particle systems is obvious, because by writing:

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_N, t) = R(\mathbf{x}_1, \ldots \mathbf{x}_N, t)\mathrm{e}^{(i/\hbar)S(\mathbf{x}_1,\ldots,\mathbf{x}_N,t)} \tag{5}$$

and substituing it in the Schrödinger equation, we get

$$Q = \sum_{i=1}^{N} -\frac{\hbar^2 \nabla_i^2 R}{2m_i R} \tag{6}$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \left(\frac{\nabla_i S(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)}{m_i}\right)_{\mathbf{X}_j = \mathbf{x}_j(t)}. \tag{7}$$

From these equations we see that the dynamics of any particle of a system is related to that of its other particles instantaneously [1,2]. In Bohmian quantum mechanics, the non-locality is a principal characteristic which affects the evolution of the system through quantum potential [1,2]. In the standard quantum mechanics, this characteristic is not obvious because it does not have a deterministic formalism, and dynamical quantities like energy and momentum are only operators in a Hilbert space. Bohm showed that it is possible to have a quantum mechanics with real dynamical variables and with non-locality. In other words, the cost of keeping reality is the loss of locality.

According to the classical field theory and special relativity, signal transition cannot take place with velocities greater than the speed of light, i.e. locality forbids any instantaneous relation among particles. On the other hand, in our usual experiments we observe that particles affect each other by sending signals, but practically we do not detect non-locality in daily experiences. At the quantum mechanical level, however, this postulate (locality) breaks down and opens new windows for understanding our Universe.

2.2 *Relativistic quantum potential for a spinless particle*

Following Bohm, we substitute the polar form of the wave function into the Klein–Gordon equation to derive the quantum mechanical Hamilton–Jacobi equation for a relativistic spinless particle:

$$\partial_\mu S \partial^\mu S = m^2(1+Q) \tag{8}$$

$$Q = \frac{\hbar^2}{m^2}\frac{\Box R}{R}. \tag{9}$$

Equation (8) indicates that the rest mass of the particle is not a constant in the rest frame of the particle; rather, it depends on the quantum potential. This is a novel feature that is in conflict with the special relativity. The particle 4-momentum is obtained from

$$p^\mu = M u^\mu = m(\sqrt{1+Q})u^\mu = -\partial^\mu S \tag{10}$$

and the 4-current of the particle and its continuity equation are

$$j^\mu = -\frac{R^2}{m}\partial^\mu S, \quad \partial_\mu j^\mu = 0. \tag{11}$$

One problem with this method is with the expression $j^\mu j_\mu$ which is not always positive definite; thus it can cause some ambiguities about particle trajectories. The 3-velocity of a particle is obtained from

$$v^i(\mathbf{x}, t) = \frac{j^i}{j^0}, \quad i = 1, 2, 3. \tag{12}$$

Density j^0 can take positive or negative values and this causes some ambiguities about particle trajectories. Some attribute such ambiguity to the particle creation and annihilation, if we do not use a field formalism, and they attribute trajectories to relativistic particles (see [9–11]).

2.3 *Bohmian field theory*

Physicists usually use field theory to overcome the problems of relativistic quantum mechanics. In the Bohmian quantum mechanics, it is possible to consider a field as beables, a terminology which was introduced by John Bell to refer to quantum entities. In this view, the evolution of the field is described by a super wave function or a functional in a deterministic manner (for details, see [1,2,4,9,12–14]).

Fermions have no satisfactory treatment in the Bohmian quantum mechanics yet. Therefore, we do not consider fermions in this paper. For the study about fermions refs [2,6–8] are recommended.

3. Geodesics equation and conformal transformations

3.1 *Why geometry?*

From general relativity, we know that the motion of a particle is affected by space-time curvature and that curvature can be described by a metric in a Riemannian manifold. As the dynamical behaviour of a particle is determined by the space-time geometry, we expect that the quantum mechanical behaviour of a particle also is due to a special aspect of space-time geometry.

3.2 *Why conformal transformations?*

Usually, transformations in relativity are coordinate transformations, through which the properties of the metric is studied. Unlike these transformations, there is another type of transformation called conformal transformation, through which all the metric is multiplied by a factor which is in general a function of coordinates on the manifold. The conformal transformations preserve the angle between two lines on the manifold. Thus, under these transformations, infinitesimal shapes are preserved (for details, see [15,16]). As all the metric is multiplied by the point-dependent factor, we expect these transformations transform the metric to a new metric and produce some new physics, i.e., we expect that such transformations alter the dynamics of a particle in a different way. A precise demonstration of this assertion is very difficult and requires some very deep investigations. We want to highlight only those concepts that lead us to preliminary results and can be a starting point for further research in the light of this view.

A conformal transformation on a metric is defined as

$$g_{\mu\nu} \longrightarrow \mathrm{e}^{\chi(x)} g_{\mu\nu} \simeq (1+\chi(x)) g_{\mu\nu} = \Omega^2(x) g_{\mu\nu} = \tilde{g}_{\mu\nu}, \quad \Omega^2(x) > 0 \tag{13}$$

and for the inverse metric we have

$$g^{\mu\nu} \longrightarrow \Omega^{-2}(x) g^{\mu\nu}. \tag{14}$$

The determinant of the metric, on a D-dimensional manifold, changes in the form:

$$\sqrt{-g} \longrightarrow \Omega^D \sqrt{-g}. \tag{15}$$

The Christoffel symbol also changes under this transformation:

$$\Gamma^{\mu}_{\nu\lambda} \longrightarrow \Gamma^{\mu}_{\nu\lambda} + \frac{1}{\Omega}\left(\delta^{\mu}_{\nu}\partial_{\lambda}\Omega + \delta^{\mu}_{\lambda}\partial_{\nu}\Omega - g_{\nu\lambda}g^{\mu\rho}\partial_{\rho}\Omega\right). \tag{16}$$

The geodesic equation for the initial metric is

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\nu\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau} = 0 \tag{17}$$

and after the conformal transformation, the geodesic equation changes to

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \Bigg(\Gamma^{\mu}_{\nu\lambda} + \frac{1}{2}\delta^{\mu}_{\nu}\partial_{\lambda}\ln(\Omega^2) + \frac{1}{2}\delta^{\mu}_{\lambda}\partial_{\nu}\ln(\Omega^2) - \frac{1}{2}g_{\nu\lambda}g^{\mu\rho}\partial_{\rho}\ln(\Omega^2)\Bigg)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau} = 0. \tag{18}$$

Now, we should explain the extra terms in this equation and attribute physical concepts to these extra terms. Naturally, we do not detect effects due to these extra terms at a classical or large-scale level. Thus, these terms should approach zero at this level. If we take $\Omega^2 = 1$, the extra terms become zero and the conformal transformation approaches identity. This takes place at large-scale dimensions. In a Minkowskian space-time, or a flat space-time, $\Gamma^{\mu}_{\nu\lambda} = 0$ everywhere, and relation (17) reduces to the free particle equation ($\mathrm{d}^2x^{\mu}/\mathrm{d}\tau^2 = 0$). But, for eq. (18) this is not true, because the extra term due to conformal transformation is not identically zero. This means that in spite of the vanishing of $\Gamma^{\mu}_{\nu\lambda}$, there is a background force that always accompanies the particle and affects its motion. It is better to write the free-particle condition as

$$\left(\Gamma^{\mu}_{\nu\lambda} + \frac{1}{2}\delta^{\mu}_{\nu}\partial_{\lambda}\ln(\Omega^2) + \frac{1}{2}\delta^{\mu}_{\lambda}\partial_{\nu}\ln(\Omega^2) - \frac{1}{2}g_{\nu\lambda}g^{\mu\rho}\partial_{\rho}\ln(\Omega^2)\right) = 0. \tag{19}$$

As a case of non-vanishing $\Gamma^{\mu}_{\nu\lambda}$, we consider a spherical coordinate system. In such a system, the components of $\Gamma^{\mu}_{\nu\lambda}$ are related to the particle acceleration and terms due to a conformal transformation are equivalent to components of $\Gamma^{\mu}_{\nu\lambda}$. Relation (19) says that extra terms balance curvature terms. There is no absolute equilibrium except for the case of a flat space-time, where the extra terms vanish in special circumstances. But in the curved space-time, all terms are collectively zero for a free particle. From the above equations, it is obvious that the derivatives of $\ln(\Omega^2)$ is proportional to a background force, as in the mechanical sense, $\Gamma^{\mu}_{\nu\lambda}$ is proportional to force. Thus, we guess that the part of Ω^2, i.e. $\chi(x)$, which is the second term in the expansion of conformal transformation in relation (13), is proportional to a background potential.

Now, we can almost be convinced that conformal transformations can produce a new physics. In the following sections we show how this transformation affects the dynamics and energy of a relativistic spinless particle.

4. Deriving a formula for the background potential

In the classical relativistic mechanics there is a Hamilton–Jacobi equation, for a particle of rest mass m, in the form

$$\partial_{\mu}S\partial^{\mu}S = m^2 \tag{20}$$

with a continuity equation:

$$\partial_\mu(\rho\partial^\mu S) = 0. \tag{21}$$

In a hydrodynamical approach, we can take a Lagrangian density in the following form:

$$\mathcal{L} = \rho(\partial_\mu S\partial^\mu S - m^2), \tag{22}$$

where $\rho(x)$ is the density of continuous fluid and S is the Hamilton's principal function for this distribution. By using variational method for the following action:

$$\mathcal{A} = \int \rho(\partial_\mu S\partial^\mu S - m^2)\mathrm{d}^4x \tag{23}$$

eqs (20) and (21) are obtained.

We expect the presence of a background field to be equivalent to a conformal transformation on the metric, which modifies relation (20) to the form:

$$\Omega^{-2}g^{\mu\nu}\partial_\nu S\partial_\mu S = m^2 \tag{24}$$

or

$$\partial_\mu S\partial^\mu S = m^2\Omega^2 \tag{25}$$

or

$$\partial_\mu S\partial^\mu S = m^2(1 + \chi(x)). \tag{26}$$

The actual agent of this force is unknown to us and we cannot, at present, express it as a function of the space-time coordinates. We deduce that the background potential can be an appropriate function of matter distribution ρ and its derivatives because, for big-scale dimensions (relativistic classical world), the average of the background potential should approach zero. On the other hand, because Hamilton–Jacobi formalism contains fields ρ and S, we expect the background potential to be a functional of ρ and its derivatives. We are not interested in taking the potential to be a function of S and its derivatives, because we do not believe that it is a velocity-dependent potential. This potential does not affect particles via a special charge and we can only detect its effects through the changes in the distrubution of matter density. Thus, relation (26) takes the form

$$\partial_\mu S\partial^\mu S - m^2 - \chi[\rho, \partial_\mu\rho, \partial_\mu\partial_\nu\rho, \ldots] = 0. \tag{27}$$

But, we can rewrite the modified action as

$$\mathcal{A} = \int \rho\big(\partial_\mu S\partial^\mu S - m^2 - \chi[\rho, \partial_\mu\rho, \partial_\mu\partial_\nu\rho, \ldots]\big)\,\mathrm{d}^4x \tag{28}$$

and, it is suitable to define $\chi = m^2 Q$.

Now, we minimize the above action with respect to ρ, to find a correct form for the background potential as a function of ρ and its derivatives.

$$\delta_\rho\mathcal{A} = \delta_\rho \int \rho\big(\partial_\mu S\partial^\mu S - m^2 - \chi[\rho, \partial_\mu\rho, \partial_\mu\partial_\nu\rho, \cdots]\big)\,\mathrm{d}^4x = 0. \tag{29}$$

On the other hand, we can suggest a general form for the background potential by using physical arguments. Then, we put that general form in modified action relation and minimize it to get the correct form of the background potential. In the following, we suggest a form for the case of a flat space-time, without losing generality, and then explain our reasons for this choice.

$$\chi = \lambda(R)^m(\partial_\mu R)^n(\partial_\mu\partial^\mu R)^p = \lambda((\rho)^r)^m(\partial_\mu(\rho)^r)^n(\partial_\mu\partial^\mu(\rho)^r)^p. \tag{30}$$

The factor λ is a constant. The variable R is the amplitude of the background field. We assume that there is a relation of the form $\rho^r = R(x,t)$, between the amplitude of the background field and the distribution function of the matter field, and we obtain the value of r. As χ is a scalar under local Lorentz transformations, we cannot include terms such as $\partial_\mu\partial^\nu$ or $\partial_\mu\partial_\nu$ in eq. (30). Although it is possible to take higher order derivatives, we choose a simple form and investigate its consequences. The power n should be even, otherwise, χ cannot be a scalar quantity. The power p can be even or odd because the term in parenthesis is always a scalar. For the choice of power m, we use the vital assumption that the background potential does not decrease with distance necessarily. This assumption persuades us to take m to be a negative integer. In other words, if we multiply the density ρ by a real factor α, the value of the potential would not change at any point. This is what we expect from a background force that always accompanies a particle. By this demand, we are attributing non-local or wholeness properties to this background potential. Naturally, this condition becomes weak in purely relativistic classical situations. Thus

$$\chi[\rho' = \alpha\rho] = \chi[\rho] \Longrightarrow \alpha^{r(m+n+p)} = 1, \quad r \neq 0. \tag{31}$$

From this condition we obtain a helpful condition for selecting powers in relation (4):

$$m + n + p = 0, \tag{32}$$

where m is a negative integer, and n and p are positive integers. The simplest choice is $m = -1$, $n = 0$ and $p = 1$. This choice satisfies condition (32). Other choices are possible, but this is the simplest choice. In general, we can write such potentials in the Taylor expansion form that has been studied for the non-relativistic case in refs [17,18]. Using this choice, we have

$$\chi = \lambda\rho^{-r}\partial_\mu\partial^\mu\rho^r. \tag{33}$$

After expanding this relation, we get

$$\chi = \lambda\left(r(r-1)\frac{(\partial_\mu\rho)^2}{\rho^2} + r\frac{\partial_\mu\partial^\mu\rho}{\rho}\right). \tag{34}$$

Every term in parenthesis satisfies condition (32) separately. For the moment, we assume that space-time is flat, but this has no effect on our conclusion. By variation with respect to ρ, we get from eq. (28) for the flat space-time, the following equation:

$$\begin{aligned}&\left[\partial_\mu S\partial^\mu S - m^2 - \chi[\rho]\right]\\&\quad + \rho\frac{\partial\chi}{\partial\rho} - \partial_\mu\left(\rho\frac{\partial\chi}{\partial(\partial_\mu\rho)}\right) + \partial_\mu\partial^\mu\left(\rho\frac{\partial\chi}{\partial(\partial_\mu\partial^\mu\rho)}\right) = 0.\end{aligned} \tag{35}$$

The expression in the bracket vanishes because of eq. (27). Thus, we get

$$\rho\frac{\partial\chi}{\partial\rho} - \partial_\mu\left(\rho\frac{\partial\chi}{\partial(\partial_\mu\rho)}\right) + \partial_\mu\partial^\mu\left(\rho\frac{\partial\chi}{\partial(\partial_\mu\partial^\mu\rho)}\right) = 0. \tag{36}$$

By substituting eq. (34) into this equation, and after some calculations, we get

$$\frac{\partial_\mu\partial^\mu\rho}{\rho} + 2(r-1)\frac{\partial_\mu\partial^\mu\rho}{\rho} = 0 \Longrightarrow r = \frac{1}{2}. \tag{37}$$

This is an interesting result, because we shall have $\rho = R^2$, which reminds us of the statistical aspects of quantum mechanics. We introduced the function ρ as the density function of the matter field and we did not attribute statistical features to it from the beginning. Now, the modified Hamilton–Jacobi equation (27) takes the form

$$\partial_\mu S\partial^\mu S - m^2 - \lambda\frac{\partial^\mu\partial_\mu\sqrt{\rho}}{\sqrt{\rho}} = 0. \tag{38}$$

If we take $\lambda = \hbar^2$, the last term in the above equation takes exactly the form of Bohmian quantum potential. It is possible to get higher-order terms of the background potential which cannot be obtained by Bohm's original method, i.e, by substituting the polar form of the guidance wave in the Schrödinger equation, but we shall not follow this way. We shall also deduce that this background field is the same as the guidance wave function in Bohmian approach.

The continuity equation in the quantum case has the same form as in the classical case, with the difference that S is now, according to eq. (38), related to ρ. The form of the background potential is similar to that of Bohm:

$$Q = \frac{\chi}{m^2} = \hbar^2\frac{\partial^\mu\partial_\mu\sqrt{\rho}}{m^2\sqrt{\rho}}. \tag{39}$$

Hereafter, we use the word 'quantum potential' instead of background potential. The interesting point in our approach is that this result is obtained by classical methods and we did not use any quantum mechanical postulates and it is capable of generalization to higher order of potentials. If we had obtained $r \neq 1/2$, we would have not obtained Bohmian form of quantum potential as a function of ρ and its derivatives in relation (39).

Our analysis can be done for a non-flat space-time. The Hamilton–Jacobi equation in a curved space-time is

$$g^{\mu\nu}\nabla_\mu S\nabla_\nu S = m^2. \tag{40}$$

After applying conformal transformation, we have

$$\nabla_\mu S\nabla^\mu S = m^2\Omega^2 = m^2(1 + \chi_{\text{curved}}). \tag{41}$$

We can write an action for a non-flat space-time as

$$\mathcal{A}_{\text{curved}} = \int \rho\left(\nabla_\mu S\nabla^\mu S - m^2 - \chi(\rho, \nabla_\mu\rho, \nabla_\mu\nabla_\nu\rho, \ldots)\right)\sqrt{g}\,\mathrm{d}^4x. \tag{42}$$

If we represent the integrand by $\mathcal{F}$, then through Euler–Lagrange method we get

$$\frac{\partial \mathcal{F}}{\partial \rho} - \nabla_\mu \left(\frac{\partial \mathcal{F}}{\partial (\nabla_\mu \rho)} \right) + \nabla_\mu \nabla^\mu \left(\frac{\partial \mathcal{F}}{\partial (\nabla_\mu \nabla^\mu \rho)} \right) = 0 \tag{43}$$

and we shall have

$$\rho \frac{\partial \chi}{\partial \rho} - \nabla_\mu \left(\rho \frac{\partial \chi}{\partial (\nabla_\mu \rho)} \right) + \nabla_\mu \nabla^\mu \left(\rho \frac{\partial \chi}{\partial (\nabla_\mu \nabla^\mu \rho)} \right) = 0. \tag{44}$$

This is because for an action for matter in a non-flat space-time we have in general:

$$S = \int \sqrt{g} \mathrm{d}^4 x \mathcal{L}. \tag{45}$$

When we take variation with respect to the matter field, we do not consider the determinant $\sqrt{g}$. Now we propose the potential to be in the form

$$\chi = \lambda((\rho)^r)^m (\nabla_\mu (\rho)^r)^n (\nabla_\mu \nabla^\mu (\rho)^r)^p. \tag{46}$$

In the simplest form $m = -1$, $n = 0$ and $p = 1$ and when the potential is scalar, we are not worried about $(\nabla_\mu \nabla^\mu (\rho)^r)$, because it becomes

$$\frac{\partial_\mu \sqrt{g}}{\sqrt{g}} \partial^\mu (\rho^r) + \Box (\rho^r)$$

which is a scalar. Finally, this analysis, in its simplest form, leads to

$$\chi = \lambda \frac{\nabla_\mu \nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \tag{47}$$

which is the background potential, or the quantum potential, in a non-flat space-time.

4.1 *Obtaining background potential by the concept of averaging*

As at the large-scale level there is no conformal effects in our experience and physics behaves in a classical manner, we deduce that the average of the background potential should approach zero at large-scale dimensions.

$$\bar{\chi}(x)_{\text{volume}\to\infty} = \int_{\text{volume}\to\infty} \rho \chi(x) \sqrt{-g} \mathrm{d}^4 x \to 0, \tag{48}$$

The vanishing of the average of background force at large scales is also possible:

$$\bar{f}_{\mu\,\text{volume}\to\infty} = \int_{\text{volume}\to\infty} \sqrt{-g} \mathrm{d}^4 x \rho \partial_\mu \chi(x) \to 0. \tag{49}$$

We have deduced the relation of the background potential to the distribution function of background field and its derivatives, from our demand for the vanishing of the averages at large scales. We represent such a background field by $\psi(x)$ and it gives some non-classical features to the dynamics of the particle through a potential. We only use the idea that when volume or surface tends to large-scale dimensions, non-classical behaviours reduce and local or relativistic classic behaviours increase. Furthermore, we prefer to work with force instead of potential, because there may be circumstances in which the average of the potential at large scales is a constant, but the average of the force is zero. So, from eqs (49) and (33) we have

$$\bar{f}_\nu = \lambda \int \mathrm{d}^4 x \rho \partial_\nu \left(\frac{\partial_\mu \partial^\mu \rho^r}{\rho^r} \right) = 0 \tag{50}$$

and upon integration by parts, we have

$$\bar{f}_\nu = \left(\rho^{(1-r)} \partial_\mu \partial^\mu \rho^r \right)_{(\text{surface} \to \infty)} - \int \partial_\nu \rho \frac{\partial_\mu \partial^\mu \rho^r}{\rho^r} \mathrm{d}^4 x. \tag{51}$$

We expect that at large distances, background potentials behave like relativistic classical potentials, which affect in a restricted area. This means that $\rho \to 0$ when surface $\to \infty$. This has two results: first, the surface integral vanishes; second, $1 - r > 0$ or $r < 1$; so $0 < r < 1$.

At this stage, only the second integral remains, which we represent by I:

$$I = \int \mathrm{d}^4 x \frac{\partial_\mu \partial^\mu \rho^r}{\rho^r} \partial_\nu \rho = \int \mathrm{d}^4 x \frac{\partial_\mu (r \rho^{r-1} \partial^\mu \rho)}{\rho^r} \partial_\nu \rho(x). \tag{52}$$

Now, we evaluate this integral at large scales, and to perform integration by parts, we put

$$\mathrm{d}v = \partial_\mu (r \rho^{r-1} \partial^\mu \rho) \mathrm{d}^4 x, \tag{53}$$

$$u = \frac{\partial_\nu \rho}{\rho^r}. \tag{54}$$

Then, we have

$$I = \frac{r}{\rho} (\partial_\mu \rho)^2 - \int \mathrm{d}^4 x \rho^{r-1} \partial^\mu (r \rho^{-r} \partial_\mu \rho) \partial_\nu \rho = \frac{r}{\rho} (\partial_\mu \rho)^2 - I'. \tag{55}$$

As in the case of our previous discussion, we should consider $\rho \to 0$ at large scales, and this allows us to avoid ambiguity. Furthermore, by the use of L'Hopital's rule, we should impose the following condition:

$$\left((\partial_\mu \rho)_{(\text{surface} \to \infty)} \right) \to 0. \tag{56}$$

This condition is understandable, because one of the features of local or classical potentials is the vanishing of the derivative of the potentials at infinity. Now, by using Hopital theorem and differentiation with respect to ρ (not x^μ), we see that the limit of $\rho/(\partial_\mu \rho)^2$ tends to zero at large scales.

Now we evaluate the integral of (55), by comparing I' with eq. (52). This leads to

$$\rho^{-r} = \rho^{r-1} \Longrightarrow r = \frac{1}{2} \quad \text{and} \quad I = 0. \tag{57}$$

Thus, the vanishing of the average of background force at large scales occurs only for $r = 1/2$. If we had a non-zero average force, i.e. a background or quantum force at large scales, the value of r could not be $1/2$. On the other hand, we can deduce that if $r \neq 1/2$ and consequently $\rho \neq R^2$, quantum forces could be detectable at large-scale dimensions.

This means that the relation $\rho = R^2$ is the consequence of having only local effects at large-scale dimensions. From the mathematical point of view, any r can occur, but the physics we observe selects the value of r. One of the advantages of our approach is the appearance of quantum and classical worlds together, i.e. we can claim that the local relativistic classical world is the average of a quantum world.

5. Energy considerations

From the relativistic classical mechanics, we have $\partial^\mu S \partial_\mu S = E^2 - \mathbf{p}^2$. Substituting this relation into eq. (38), we obtain

$$E^2 = \mathbf{p}^2 + m^2 + \hbar^2 \frac{\partial^\mu \partial_\mu R}{R} \tag{58}$$

or

$$E^2 = \mathbf{p}^2 + m^2(1 + Q). \tag{59}$$

Now, we have an important result for the energy of a particle moving on a generalized geodesic, where Q is a background or quantum potential. One of the important results in our approach is that quantum world is along the local world, which is obvious from relation (59) where the background term is added to the classical quantities m and $\mathbf{p}$ [19]. In the standard quantum mechanics, physical quantities are operators in a Hilbert space and their eigenvalues coincide with experimental results. The uncertainty that governs in the standard quantum mechanics appears here in a different context. The $\mathbf{p}$ that appeared in relation (59) is not the standard quantum mechanical operator $\hat{\mathbf{p}}$; rather, the quantum mechanical operator $\hat{\mathbf{p}}$ is hidden in the third term of relation (59).

As R is the amplitude of the background field, we can write R multiplied by a phase factor as $R\exp(if(x)) = \psi$, to obtain a polar form for $\psi(x)$. From eqs (38) and (21), we deduce that $f = S/\hbar$, because by this consideration, we can derive the famous Klein–Gordon equation by using eqs (38) and (21) and $\psi(x) = R\exp(i(S/\hbar))$:

$$(\Box + m^2)\psi(x) = 0. \tag{60}$$

In fact, this is the equation of evolution of the background field ψ which guides spinless relativistic particles with mass m. Although we admit that the physical reality of the background field ψ is not shown to us, the belief in determinism and causality is more appealing to us than the belief in an inherent uncertainty and disorder in the Universe. This means that we have obtained Bohmian form of wave function $\psi = R\mathrm{e}^{iS/\hbar}$ in a new way, i.e. it is not taken as a postulate from the beginning.

From classical mechanics, we can write $S = \mathbf{p} \cdot \mathbf{x} - Et$ for a plane wave, where S is the action or Hamilton's principal function. But, from the aforementioned arguments,

$$\psi = R\exp\left(\frac{iS}{\hbar}\right) = R\exp\left(\frac{i(\mathbf{P} \cdot \mathbf{x} - Et)}{\hbar}\right) = R\exp(ip_\mu x^\mu).$$

On the other hand, for the argument of a plane wave we have: $i(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Thus, we conclude that

$$E = \hbar\omega \quad \text{and} \quad \mathbf{p} = \hbar\mathbf{k}.$$

These are the fundamental relations of quantum mechanics which we have obtained through our approach.

In the following, we demonstrate that the results of our approach is compatible with those of the standard quantum mechanics. This is done through an example in the non-relativistic limit.

In non-relativistic limit, the difference of total energy and rest energy, i.e. kinetic energy, is very small. Thus, by expanding relation (59) we get

$$K = \frac{\mathbf{p}^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R(x,t)}{R(x,t)}. \tag{61}$$

The second term of this equation can be seen as a background kinetic energy and is always non-zero, unlike the plane-wave case, where the boundaries are at infinity. In the standard quantum mechanics, this part of energy comes from the uncertainty principle. We consider a situation which is meaningless in the standard quantum mechanics, i.e, $\mathbf{p} = 0$. We bind such a particle, which is at rest, ($\mathbf{p} = 0$), by the conditions

$$R(x) = R(x+L) = 0 \quad \text{and} \quad \frac{\partial R}{\partial t} = 0.$$

Thus, background kinetic energy in one dimension is obtained from

$$K = -\frac{\hbar^2}{2mR(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}R(x). \tag{62}$$

According to the boundary conditions and Fourier theorem a suitable solution for R is $\sin(n\pi x/L)$. Thus

$$\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} = -\frac{n^2\pi^2}{L^2}R$$

and the background kinetic energy is

$$K = \frac{\hbar^2\pi^2 n^2}{2mL^2}, \quad n = 1, 2, 3, \ldots. \tag{63}$$

These are solutions of an infinite potential well in the standard quantum mechanics. So, unlike standard quantum mechanics, a particle can be at rest in a potential well and its energy can be produced by a background potential. Moreover, in relation (61) a classical term is added to a quantum mechanical term and this is a feature that does not appear in the standard approach of quantum mechanics. We have obtained this result by classical considerations and by the assumption of the existence of a background field, without using Schrödinger's equation.

The relativistic correction that is customary in the standard quantum mechanics can be seen here. From eq. (59) for a particle at rest, this correction is

$$K = -\frac{1}{2}\frac{\hbar^2\nabla^2 R}{m^2 R} + \frac{\hbar^4}{8}\frac{\nabla^2(\nabla^2 R)}{m^3 R}. \tag{64}$$

Now, by a comparison with the standard quantum mechanics and by taking $\nabla \propto \hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ is the momentum operator in the standard quantum mechanics, it is obvious that the second term is the relativistic correction in the standard quantum mechanics. Our aim of giving this example is not only to demonstrate the compatibility of results in the two approaches, but also to show that the quantum world is comprehensible if we impose conformal transformations on classical mechanics. This result was obtained by a background potential approach and we have not used Schrödinger's equation to obtain it. This shows that quantum potential is more fundamental than wave equation and that Bohm's original approach to obtain quantum potential is not the only possible way. Thus criticisms about the fictitious nature of quantum potential is removed by our approach.

6. Conclusion

In fact, our research contains two parts: (1) highlighting the physical significance of conformal transformations and showing how these transformations lead us to a new concept, (2) deriving the effects of these transformations by variational method, which is stronger than that of Bohm's approach. As the dynamics of a particle is determined by geometry of space-time, we believe that the quantum behaviour of a particle is due to a special aspect of space-time and the transformations that can give quantum features to a metric are conformal transformations. In this way, we tried to show that conformal transformations are related to quantum mechanical behaviour of particles and can be seen as a new aspect of the geometry. But, this is only the first step and requires more effort. We started by using conformal transformation because it is not a coordinate transformation and transforms a metric to a new metric at every point by multiplying the metric by a point-dependent factor. By imposing this idea on particle geodesics, we found that there is a force that always accompanies the particle. Under these circumastances, we obtained the form of this potential through two approaches. First, by minimizing the classical Hamilton–Jacobi equation on which a conformal transformation had been exerted; and second, by considering the fact that at large scales we do not detect background or wholeness effects and the world behaves locally. Then, we obtained the Klein–Gordon equation and the important relations: $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$ through our approach. In obtaining the potential we did not use any quantum mechanical approach; rather, we started by conformal transformations and combined it with the relativistic classical mechanics. If $r \neq 1/2$, the part of the world which we consider at large scales could show quantum mechanical effects. By introducing a background potential, in a specific way, we can take care of wholeness, seen at the microlevel. This is a sort of ethereal potential.

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[19] Here by classical quantities we mean quantities that can be defined without considering background field $\psi(x)$. The quantity Q is defined by using the background field $\psi(x)$; But m and $\mathbf{p}$ are definable as in the classical world.