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# Moving potential for Dirac and Klein–Gordon equations

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**Abstract.** Using the Lorentz transformation, the Klein–Gordon and Dirac equations with moving potentials are reduced to one standard where the potential is time-independent. As application, the reflection and transmission coefficients are determined by considering the moving step with a constant velocity $v$. It has been found that $R \pm T = 1$ only at $x = v$t. The problem of massless (2+1) Dirac particle is also considerered.

**Keywords.** Klein–Gordon equation; Dirac equation; moving potential; moving step; reflection and transmission coefficients.

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## 1. Introduction

For a certain form of potential, in quantum mechanics, different methods were developed where the so-called special functions play central roles when the solutions are analytic. If, in addition the potentials are time-dependent, to obtain a solution becomes more difficult and the use of approximate solution becomes inevitable. Extensive effort has been made to obtain exact solutions of the time-dependent systems using different methods, for example: path integral [1,2], second quantization [3], Heisenberg representation [4], the standard method of separation of variables [5], the supersymmetry method [6] and invariant method [7]. However, certain time-dependent systems have been the subject of great interest especially those that admit invariants, and the exact solutions are obtained by using auxiliary equations and by determining a certain phase [8–11].

For relativistic systems with variable boundary conditions, it is known that there is a problem of separability of space and time. Note that the first model with a boundary condition variable studied in the context of cosmic radiation is due to Fermi [12] and according to our knowledge the mathematical treatment of relativistic problems with moving boundaries is given by Moore [13], the effect of acceleration of a single mirror on the production of particles has been considered in [14–16] and the case of two or more boundaries, one of which moves with a constant relativistic velocity, has been analysed in [17,18]. For non-relativistic quantum systems with moving boundary conditions, many researchers found the exact solution in some specific cases [19–21].

In this paper, we propose to study another relativistic time-dependent system: it is about the Klein–Gordon and Dirac particle submitted to a potential $V(x - vt)$ and which moves with a constant velocity $v$. For this, the Lorentz transformation $(x, t) \rightarrow (y, \tau)$ is used to recast the original problem to a problem with time-independent system and as an illustration, the moving step $V_0\theta(x - vt)$ is used to determine the transmission and reflection coefficients via density $\bar{\rho}$ and current $\bar{\mathcal{J}}$ derived in the new coordinate system $(y, \tau)$.

## 2. Method for $V(x - vt)$

### 2.1 *KG particle*

For particles of spin-0, the Klein–Gordon equation with the moving potential is given as

$$\left[\left(\frac{i\partial}{\partial t} - V(x - vt)\right)^2 + \frac{\partial^2}{\partial x^2} - m^2\right]\psi = 0, \tag{1}$$

where $v$ is the velocity. A standard approach is to transform the KG equation with moving potential to a similar equation but with a nonmoving potential and thus the solution of time-dependent problem can be deduced due to the Lorentz invariance. Let us use the Lorentz transformation $(x, t) \rightarrow (y, \tau)$ defined as follows:

$$y = \lambda(x - vt), \qquad \tau = \lambda(t - vx) \quad \text{with } \lambda = \frac{1}{\sqrt{1 - v^2}}. \tag{2}$$

We then obtain a new Klein–Gordon equation with a time-independent potential

$$\left[\left(i\frac{\partial}{\partial \tau} - \lambda V\left(\frac{y}{\lambda}\right)\right)^2 + \frac{\partial^2}{\partial y^2} - m^2\right]\bar{\psi} = 0, \tag{3}$$

after following the Lorentz transformation by another transformation on the solution $\psi \rightarrow \bar{\psi}$ defined by

$$\psi = \mathrm{e}^{-iv\lambda \int^y \mathrm{d}u V(u/\lambda)}\bar{\psi}, \tag{4}$$

in order to have the same form of potential $V$ in the KG equation.

Now, let us consider the continuity equations. In coordinate system $(y, \tau)$ we have

$$\frac{\partial}{\partial \tau}\bar{\rho} + \frac{\partial}{\partial y}\bar{\mathcal{J}} = 0, \tag{5}$$

where

$$\bar{\mathcal{J}} = \bar{\psi}^*\frac{\partial}{\partial y}\bar{\psi} - \bar{\psi}\frac{\partial}{\partial y}\bar{\psi}^* \quad \text{and} \quad \bar{\rho} = \bar{\psi}\frac{\partial}{\partial \tau}\bar{\psi}^* - \bar{\psi}^*\frac{\partial}{\partial \tau}\bar{\psi} - 2i\lambda V\left(\frac{y}{\lambda}\right)\left|\bar{\psi}\right|^2, \tag{6}$$

are respectively the current density (of charge) and the density (of charge). By using transformations (2), the passage to coordinate systems $(x, t)$ is simple and the continuity equations become

$$\frac{\partial}{\partial t}\left(\bar{\rho}+v\bar{\mathcal{J}}\right)+\frac{\partial}{\partial x}\left(\bar{\mathcal{J}}+v\bar{\rho}\right)=0, \tag{7}$$

i.e. of the form

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}\mathcal{J}=0. \tag{8}$$

Thus, the passage $(\rho, \mathcal{J}) \rightarrow \left(\bar{\rho}, \bar{\mathcal{J}}\right)$ is given by the relationships

$$\rho=\bar{\rho}+v\bar{\mathcal{J}} \quad \text{and} \quad \mathcal{J}=\bar{\mathcal{J}}+v\bar{\rho}. \tag{9}$$

## 2.2 *Dirac particle*

For a spin-$\frac{1}{2}$ particle that interacts with moving potential in (1+1) dimension, its movement is described by the following Dirac equation:

$$\left[\sigma^3\left(i\frac{\partial}{\partial t}-V(x-vt)\right)-\sigma^2\frac{\partial}{\partial x}-m\right]\psi=0. \tag{10}$$

With the same Lorentz transformation (2) as that used in the case of spin-0, we first obtain the following equation:

$$\left[\lambda\left(\sigma^3-iv\sigma^2\right)\left(i\frac{\partial}{\partial\tau}\right)-\sigma^3V\!\left(\frac{y}{\lambda}\right)+\lambda\left(i\sigma^2-v\sigma^3\right)\left(i\frac{\partial}{\partial y}\right)-m\right]\psi=0, \tag{11}$$

where the potential is time-independent but the equation does not have the same form yet as that of the initial equation. Then, with the help of rotation of an 'angle' $\varphi$ in the plane $(y, \tau)$ such as $\tanh\varphi = v$ and followed by the transformation on solution $\psi \rightarrow \bar{\psi}$ defined by

$$\psi=\mathrm{e}^{-iv\lambda\int^{y}\mathrm{d}u\theta(u)}S\Phi, \tag{12}$$

where

$$S=\mathrm{e}^{(\varphi/2)\gamma^0\gamma^1}=\cosh\frac{\varphi}{2}+\gamma^0\gamma^1\sinh\frac{\varphi}{2}, \tag{13}$$

is the rotation matrix, we obtain in the system$(y, \tau)$, the following Dirac equation:

$$\left[\sigma^3\left(i\frac{\partial}{\partial\tau}-\lambda V\!\left(\frac{y}{\lambda}\right)\right)-\sigma^2\frac{\partial}{\partial y}-m\right]\bar{\psi}=0, \tag{14}$$

which has the same form as that of the initial equation and where the potential is time-independent. In the coordinate system $(y, \tau)$, the expression of density $\bar{\rho}$ and current $\bar{\mathcal{J}}$ are given by

$$\bar{\rho}=\bar{\psi}^{+}\bar{\psi}, \qquad \bar{\mathcal{J}}=\bar{\psi}^{+}\sigma_x\bar{\psi}, \tag{15}$$

and the passage $(\rho, \mathcal{J}) \rightarrow \left(\bar{\rho}, \bar{\mathcal{J}}\right)$ remaining obviously the same as the one relating to the case of spin-0.

Let us examine our method by elaborating an explicit example. As application, let us choose the step potential $V(x-vt) = V_0\theta(x-vt)$. Thanks to the propriety of the function of Heaviside, the potential keeps the same form step $V(y/\lambda) = V_0\theta(y/\lambda) = V_0\theta(y)$ in the system $(y, \tau)$.

## 3. Application: Moving step

### 3.1 *KG particle*

In the system $(y, \tau)$ it is known that the solutions in the two regions are the following

$$\begin{cases} \bar{\psi}_{\mathrm{I}}(y, \tau) = \mathrm{e}^{-i\varepsilon\tau}\left[\mathrm{e}^{ipy} + \dfrac{p-q}{p+q}\mathrm{e}^{-ipy}\right], & \text{for } y < 0 \\ \bar{\psi}_{\mathrm{II}}(y; t) = \dfrac{2p}{p+q}\mathrm{e}^{-i(\varepsilon\tau - qy)}, & \text{for } y > 0 \end{cases}, \tag{16}$$

where

$$p^2 = \varepsilon^2 - m^2, \quad q^2 = (\varepsilon - \lambda_0)^2 - m^2 \quad \text{and} \quad \lambda_0 = \lambda V_0, \tag{17}$$

and with the definition of current, it is easy to see that the total current density when $y < 0$ is equal to

$$\bar{\mathcal{J}}_{\mathrm{I}} = 2ip\left(1 - \left|\frac{p-q}{p+q}\right|^2\right), \tag{18}$$

i.e. equal to the difference of incident current density $\bar{\mathcal{J}}_{\mathrm{inc}} = -2ip$ and reflected current density $\left|\bar{\mathcal{J}}_{\mathrm{ref}}\right| = 2ip\left|(p-q)/(p+q)\right|^2$, whereas when $y > 0$, the transmitted current is

$$\bar{\mathcal{J}}_{\mathrm{tra}} = i\left(q + q^*\right)\left|\frac{2p}{p+q}\right|^2 \mathrm{e}^{i(q-q^*)y}. \tag{19}$$

Note that the energy $\varepsilon$ in the system $(y, \tau)$ is the same or conserved for the incident, reflected and transmitted particles in the two regions I and II.

Using the expressions (6) and (16), the densities in regions I and II are respectively

$$\begin{cases} \bar{\rho}_{\mathrm{I}}(y) = 2i\varepsilon\left(1 + \left|\dfrac{p-q}{p+q}\right|^2 + \dfrac{p-q}{p+q}\mathrm{e}^{-2ipy} + \dfrac{p-q^*}{p+q^*}\mathrm{e}^{2ipy}\right), & \text{for } y<0 \\ \bar{\rho}_{\mathrm{II}}(y) = 2i\left|\dfrac{2p}{p+q}\right|^2(\varepsilon - \lambda_0), & \text{for } y>0 \end{cases}, \tag{20}$$

where the continuity

$$\bar{\rho}_{\mathrm{I}}(y=0) = \bar{\rho}_{\mathrm{II}}(y=0) \ \text{ at y} = 0$$

or

$$\varepsilon\left(1 + \left|\frac{p-q}{p+q}\right|^2 + \frac{p-q}{p+q} + \frac{p-q^*}{p+q^*}\right) = \left|\frac{2p}{p+q}\right|^2(\varepsilon - \lambda_0)$$

is satisfied.

Due to the invariance of KG, the solution in the system of coordinates$(x, t)$, can be deduced and its expression in the two regions is as follows:

$$\begin{aligned} \psi_{\rightarrow} = &\left\{\theta(vt - x)\left[\mathrm{e}^{i\lambda[-(\varepsilon+vp)t+(p+v\varepsilon)x]} + \left(\frac{p-q}{p+q}\right)\mathrm{e}^{-i\lambda[(\varepsilon-vp)t+(p-v\varepsilon)x]}\right]\right. \\ &\left. +\theta(x - vt)\left(\frac{2p}{p+q}\right)\mathrm{e}^{i\lambda[-(\varepsilon+v(q-v\lambda_0))t+(q+v(\varepsilon-\lambda_0))x]}\right\}. \end{aligned} \tag{21}$$

Note that the energy in the system $(x, t)$ is not the same or not conserved in regions I and II because we have a time-dependent problem. The current densities are

$$\begin{cases} \mathcal{J}_{\mathrm{I}} = \bar{\mathcal{J}}_{\mathrm{I}} + v\bar{\rho}_{\mathrm{I}} = 2i\left\{ p + v\varepsilon - \left|\dfrac{p-q}{p+q}\right|^2 (p - v\varepsilon) \right. \\ \qquad\qquad \left. + v\varepsilon\left[\dfrac{p-q^*}{p+q^*}\mathrm{e}^{2i\lambda p(x-vt)} + \dfrac{p-q}{p+q}\mathrm{e}^{-2i\lambda p(x-vt)}\right]\right\}, \\ \mathcal{J}_{\mathrm{II}} = \bar{\mathcal{J}}_{\mathrm{II}} + v\bar{\rho}_{\mathrm{II}} = i\left|\dfrac{2p}{p+q}\right|^2 \left[\left(q^* + q\right) + 2v\left(\varepsilon - \lambda_0\right)\right]\mathrm{e}^{i\lambda(q-q^*)(x-vt)} \end{cases} \tag{22}$$

where the current density in region I can be separated into two components of the incident and the reflected current densities as follows:

$$\bar{\mathcal{J}}_{\mathrm{inc}} = 2i(p + v\varepsilon)$$

and

$$\bar{\mathcal{J}}_{\mathrm{ref}} = -2i\left\{\left|\frac{p-q}{p+q}\right|^2 (p - v\varepsilon) - v\varepsilon\left[\frac{p-q^*}{p+q^*}\mathrm{e}^{2i\lambda p(x-vt)} + \frac{p-q}{p+q}\mathrm{e}^{-2i\lambda p(x-vt)}\right]\right\}. \tag{23}$$

Then, the reflection coefficient $\mathcal{R}$ and transmission coefficients $\mathcal{T}$ can be determined by distinguishing three cases for value of $V_0$.

(a) $V_0 < (\varepsilon - m)\sqrt{1 - v^2}$ , $q$ is real and positive

$$\begin{cases} \mathcal{R} = \left(\dfrac{p-q}{p+q}\right)^2 - \dfrac{2v\varepsilon}{p+v\varepsilon}\dfrac{p-q}{p+q}\left\{\dfrac{p-q}{p+q} + \cos(2\lambda p\,(x - vt))\right\} \\ \mathcal{T} = \left(\dfrac{2p}{p+q}\right)^2 \dfrac{q + v(\varepsilon - \lambda_0)}{p + v\varepsilon} \end{cases}, \tag{24}$$

we have only $\mathcal{R} + \mathcal{T}|_{x=vt} = 1 \forall v$. For $v = 0$, $\mathcal{R} = ((p-q)/(p+q))^2$, $\mathcal{T} = (q/p)\,(2p/(p+q))^2$, $\mathcal{R} + \mathcal{T} = 1$ is obviously satisfied.

(b) $(\varepsilon - m)\sqrt{1 - v^2} < V_0 < (\varepsilon + m)\sqrt{1 - v^2}$, $q = i\alpha$ is pure imaginary

$$\begin{cases} \mathcal{R} = 1 - \dfrac{2v\varepsilon}{p+v\varepsilon}\left\{1 + \dfrac{p^2-\alpha^2}{p^2+\alpha^2}\cos(2\lambda p\,(x - vt)) \right. \\ \qquad\qquad \left. - \dfrac{2p\alpha}{p^2+\alpha^2}\sin\left(2\lambda p\,(x - vt)\right)\right\} \\ \mathcal{T} = \dfrac{4vp^2}{p^2+\alpha^2}\dfrac{\varepsilon - \lambda_0}{p + v\varepsilon}\mathrm{e}^{-2\lambda\alpha(x-vt)} \end{cases}, \tag{25}$$

and we can verify that $\mathcal{R} + \mathcal{T}|_{x=vt} = 1$ is satisfied only at $x = vt$ and we remark that in this case $\mathcal{T} \neq 0$. For $v = 0$, we have $\mathcal{R} = 1$, $\mathcal{T} = 0$.

(c) $V_0 > (\varepsilon - m)\sqrt{1 - v^2}$, $q$ is real

$$\begin{cases} \mathcal{R} = \left(\dfrac{p+q}{p-q}\right)^2 - \dfrac{2v\varepsilon}{p+v\varepsilon}\dfrac{p+q}{p-q}\left\{\dfrac{p+q}{p-q} + \cos(2\lambda p(x - vt))\right\} \\ \mathcal{T} = \dfrac{q - v\left(\varepsilon - \lambda_0\right)}{p + v\varepsilon}\left(\dfrac{2p}{p-q}\right)^2 \end{cases}. \tag{26}$$

In this case, we can verify that $\mathcal{R} - \mathcal{T}|_{x=vt} = 1$ ( $\mathcal{R} > 1$ and $\mathcal{T} < 0$), i.e. this result is known: it exhibits non-classical behaviours. With the help of antiparticles, this result can be reinterpreted: consequently there is no paradox (Klein).

For $v = 0$, $\mathcal{R} = ((p+q)/(p-q))^2$ and $\mathcal{T} = (q/p)\,(2p/(p-q))^2$ we have $\mathcal{R} - \mathcal{T} = 1$ as it should be.

### 3.2 *Dirac particle*

In the case of Dirac particles with the moving step, it is easy to obtain in coordinate system $(y;\tau)$ the solution which is

$$\begin{cases} \bar{\psi}_{\mathrm{I}}(y;\tau) = \mathrm{e}^{-i\varepsilon\tau}\left[\begin{pmatrix} 1 \\ \frac{p}{\varepsilon+m}\end{pmatrix}\mathrm{e}^{ipy} + \dfrac{1-\kappa}{1+\kappa}\begin{pmatrix} 1 \\ \frac{-p}{\varepsilon+m}\end{pmatrix}\mathrm{e}^{-ipy}\right], & \text{for } y < 0 \\ \bar{\psi}_{\mathrm{II}}(y;\tau) = \dfrac{2}{1+\kappa}\begin{pmatrix} 1 \\ \frac{q}{\varepsilon-\lambda_0+m}\end{pmatrix}\mathrm{e}^{-i(\varepsilon\tau - qy)}, & \text{for } y > 0 \end{cases}, \tag{27}$$

where $\kappa = (q/p)((\varepsilon+m)/(\varepsilon-\lambda_0+m))$ and also the current densities in each region

$$\bar{\mathcal{J}}_{\mathrm{I}} = \frac{2p}{\varepsilon+m}\left(1-|B|^2\right) \quad \text{and} \quad \bar{\mathcal{J}}_{\mathrm{II}} = \frac{q+q^*}{(\varepsilon-\lambda_0)+m}|C|^2\,\mathrm{e}^{-i(q^*-q)y}, \tag{28}$$

with $B = (1-\kappa)/(1+\kappa)$ and $C = 2/(1+\kappa)$.

Note that the energy $\varepsilon$ in system $(y,\tau)$ is also conserved in regions I and II.

The current density in region I is the sum of the two current densities (incident +reflected)

$$\bar{\mathcal{J}}_{\mathrm{inc}} = \frac{2p}{\varepsilon+m}, \quad \bar{\mathcal{J}}_{\mathrm{ref}} = -\frac{2p}{\varepsilon+m}|B|^2. \tag{29}$$

As the density in the two regions are

$$\begin{cases} \bar{\rho}_{\mathrm{I}} = \left\{\left(1+\dfrac{p^2}{(\varepsilon+m)^2}\right)\left(1+|B|^2\right) + \left(1-\dfrac{p^2}{(\varepsilon+m)^2}\right)\left(B\mathrm{e}^{-2ipy} + B^*\mathrm{e}^{2ipy}\right)\right\} \\ \bar{\rho}_{\mathrm{II}} = \left\{1 + \left|\dfrac{q}{(\varepsilon-\lambda_0)+m}\right|^2\right\}|C|^2\,\mathrm{e}^{i(q-q^*)y} \end{cases}, \tag{30}$$

at $y = 0$, $\bar{\rho}_{\mathrm{I}}(y=0) = \bar{\rho}_{\mathrm{II}}(y=0)$, i.e. the continuity at $y = 0$ is well satisfied.

Let us return to the system of coordinates $(x,t)$. The current densities in the two regions $x < vt$ and $x > vt$, are

$$\begin{cases} \mathcal{J}_{\mathrm{I}} = \dfrac{2p}{\varepsilon+m}\left(1-|B|^2\right) + v\left\{\left(1+|B|^2\right)\left(1+\dfrac{p^2}{(\varepsilon+m)^2}\right)\right. \\ \qquad \left. + \left(1-\dfrac{p^2}{(\varepsilon+m)^2}\right)\left(B\mathrm{e}^{-2i\lambda p(x-vt)} + B^*\mathrm{e}^{2i\lambda p(x-vt)}\right)\right\}. \\ \mathcal{J}_{\mathrm{II}} = \left\{\dfrac{q+q^*}{(\varepsilon-\lambda)+m} + v\left(1+\left|\dfrac{q}{(\varepsilon-\lambda_0)+m}\right|^2\right)\right\} \\ \qquad \times |C|^2\,\mathrm{e}^{i\lambda(q-q^*)(x-vt)} \end{cases} \tag{31}$$

Current densities in region I can be separated into two components of the incident and the reflected current densities as follows:

$$\begin{cases} \mathcal{J}_{\rm inc} = \dfrac{2p}{\varepsilon+m} + v\left(1+\dfrac{p^2}{(\varepsilon+m)^2}\right) \\ |\mathcal{J}_{\rm ref}| = |B|^2\left(\dfrac{2p}{\varepsilon+m} - v\left(1+\dfrac{p^2}{(\varepsilon+m)^2}\right)\right) \\ -v\left(1-\dfrac{p^2}{(\varepsilon+m)^2}\right)\left(B\mathrm{e}^{-2i\lambda p(x-vt)} + B^*\mathrm{e}^{2i\lambda p(x-vt)}\right). \end{cases} \tag{32}$$

A simple calculation allows to obtain respectively the reflection coefficient and the transmission coefficient by considering the following cases:

(a) $V_0 < (\varepsilon - m)\sqrt{1-v^2}$ , $q$ is real positive

$$\begin{cases} \mathcal{R} = \dfrac{1-\kappa}{1+\kappa}\left\{\dfrac{1-\kappa}{1+\kappa} - \dfrac{2v\left[\frac{1-\kappa}{1+\kappa}\left(1+\frac{p^2}{(\varepsilon+m)^2}\right) + \cos(2\lambda p\,(x-vt))\left(1\frac{p^2}{(\varepsilon+m)^2}\right)\right]}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)}\right\} \\ \mathcal{T} = \dfrac{4}{(1+\kappa)^2}\dfrac{\frac{2q}{\varepsilon-\lambda_0+m} + v\left(1+\frac{q^2}{(\varepsilon-\lambda_0+m)^2}\right)}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)} \end{cases} \tag{33}$$

and we have $\mathcal{R}+\mathcal{T}|_{x=vt} = 1$ at $x = vt$. For $v = 0$, $\mathcal{R} = (1-\kappa)/(1+\kappa)$, $\mathcal{T} = 4\kappa/(1+\kappa)$ and the condition $\mathcal{R}+\mathcal{T} = 1$ is satisfied.

(b) $(\varepsilon - m)\sqrt{1-v^2} < V_0 < (\varepsilon + m)\sqrt{1-v^2}$, $q = i\beta$ is pure imaginary

$$\begin{cases} \mathcal{R} = 1 - 2v\left\{\dfrac{1+\frac{p^2}{(\varepsilon+m)^2}}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)} + \dfrac{1-\frac{p^2}{(\varepsilon+m)^2}}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)}\right. \\ \qquad \left.\times\left[\dfrac{1-\alpha^2}{1+\alpha^2}\cos(2\lambda p(x-vt)) - \dfrac{2\alpha}{1+\alpha^2}\sin(2\lambda p\,(x-vt))\right]\right\}, \\ \mathcal{T} = \dfrac{4v}{1+\alpha^2}\dfrac{1+\frac{\beta^2}{(\varepsilon-\lambda_0+m)^2}}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)}\mathrm{e}^{-2\beta\lambda(x-vt)} \end{cases} \tag{34}$$

with $\alpha = (\beta/p)[(\varepsilon+m)/(\varepsilon-\lambda_0+m)]$. We can verify that $\mathcal{R}+\mathcal{T} = 1$ at the boundary at $x = vt$. We note that the transmitted coefficient $T|_{x=vt} \neq 0$ and $R|_{x=vt} < 1$. For $v = 0$, $\mathcal{T} = 0$ and $\mathcal{R} = 1$.

(c) $V_0 > (\varepsilon + m)\sqrt{1-v^2}$, $q$ is real. The reflected and transmitted coefficients take the following form:

$$\begin{cases} \mathcal{R} = \left(\dfrac{1+\kappa}{1-\kappa}\right)^2 - 2v\dfrac{1+\kappa}{1-\kappa}\left\{\dfrac{1+\frac{p^2}{(\varepsilon+m)^2}}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)}\right. \\ \qquad \left. + \dfrac{1-\frac{p^2}{(\varepsilon+m)^2}}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)}\cos(2\lambda p\,(x-vt))\right\}. \\ \mathcal{T} = \dfrac{4}{(1-\kappa)^2}\dfrac{\frac{2q}{\varepsilon-\lambda+m} - v\left(1+\frac{q^2}{(\varepsilon-\lambda_0+m)^2}\right)}{\frac{2p}{\varepsilon+m} + v\left(1+\frac{p^2}{(\varepsilon+m)^2}\right)} \end{cases} \tag{35}$$

We have $\mathcal{R} - \mathcal{T}|_{x=vt} = 1$ and in this case there is a paradox which can be explained by the introduction of the antiparticle (positron). For $v = 0$, $\mathcal{R} = ((1+\kappa)/(1-\kappa))^2$, $\mathcal{T} = 4\kappa/(1-\kappa)^2$, $R - T = 1$.

Finally, the expression of the wave function can be written in compact form for the two regions

$$
\begin{aligned}
\psi_{\rightarrow} = \mathrm{e}^{(\varphi/2)\sigma^1} \Bigg\{ & \theta(vt-x) \Bigg[ \begin{pmatrix} 1 \\ \frac{p}{\varepsilon+m} \end{pmatrix} \mathrm{e}^{i\lambda[-(\varepsilon+vp)t+(p+v\varepsilon)x]} \\
& + \frac{1-\kappa}{1+\kappa} \begin{pmatrix} 1 \\ \frac{-p}{\varepsilon+m} \end{pmatrix} \mathrm{e}^{-i\lambda[(\varepsilon-vp)t+(p-v\varepsilon)x]} \Bigg] \\
& + \theta(x-vt) \frac{2}{1+\kappa} \begin{pmatrix} 1 \\ \frac{q}{\varepsilon-\lambda_0+m} \end{pmatrix} \\
& \times \mathrm{e}^{i\lambda[-(\varepsilon+v(q-v\lambda_0))t+(q+v(\varepsilon-\lambda_0))x]} \Bigg\}.
\end{aligned}
\tag{36}
$$

## 4. Conclusion

In this paper, it has been shown that the treatment of Klein–Gordon and Dirac equations for time-dependent potential is in general not direct. Also, a method which treats the problem of moving potential with a constant velocity $v$ is proposed. This method gives the Dirac and KG equations with a nonmoving potential by using the Lorentz transformations. As an illustration, the moving step has been considered and the reflection and transmission coefficients have been determined and the expressions obtained are functions of quantity $x - vt$. The main result is that we have obtained $\mathcal{R} \pm \mathcal{T} = 1$ only at the frontier which separates the two regions ($x = vt$) and this is because of the fact that the reflected current considered is the difference of total current of region I and the incident current ($\mathcal{J}_{\text{ref.}} = \mathcal{J}_{\text{I}} - \mathcal{J}_{\text{inc}}$).

However, we have only treated the problem at $(1+1)D$. For example, if we consider the movement of (2+1) massless relativistic particle of spin-$\frac{1}{2}$ with the moving step, described by the following Dirac equation:

$$
\left[\sigma^3\left(i\frac{\partial}{\partial t} - V_0\theta\,(r-vt)\right) - \sigma^2\frac{\partial}{\partial x} + \sigma^1\frac{\partial}{\partial y}\right]\Psi\,(t,x,y) = 0,
\tag{37}
$$

where $r = \sqrt{x^2+y^2}$ is the radial distance, the approach is as follows: first, it is appropriate for this form of potential to use the polar coordinates $(r, \varphi)$ and to modify $\Psi \rightarrow \Psi'$ as follows:

$$
\Psi = \frac{1}{\sqrt{r}}\mathrm{e}^{\sigma^2\sigma^1(\phi/2)}\Psi'
\tag{38}
$$

after a rotation of an angle $\phi$ around OZ, the Dirac equation then becomes

$$
\left[\sigma^3\left(i\frac{\partial}{\partial t} - V_0\theta\,(r-vt)\right) - \sigma^2\frac{\partial}{\partial r} + \sigma^1\frac{1}{r}\frac{\partial}{\partial \phi}\right]\Psi'\,(t,r,\phi) = 0.
\tag{39}
$$

Because of $\mathrm{e}^{i\ell\phi}$ ($\ell = 0, \pm1, \pm2, \ldots$), the operator $L_z = (1/i)(\partial/\partial\phi)$ is eliminated and then with a Lorentz transformation $(r,t) \rightarrow (R,\tau)$ and after a change, $\Psi' \rightarrow \bar{\Psi}$ is defined by

$$
\Psi' \rightarrow \mathrm{e}^{i\ell\phi - iv\lambda_0\int^R \theta(u)\mathrm{d}u}\mathrm{e}^{(i\varphi/2)\sigma^3\sigma^2}\bar{\Psi}\,(t,R)
\tag{40}
$$

with $\tanh\varphi = v$, the Dirac equation takes the following new form:

$$\begin{cases} \left[\sigma^3\left(i\dfrac{\partial}{\partial\tau} - \lambda_0\theta\,(R)\right) - \sigma^2\dfrac{\partial}{\partial R} + i\sigma^1 U_\ell\,(R,\tau)\right]\bar{\Psi}_\ell\,(t,R) = 0 \\ U_\ell\,(R - v\tau) = \dfrac{\ell}{\lambda\,|R - v\tau|} \end{cases}, \tag{41}$$

where now the step potential is time-independent by noting that there is apparition a time-dependant term of Coulomb type. Equation (40) cannot be separated because the function $U_\ell$ depends on $R$ and $\tau$.

However, for $v \ll 1$,

$$\frac{\ell}{\lambda\,|R - v\tau|} \simeq \frac{\ell}{\lambda\,|R|} + v\tau\frac{\ell}{\lambda R^2}(\text{with } \ell \neq 0),$$

the equation with step + Coulomb is soluble in principle, the time-dependent term $v\tau(\ell/\lambda R^2)$ can be considered as a perturbation.

In contrast, for $\ell = 0$, the equation

$$\left[\sigma^3\left(i\frac{\partial}{\partial\tau} - \lambda_0\theta(R)\right) - \sigma^2\frac{\partial}{\partial R}\right]\bar{\Psi}_{\ell=0} = 0, \tag{42}$$

is analytically soluble. Indeed, in system $(R, \tau)$ the solution is

$$\bar{\Psi}_{\ell=0} = \mathrm{e}^{-i\varepsilon\tau}\left[\theta(-R)\begin{pmatrix}1\\1\end{pmatrix}\mathrm{e}^{ipR} + \theta(R)\begin{pmatrix}1\\1\end{pmatrix}\mathrm{e}^{iqR}\right], \tag{43}$$

where $p^2 = \varepsilon^2$and $q^2 = (\varepsilon - \lambda_0)^2$ and in system $(r, \varphi, t)$ it becomes

$$\Psi_{\ell=0} = \frac{\mathrm{e}^{-i\lambda\varepsilon(t-vr)+(\varphi/2)}}{\sqrt{r}}\begin{pmatrix}\mathrm{e}^{i\phi/2}\\ \mathrm{e}^{-i\phi/2}\end{pmatrix}\left[\theta(vt - r)\,\mathrm{e}^{i\lambda p(r-vt)} + \theta\,(r + vt)\,\mathrm{e}^{i\lambda(q - v\lambda_0)(r-vt)}\right]. \tag{44}$$

Thus, we can see that $\mathcal{R} = 0$ and $\mathcal{T} = 1\ \forall v$.

## References


[1] R P Feynman and A R Hibbs, *Quantum mechanics and path integrals* (McGraw Hill, New York, 1965)
[2] L Chetouani, L Guechi and T F Hammann, *Phys. Rev. A* **40**, 1157 (1989)
[3] H G Oh, H R Lee, T F George and C I Um, *Phys. Rev. A* **39**, 5515 (1989)
[4] J Y Ji, J K Kim and S P Kim, *Phys. Rev. A* **51**, 4268 (1995)
[5] C J Efthimiou and D Spector, *Phys. Rev. A* **49**, 2301 (1994)
[6] B F Samsonov, *J. Phys. A: Math. Gen.* **33**, 591 (2000)
[7] H R Lewis Jr and W B Riesenfeld, *J. Math. Phys.* **10**, 1458 (1969)
[8] J G Hartley and J R Ray, *Phys. Rev. A* **24**, 2873 (1981)
[9] M Maamache, *Phys. Rev. A* **52**, 936 (1995)
[10] K H Yeon, C I Um and T F George, *Phys. Rev. A* **68**, 052108 (2003)
[11] A S de Castro and A de Souza Dutra, *Phys. Rev. A* **67**, 054101 (2003)
[12] E Fermi, *Phys. Rev.* **75**, 1169 (1949)

[13] G T Moore, *J. Math. Phys.* **11**, 2679 (1970)
[14] P C W Davies and S A Fulling, *Proc. R. Soc. London A* **356**, 237 (1977)
[15] L H Ford and A Vilenkin, *Phys. Rev. D* **25**, 2569 (1982)
[16] W R Walker, *Phys. Rev. D* **31**, 767 (1984)
[17] G Petrov, *Bulg. J. Phys.* **12**, 355 (1985)
[18] G Petrov, *Rev. Roum. Phys.* **34**, 453 (1989)
[19] S W Doescher and M H Rice, *Am. J. Phys.* **37**, 1246 (1969)
[20] A J Makowski and S T Dembinski, *Phys. Lett. A* **154**, 217 (1991)
[21] T Samura and M Ohmukai, arXiv:quant-ph/0411113v1 (2004)